IMACS

Ingénierie Mathématique
& Calcul Scientifique

rapid

Hybrid H-matrix — FMM solvers for wave equations

Toufic Abboud

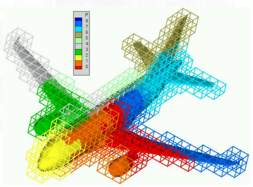
Conference in honor of Abderrahmane Bendali

Pau, 12-14 December 2017

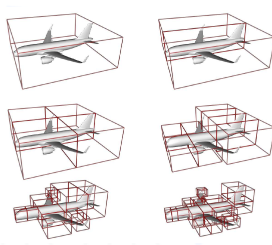
Merci Abderrahmane !

BEM

- Boundary Element Method, also known as Method of Moment (MoM) is the method of choice for many Electromagnetