Some (new) results on least-square Petrov-Galerkin projection for parametric PDEs

Tommaso Taddei, Inria Team MEMPHIS

Aria Seminar, May 23th 2024 (from Bordeaux)

Collaborators: C Farhat (Stanford), A Iollo (Inria)
H Telib (Optimad srl)

Inria

Institut de Mathématiques de Bordeaux

Objective: model order reduction (MOR) of parametric systems

MOR methods aim to reduce the *marginal* cost of the solution to the parametric problem: find $u_\mu \in \mathcal{X} \, : \, R_\mu(u_\mu, v) = 0 \;\; \forall \, v \in \mathcal{Y}$.

- $R_\mu : \mathcal{X} \to \mathcal{Y}'$ residual of the equations;
- $\mu \in \mathcal{P}$ vector of model parameters;
- $\mathcal{X}, \mathcal{Y}$ suitable Hilbert spaces defined on the domain Ω;
- $\mathcal{M} := \{u_\mu : \mu \in \mathcal{P}\} \subset \mathcal{X}$ solution manifold.

Objective: model order reduction (MOR) of parametric systems

MOR methods aim to reduce the *marginal* cost of the solution to the parametric problem: find $u_\mu \in \mathcal{X} \, : \, R_\mu(u_\mu, v) = 0 \;\; \forall \, v \in \mathcal{Y}$.

- $R_\mu : \mathcal{X} \to \mathcal{Y}'$ residual of the equations;
- $\mu \in \mathcal{P}$ vector of model parameters;
- $\mathcal{X}, \mathcal{Y}$ suitable Hilbert spaces defined on the domain Ω;
- $\mathcal{M} := \{u_\mu : \mu \in \mathcal{P}\} \subset \mathcal{X}$ solution manifold.

Aim of the talk:

1. review least-square Petrov-Galerkin (LSPG) projection;
2. discuss the choice of the weighting norm in residual minimization.

Limitations: focus on steady-state problems; little (or no) discussion on hyper-reduction, *a posteriori* error estimation.

Definitions and notation

We introduce the bases $\{\xi_i\}_{i=1}^N$ and $\{\nu_i\}_{i=1}^N$ of $\mathcal{X}$ and $\mathcal{Y}$.

Definitions and notation

We introduce the bases $\{\xi_i\}_{i=1}^N$ and $\{\nu_i\}_{i=1}^N$ of $\mathcal{X}$ and $\mathcal{Y}$.

- $u \in \mathcal{X}, v \in \mathcal{Y} \leftrightarrow \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, $u = \sum_{i=1}^{N} (\boldsymbol{u})_i \xi_i, v = \sum_{i=1}^{N} (\boldsymbol{v})_i \nu_i$;
- $F \in \mathcal{Y}' \leftrightarrow \boldsymbol{F} \in \mathbb{R}^N$, $F(v) = \boldsymbol{F}^\top \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in \mathbb{R}^N$;

We introduce the bases $\{\xi_i\}_{i=1}^N$ and $\{\nu_i\}_{i=1}^N$ of $\mathcal{X}$ and $\mathcal{Y}$.

- $u \in \mathcal{X}, v \in \mathcal{Y} \leftrightarrow \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, $u = \sum_{i=1}^N (\boldsymbol{u})_i \xi_i, v = \sum_{i=1}^N (\boldsymbol{v})_i \nu_i$;
- $F \in \mathcal{Y}' \leftrightarrow \boldsymbol{F} \in \mathbb{R}^N$, $F(v) = \boldsymbol{F}^\top \boldsymbol{v} \quad \forall \boldsymbol{v} \in \mathbb{R}^N$;
- $(\cdot,\cdot)_\mathcal{X}, (\cdot,\cdot)_\mathcal{Y} \leftrightarrow \boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{N\times N}$,
$$(u,v)_\mathcal{X} = \boldsymbol{v}^\top \boldsymbol{X} \boldsymbol{u}, \quad (w,z)_\mathcal{Y} = \boldsymbol{z}^\top \boldsymbol{Y} \boldsymbol{w};$$
- $R_\mu(u_\mu, v) = 0 \ \ \forall v \in \mathcal{Y} \leftrightarrow \boldsymbol{R}_\mu(\boldsymbol{u}_\mu) = 0$;
- $DR_\mu[u_\mu] : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \leftrightarrow \boldsymbol{J}_\mu[\boldsymbol{u}_\mu] \in \mathbb{R}^{N\times N}$.

Definitions and notation

We introduce the bases $\{\xi_i\}_{i=1}^N$ and $\{\nu_i\}_{i=1}^N$ of $\mathcal{X}$ and $\mathcal{Y}$.

- $u \in \mathcal{X}, v \in \mathcal{Y} \leftrightarrow \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, $u = \sum_{i=1}^N (\boldsymbol{u})_i \xi_i, v = \sum_{i=1}^N (\boldsymbol{v})_i \nu_i$;
- $F \in \mathcal{Y}' \leftrightarrow \boldsymbol{F} \in \mathbb{R}^N$, $F(v) = \boldsymbol{F}^\top \boldsymbol{v} \quad \forall \boldsymbol{v} \in \mathbb{R}^N$;
- $(\cdot,\cdot)_\mathcal{X}, (\cdot,\cdot)_\mathcal{Y} \leftrightarrow \boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{N\times N}$,
  $$(u,v)_\mathcal{X} = \boldsymbol{v}^\top \boldsymbol{X} \boldsymbol{u}, \quad (w,z)_\mathcal{Y} = \boldsymbol{z}^\top \boldsymbol{Y} \boldsymbol{w};$$
- $R_\mu(u_\mu, v) = 0 \ \ \forall v \in \mathcal{Y} \leftrightarrow \boldsymbol{R}_\mu(\boldsymbol{u}_\mu) = 0$;
- $DR_\mu[u_\mu] : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \leftrightarrow \boldsymbol{J}_\mu[\boldsymbol{u}_\mu] \in \mathbb{R}^{N\times N}$.

- The choice of $\boldsymbol{X}$ reflects the target norm of interest (L^2, H^1,...).
- The choice of $\boldsymbol{Y}$ depends on $\boldsymbol{X}$ and on the problem of interest.

topic of the talk

Relevant identities

- The dual norm of the functional F satisfies

$$\|F\|_{\mathcal{Y}'} := \sup_{v \in \mathcal{Y}} \frac{F(v)}{\|v\|_{\mathcal{Y}}} = \sqrt{\boldsymbol{F}^\top \boldsymbol{Y}^{-1} \boldsymbol{F}}.$$

- Given $F \in \mathcal{Y}'$, there exists a unique $f \in \mathcal{Y}$ s.t.

$$(f, v)_{\mathcal{Y}} = F(v) \;\; \forall\, v \in \mathcal{Y} \leftrightarrow \boldsymbol{f} = \boldsymbol{Y}^{-1} \boldsymbol{F}$$

f is called *Riesz representation* of F.

- Let $\mathcal{W} \subset \mathcal{Y}$ be a subspace with orthonormal basis $\psi_1, \ldots, \psi_m$.

1. $\|F\|_{\mathcal{W}'} := \sup_{v \in \mathcal{W}} \frac{F(v)}{\|v\|_{\mathcal{Y}}} = \|\boldsymbol{W}^\top \boldsymbol{F}\|_2$ with $\boldsymbol{W} = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_m]$.
2. $\|F\|^2_{\mathcal{W}'} + \inf_{v \in \mathcal{W}} \|f - v\|^2_{\mathcal{Y}} := \|F\|^2_{\mathcal{Y}'}$.

Least-square Petrov-Galerkin formulation

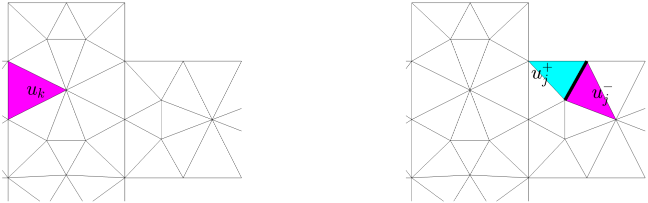
Preliminary (I): residual decomposition

Consider the FE mesh $\mathcal{T}_{\rm hf}$ of Ω with elements $\{\mathtt{D}_k\}_{k=1}^{N_{\rm e}}$, nodes $\{x_j^{\rm hf}\}_{j=1}^{N_{\rm v}}$ and facets $\{\mathtt{F}_j\}_{j=1}^{N_{\rm f}}$.

The residual can be decomposed as follows:

$$\mathfrak{R}(u,v) = \sum_{k=1}^{N_{\rm e}} r_k^{\rm e}(u_k, v_k) + \sum_{j=1}^{N_{\rm f}} r_j^{\rm f}(u_j^+, u_j^-, v_j^+, v_j^-),$$

where $\{r_k^{\rm e}\}_k$ and $\{r_j^{\rm f}\}_j$ are local operators.



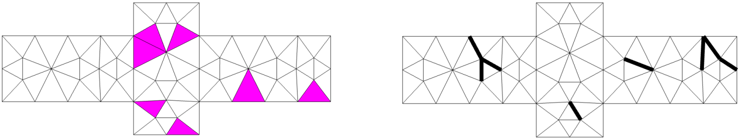
The decomposition is consistent with typical FE assembly routines.

Preliminary (II): weighted residual

Residual evaluation is accelerated by resorting to a reduced quadrature:

$$\mathfrak{R}_{\mu}^{\rm eq}(u,v) = \sum_{k=1}^{N_{\rm e}} \rho_k^{\rm e,eq} r_k^{\rm e}(u_k, v_k) \; + \; \sum_{j=1}^{N_{\rm f}} \rho_j^{\rm f,eq} r_j^{\rm f}(u_j^+, u_j^-, v_j^+, v_j^-),$$

where $\boldsymbol{\rho}^{\rm e,eq} \in \mathbb{R}_+^{N_{\rm e}}, \boldsymbol{\rho}^{\rm f,eq} \in \mathbb{R}_+^{N_{\rm f}}$ are sparse.



Refs: *reduced integration domain* (Ryckelynck, 2005); *mesh sampling and weighting* (Farhat et al., 2015); *empirical cubature* (Hernández et al., 2017); *empirical quadrature procedures* (Yano, Patera, 2019).

Remark: several alternatives are available (e.g., elementwise residual; pointwise residual).

We introduce the low-rank approximation: $\widehat{u}_\mu = \bar{u}_\mu + Z\widehat{\alpha}_\mu$.

- the offset $\bar{u}_\mu$ handles boundary conditions;
- the operator $Z = [\zeta_1, \ldots, \zeta_n] : \mathbb{R}^n \to \mathcal{X}$ is linear.

Our point of departure is the minimum residual formulation:

$$\min_{\alpha \in \mathbb{R}^n} \left(\sup_{v \in \mathcal{Y}} \frac{R_\mu(\bar{u}_\mu + Z\alpha, v)}{\|v\|_{\mathcal{Y}}} \right) = \min_{\alpha \in \mathbb{R}^n} \| \boldsymbol{Y}^{-1/2} \boldsymbol{R}_\mu (\bar{\boldsymbol{u}}_\mu + \boldsymbol{Z}\alpha) \|_2.$$

Refs: Maday et al., 2003; Yano, 2014 (*minimum residual MOR*); Grimberg et al., 2021 (*reduced quadrature rules*).

Minimum residual formulation

We introduce the low-rank approximation: $\widehat{u}_\mu = \bar{u}_\mu + Z\widehat{\alpha}_\mu$.

- the offset $\bar{u}_\mu$ handles boundary conditions;
- the operator $Z = [\zeta_1, \ldots, \zeta_n] : \mathbb{R}^n \to \mathcal{X}$ is linear.

Our point of departure is the minimum residual formulation:

$$\min_{\alpha \in \mathbb{R}^n} \left(\sup_{v \in \mathcal{Y}} \frac{R_\mu(\bar{u}_\mu + Z\alpha, v)}{\|v\|_{\mathcal{Y}}} \right) = \min_{\alpha \in \mathbb{R}^n} \| \mathbf{Y}^{-1/2} \mathbf{R}_\mu (\bar{\mathbf{u}}_\mu + \mathbf{Z}\alpha) \|_2.$$

- The ROM is online-efficient only for parametrically-affine problems.
- Application of reduced quadrature rules is possible if $\mathbf{Y}$ is diagonal.

Refs: Maday et al., 2003; Yano, 2014 (*minimum residual MOR*); Grimberg et al., 2021 (*reduced quadrature rules*).

We introduce two approximations:

1. we replace $\sup_{v \in \mathcal{Y}}(\cdot)$ with $\sup_{v \in \mathcal{W}}(\cdot)$ where $\mathcal{W} = \mathrm{span}\{\psi_i\}_{i=1}^{m}$.

$$\min_{\alpha \in \mathbb{R}^n} \left(\sup_{v \in \mathcal{W}} \frac{R_\mu(\bar{u}_\mu + Z\alpha, v)}{\|v\|_{\mathcal{Y}}} \right) = \min_{\alpha \in \mathbb{R}^n} \| \boldsymbol{W}^\top \boldsymbol{R}_\mu (\bar{\boldsymbol{u}}_\mu + \boldsymbol{Z}\alpha) \|_2$$

Approximate minimum residual formulation ([TT, Zhang, 2021])

We introduce two approximations:

1. we replace $\sup_{v \in \mathcal{Y}} (\cdot)$ with $\sup_{v \in \mathcal{W}} (\cdot)$ where $\mathcal{W} = \mathrm{span}\{\psi_i\}_{i=1}^m$.

$$\min_{\alpha \in \mathbb{R}^n} \left(\sup_{v \in \mathcal{W}} \frac{R_\mu(\bar{u}_\mu + Z\alpha, v)}{\|v\|_{\mathcal{Y}}} \right) = \min_{\alpha \in \mathbb{R}^n} \|\boldsymbol{W}^\top \boldsymbol{R}_\mu(\bar{\boldsymbol{u}}_\mu + \boldsymbol{Z}\alpha)\|_2$$

2. we replace the HF residual with the weighted residual

$$\min_{\alpha \in \mathbb{R}^n} \|\boldsymbol{W}^\top \boldsymbol{R}_\mu^{\mathrm{eq}}(\bar{\boldsymbol{u}}_\mu + \boldsymbol{Z}\alpha)\|_2 = \sum_{i=1}^m \left(R_\mu^{\mathrm{eq}}(\bar{u}_\mu + Z\alpha, \psi_i) \right)^2.$$

- The final ROM reads as a nonlinear least-square problem with m equations and n unknowns.
- The final ROM does not explicitly depend on $\boldsymbol{Y}$.

Choice of $\mathcal{W}$: linear analysis

Goal: exploit the linear analysis to inform the choice of $\mathcal{W}$.

Consider the problem: find $u^\star \in \mathcal{X} : \quad A(u^\star, v) = F(v) \qquad \forall\, v \in \mathcal{Y}$.

Define $\beta = \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \dfrac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$, $\gamma = \sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \dfrac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$.

Introduce $\mathcal{Z} = \mathrm{span}\{\zeta_j\}_{j=1}^{n} \subset \mathcal{X}$ and $\mathcal{W} = \mathrm{span}\{\psi_i\}_{i=1}^{m} \subset \mathcal{Y}$,

and the LSPG ROM: $\hat{u} = \arg\min_{u\in\mathcal{Z}} \sup_{v\in\mathcal{W}} \dfrac{A(u,v) - F(v)}{\|v\|_{\mathcal{Y}}}$.

Choice of $\mathcal{W}$: linear analysis

Goal: exploit the linear analysis to inform the choice of $\mathcal{W}$.

Consider the problem: find $u^\star \in \mathcal{X} : \quad A(u^\star, v) = F(v) \qquad \forall\, v \in \mathcal{Y}$.

Define $\beta = \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$, $\gamma = \sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$.

Introduce $\mathcal{Z} = \mathrm{span}\{\zeta_j\}_{j=1}^n \subset \mathcal{X}$ and $\mathcal{W} = \mathrm{span}\{\psi_i\}_{i=1}^m \subset \mathcal{Y}$,

and the LSPG ROM: $\hat{u} = \arg\min_{u\in\mathcal{Z}} \sup_{v\in\mathcal{W}} \frac{A(u,v) - F(v)}{\|v\|_{\mathcal{Y}}}$.

Given the space $\mathcal{Y}^\star = \mathrm{span}\left\{v_i : (v_i, v)_{\mathcal{Y}} = A(\zeta_i, v)\, \forall\, v \in \mathcal{Y}\right\}_{i=1}^n$,

we introduce $\delta_{n,m} = \inf_{s\in\mathcal{Y}^\star} \sup_{v\in\mathcal{W}} \frac{(s,v)_{\mathcal{Y}}}{\|s\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}}} = \inf_{s\in\mathcal{Y}^\star} \frac{\|\Pi_{\mathcal{W}} s\|_{\mathcal{Y}}}{\|s\|_{\mathcal{Y}}}$.

Choice of $\mathcal{W}$: linear analysis

Goal: exploit the linear analysis to inform the choice of $\mathcal{W}$.

Consider the problem: find $u^\star \in \mathcal{X}: \quad A(u^\star, v) = F(v) \qquad \forall\, v \in \mathcal{Y}$.

Define $\beta = \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \dfrac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$, $\gamma = \sup_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \dfrac{A(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}$.

Introduce $\mathcal{Z} = \mathrm{span}\{\zeta_j\}_{j=1}^n \subset \mathcal{X}$ and $\mathcal{W} = \mathrm{span}\{\psi_i\}_{i=1}^m \subset \mathcal{Y}$,

and the LSPG ROM: $\hat{u} = \arg\min_{u\in\mathcal{Z}} \sup_{v\in\mathcal{W}} \dfrac{A(u,v) - F(v)}{\|v\|_{\mathcal{Y}}}$.

Given the space $\mathcal{Y}^\star = \mathrm{span}\left\{ \upsilon_i : (\upsilon_i, v)_{\mathcal{Y}} = A(\zeta_i, v)\, \forall\, v \in \mathcal{Y} \right\}_{i=1}^n$,

we introduce $\delta_{n,m} = \inf_{s\in\mathcal{Y}^\star} \sup_{v\in\mathcal{W}} \dfrac{(s,v)_{\mathcal{Y}}}{\|s\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}}} = \inf_{s\in\mathcal{Y}^\star} \dfrac{\|\Pi_{\mathcal{W}} s\|_{\mathcal{Y}}}{\|s\|_{\mathcal{Y}}}$.

Error estimate: $\|\hat{u} - u^\star\|_{\mathcal{X}} \le \dfrac{\gamma}{\beta} \dfrac{1}{\delta_{n,m}} \inf_{u\in\mathcal{Z}} \|u - u^\star\|_{\mathcal{X}}$.

Choice of $\mathcal{W}$: implications of linear analysis

The space $\mathcal{W}$ should approximate $\left\{ \boldsymbol{Y}^{-1} \boldsymbol{J}_\mu[\boldsymbol{u}_\mu]\zeta_i \ : \ i = 1, \ldots, n \right\}$ for all $\mu \in \mathcal{P}$.

Practical algorithm: given the snapshots $\{u_\mu : \mu \in \mathcal{P}_{\rm train}\}$,

1. compute $\mathfrak{Y}_{\rm train} = \left\{ \boldsymbol{Y}^{-1} \boldsymbol{J}_\mu[\boldsymbol{u}_\mu]\zeta_i \ : \ i = 1, \ldots, n, \mu \in \mathcal{P}_{\rm train} \right\}$;
2. apply POD to $\mathfrak{Y}_{\rm train}$ with inner product $(\cdot, \cdot)_{\mathcal{Y}}$;
3. choose m based on an energy criterion.

Ref: Taddei, Zhang, 2021. **Related works:** Dahmen et al. M2AN, 2014; de Parga et al. CMAME, 2023.

Choice of $\mathcal{W}$: implications of linear analysis

The space $\mathcal{W}$ should approximate $\left\{ \boldsymbol{Y}^{-1} \boldsymbol{J}_{\mu}[\boldsymbol{u}_{\mu}] \zeta_i \ : \ i = 1, \ldots, n \right\}$ for all $\mu \in \mathcal{P}$.

Practical algorithm: given the snapshots $\{u_{\mu} : \mu \in \mathcal{P}_{\rm train}\}$,

1. compute $\mathfrak{Y}_{\rm train} = \left\{ \boldsymbol{Y}^{-1} \boldsymbol{J}_{\mu}[\boldsymbol{u}_{\mu}] \zeta_i \ : \ i = 1, \ldots, n, \mu \in \mathcal{P}_{\rm train} \right\}$;
2. apply POD to $\mathfrak{Y}_{\rm train}$ with inner product $(\cdot, \cdot)_{\mathcal{Y}}$;
3. choose m based on an energy criterion.

The space $\mathcal{W}$ is built during the offline stage.

$\Rightarrow$ The method can cope with arbitrary non-diagonal $\boldsymbol{Y}$.

Ref: Taddei, Zhang, 2021. **Related works:** Dahmen et al. M2AN, 2014; de Parga et al. CMAME, 2023.

A few references

Minimum residual: Maday et al., 2003.

Application to linear non-coercive problems; no hyper-reduction.

LSPG (a): Carlberg et al., 2013.

minimum residual + (D)EIM hyper-reduction; diagonal $\mathbf{Y}$.

Recently, Lindsay el al, 2022 considered weighted norms (very similar to this work!).

LSPG (b): TT, Zhang, 2021; de Parga et al., 2023.

Introduction of low-dimensional test space to cope with non-diagonal $\mathbf{Y}$.

Hyper-reduction based on reduced quadrature methods (as in Grimberg et al, 2021).

Choice of the test norm

Optimal norm

Linear problem: $u^\star \in \mathcal{X}: \quad A(u^\star, v) = F(v) \qquad \forall\, v \in \mathcal{Y}$.

LSPG ROM: $\hat{u} = \arg\min_{u \in \mathcal{Z}} \sup_{v \in \mathcal{W}} \dfrac{A(u,v) - F(v)}{\|v\|_{\mathcal{Y}}}$.

Error analysis: $\|\hat{u} - u^\star\|_{\mathcal{X}} \leq \boxed{\dfrac{\gamma}{\beta}} \dfrac{1}{\delta_{n,m}} \inf_{u \in \mathcal{Z}} \|u - u^\star\|_{\mathcal{X}}$.

Idea: choose $(\cdot,\cdot)_{\mathcal{Y}}$ to ensure $\frac{\gamma}{\beta} = 1$. $\Rightarrow$ $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}^{-1}\boldsymbol{A}^\top$.

Optimal norm

Linear problem: $u^\star \in \mathcal{X}: \quad A(u^\star, v) = F(v) \qquad \forall\, v \in \mathcal{Y}$.

LSPG ROM: $\hat{u} = \arg\min_{u \in \mathcal{Z}} \sup_{v \in \mathcal{W}} \dfrac{A(u,v) - F(v)}{\|v\|_{\mathcal{Y}}}$.

Error analysis: $\|\hat{u} - u^\star\|_{\mathcal{X}} \le \boxed{\dfrac{\gamma}{\beta}} \dfrac{1}{\delta_{n,m}} \inf_{u \in \mathcal{Z}} \|u - u^\star\|_{\mathcal{X}}$.

Idea: choose $(\cdot,\cdot)_{\mathcal{Y}}$ to ensure $\frac{\gamma}{\beta} = 1$. $\Rightarrow$ $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}^{-1}\boldsymbol{A}^\top$.

- The definition is based on the algebraic formulation

$\Rightarrow$ independent of the numerical scheme.

- The same idea has been exploited in several works.

PDE analysis: Lions, Magenes, "transposition method", 1972.

MOR: Dahmen et al, 2014; Brunken et al., 2019; Edel&Maday, 2023.

- The choice of $\boldsymbol{Y}$ cannot be extended to parametric problems.

Goal: determine the norm matrix $\boldsymbol{Y}$ for the nonlinear parametric problem: $\boldsymbol{R}_{\mu}(\boldsymbol{u}_{\mu}) = 0$.

1. **Natural norm:** $\boldsymbol{Y} = \overline{\boldsymbol{A}}\boldsymbol{X}^{-1}\overline{\boldsymbol{A}}^{\top}$, where $\overline{\boldsymbol{A}} = \boldsymbol{J}_{\bar{\mu}}[\boldsymbol{u}_{\bar{\mu}}]$.

2. **A priori choice:** exploit the variational formulation of the PDE to devise appropriate norms.

- *Incompressible flows:* H^1 norm for velocity, L^2 norm for pressure.
- *Conservation laws in divergence form* $\nabla \cdot F(u) = S(u)$*:* $\mathcal{Y} = H^1$.

Goal: determine the norm matrix $\boldsymbol{Y}$ for the nonlinear parametric problem: $\boldsymbol{R}_\mu(\boldsymbol{u}_\mu) = 0$.

1. Natural norm: $\boldsymbol{Y} = \overline{\boldsymbol{A}}\boldsymbol{X}^{-1}\overline{\boldsymbol{A}}^\top$, where $\overline{\boldsymbol{A}} = \boldsymbol{J}_{\bar{\mu}}[\boldsymbol{u}_{\bar{\mu}}]$.

Pros: exploits already-available tools of the code; provable optimality for $\mu = \bar{\mu}$.

Cons: expensive to evaluate.

2. A priori choice: exploit the variational formulation of the PDE to devise appropriate norms.

- *Incompressible flows:* H^1 norm for velocity, L^2 norm for pressure.
- *Conservation laws in divergence form* $\nabla \cdot F(u) = S(u)$*:* $\mathcal{Y} = H^1$.

Pro: less expensive to evaluate;

Cons: more involved implementation for non-standard formulations.

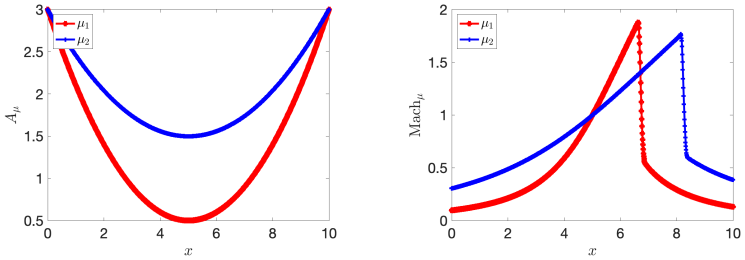
Numerical results

Model problem (I): nozzle flow

Define $u = \begin{bmatrix} A\rho \\ A\rho v \\ AE \end{bmatrix}$, $F(u) = \begin{bmatrix} A\rho v \\ A(\rho v^2 + p) \\ Au(E+p) \end{bmatrix}$ and

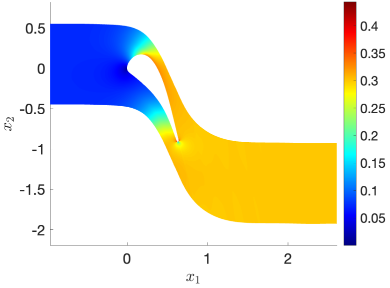
$S(u) = \begin{bmatrix} 0 \\ p\partial_x A \\ 0 \end{bmatrix}$ with $A(x) = 3 + 4(A_0 - 3)\frac{x}{L}\left(1 - \frac{x}{L}\right)$, $L = 10$.

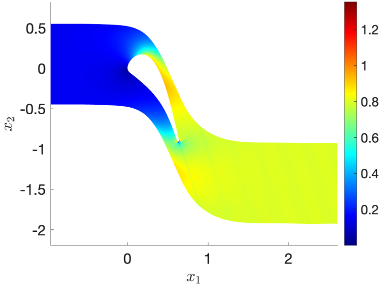
Consider the conservation law $\partial_x F(u) = S(u)$ in $(0, L)$ with parameters $\mu = [A_0, p_0]$.



Model: compressible Euler equations in transonic regime ($\gamma = 1.4$).

Parameters: distance between blades, far-field Mach number.

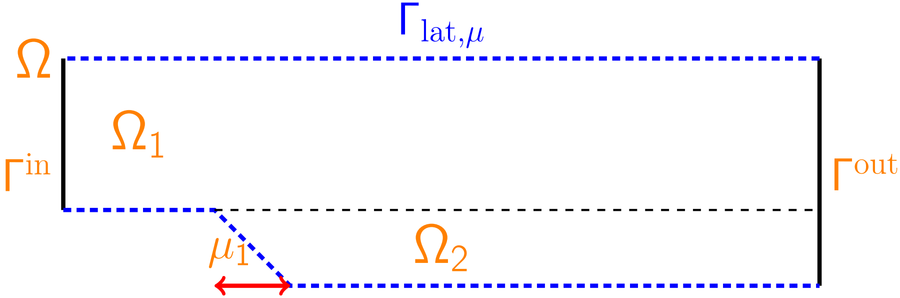




Consider the incompressible flow past a parametric backstep:

$$\begin{cases} -\nu \Delta v_\mu + (v_\mu \cdot \nabla) v_\mu + \nabla p_\mu = 0 & \text{in } \Omega_\mu \\ \nabla \cdot v_\mu = 0 & \text{in } \Omega_\mu \\ v_\mu|_{\Gamma_{\mathrm{in}}} = \mathrm{Re}\,\nu f_{\mathrm{pois}} e_1, \quad v_\mu|_{\Gamma_{\mathrm{lat},\mu}} = 0, \quad \frac{1}{\mu_2}\partial_n v_\mu - p_\mu n|_{\Gamma_{\mathrm{out}}} = 0 & \end{cases}$$

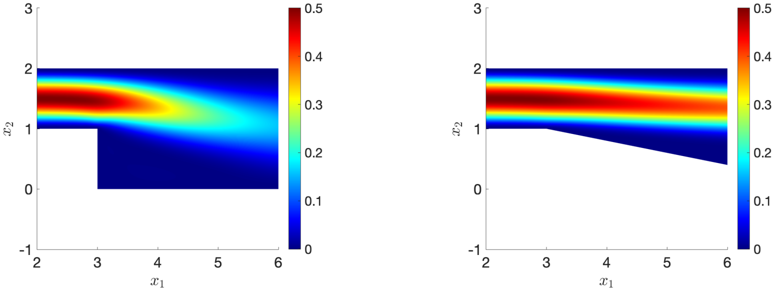
with $\nu = \frac{1}{250}$, $\mathrm{Re} \in [50, 450]$, f_{pois} parabolic Poiseuille flow, $\mu_1 \in [0, 5]$.



Consider the incompressible flow past a parametric backstep:

$$\begin{cases} -\nu\Delta v_\mu + (v_\mu \cdot \nabla) v_\mu + \nabla p_\mu = 0 & \text{in } \Omega_\mu \\ \nabla \cdot v_\mu = 0 & \text{in } \Omega_\mu \\ v_\mu|_{\Gamma_{\rm in}} = \mathrm{Re}\,\nu f_{\rm pois} e_1, \quad v_\mu|_{\Gamma_{{\rm lat},\mu}} = 0, \quad \frac{1}{\mu_2}\partial_n v_\mu - p_\mu n|_{\Gamma_{\rm out}} = 0 \end{cases}$$

with $\nu = \frac{1}{250}$, $\mathrm{Re} \in [50, 450]$, $f_{\rm pois}$ parabolic Poiseuille flow, $\mu_1 \in [0, 5]$.



Metrics

1. **Relative error:** $E_\mu = \frac{\|\widehat{u}_\mu - u_\mu\|_\mathcal{X}}{\|u_\mu\|_\mathcal{X}}$.
2. **Suboptimality index:** $\eta_\mu = \frac{\|\widehat{u}_\mu - u_\mu\|_\mathcal{X}}{\min_{\boldsymbol{\alpha}\in\mathbb{R}^n} \|\bar{u}_\mu + Z(\boldsymbol{\alpha}) - u_\mu\|_\mathcal{X}}$.
3. **Number of GNM iterations.**

1. **Relative error:** $E_\mu = \dfrac{\|\widehat{u}_\mu - u_\mu\|_{\mathcal{X}}}{\|u_\mu\|_{\mathcal{X}}}$.
2. **Suboptimality index:** $\eta_\mu = \dfrac{\|\widehat{u}_\mu - u_\mu\|_{\mathcal{X}}}{\min_{\boldsymbol{\alpha}\in\mathbb{R}^n} \|\bar{u}_\mu + Z(\boldsymbol{\alpha}) - u_\mu\|_{\mathcal{X}}}$.
3. **Number of GNM iterations.**

We consider $\mathcal{X} = L^2(\Omega)$ for compressible Euler test cases, and $\mathcal{X} = [H^1(\Omega)]^2 \times L^2(\Omega)$ for the flow past a backstep test case.

η_μ measures to what extent LSPG projection is suboptimal compared to the best-fit error.

Number of GNM iterations provides insights about the "difficulty" of the optimization problem.

Results (I): nozzle flow

L^2 error			
n_{train}	L^2	natural	H^1
5	0.0589	0.0302	0.0300
10	0.0387	0.0126	0.0128
15	0.0309	0.0071	0.0113
20	0.0336	0.0045	0.0149
25	0.0280	0.0029	0.0106
30	0.0356	0.0022	0.0080
35	0.0235	0.0012	0.0053
40	0.0305	0.0036	0.0072

suboptimality index			
n_{train}	L^2	natural	H^1
5	3.0	1.4	1.4
10	8.9	1.6	1.8
15	14.2	2.2	3.9
20	34.2	2.1	12.8
25	68.1	2.1	16.0
30	79.1	1.9	10.9
35	69.7	2.1	9.7
40	93.7	2.6	8.9

Results (I): nozzle flow

GNM iterations			
n_{train}	L^2	natural	H^1
5	6.8	10.3	7.6
10	6.4	9.2	8.2
15	6.6	21.7	6.9
20	9.6	8.8	6.5
25	7.4	9.7	8.9
30	9.4	8.6	14.0
35	9.6	6.9	8.2
40	7.6	7.2	9.5

Results (II): array of LS89 blades

L^2 error			
n_{train}	L^2	natural	H^1
5	0.0036	0.0009	0.0010
10	0.0007	0.0003	0.0004
15	0.0005	0.0002	0.0003
20	0.0003	0.0002	0.0002

suboptimality index			
n_{train}	L^2	natural	$\mathcal{X}$
5	4.9	1.0	1.2
10	3.2	1.1	1.7
15	3.1	1.1	1.6
20	2.7	1.1	1.5

GNM iterations			
n_{train}	L^2	natural	H^1
5	3.7	3.4	3.3
10	3.8	3.4	3.3
15	3.4	3.2	3.0
20	3.4	3.2	3.0

Results (III): flow past a backstep

$\mathcal{X}$ error			
n_{train}	L^2	natural	$\mathcal{X}$
5	0.2397	0.0666	0.1713
10	0.1029	0.0295	0.1076
15	0.0474	0.0128	0.0414
20	0.0125	0.0041	0.0149

suboptimality index			
n_{train}	L^2	natural	$\mathcal{X}$
5	6.9	1.4	4.1
10	6.1	1.3	5.8
15	7.2	1.3	5.1
20	3.7	1.2	4.0

GNM iterations			
n_{train}	L^2	natural	$\mathcal{X}$
5	9.2	6.8	6.6
10	7.8	6.4	6.6
15	9.9	5.0	7.3
20	5.8	4.3	5.6

Comments

- The choice of the weighting norm plays a major role in the performance of LSPG ROMs.
- Further experiments on finer grids suggest that the impact of the choice of the norm becomes more severe as we increase mesh resolution.
- The natural norm considerably outperforms the other methods for all test cases.
- For compressible flow test cases, the H^1 norm performs fairly well.

Snapping (I): presentation of the test

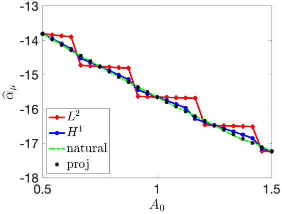
Definition: the solution to the LSPG ROM snaps for $\mu \in \mathcal{P}$ if there exists $\nu \in \mathcal{P}$ such that $u_\mu \neq u_\nu$ but $\widehat{u}_\mu = \widehat{u}_\nu$.

Informally, snapping occurs when the solution to the ROM is nearly piecewise-constant wrt the parameter.

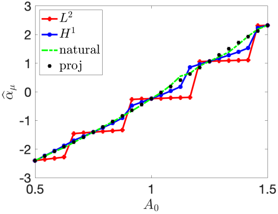
Snapping severely undermines the use of ROMs for parametric studies.

Numerical test: we consider the nozzle flow test case for $p_0 = 0.7$ and we let A_0 vary in $[0.5, 1.5]$. We consider $n_{\rm train} = n = 5$ and we study the sensitivity of the generalized coordinates with respect to A_0.

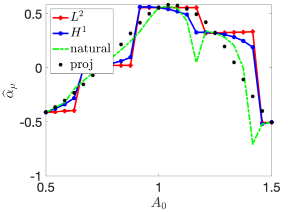
Snapping (II): results



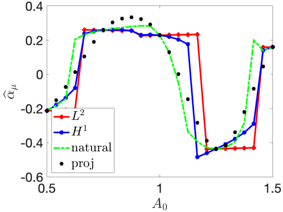
(a) $i = 1$



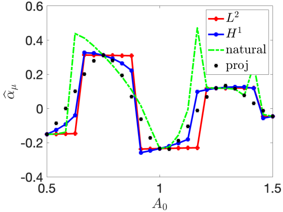
(b) $i = 2$



(c) $i = 3$

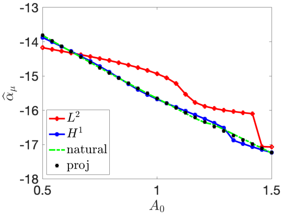


(d) $i = 4$

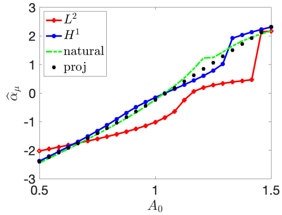


(e) $i = 5$

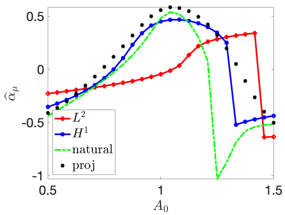
Snapping disappears if we consider a truncated basis ($n = 3 < n_{\mathrm{train}} = 5$); nevertheless, results of L^2 LSPG are very inaccurate.



(a) $i = 1$



(b) $i = 2$



(c) $i = 3$

Thank you for your attention!

For more information, visit the website:

math.u-bordeaux.fr/~ttaddei/ .

1. Taddei, Zhang; *Space-time registration-based model reduction of parameterized one-dimensional hyperbolic PDEs*, M2AN, 2021.

2. Farhat, Iollo, Taddei, Telib. *in preparation*.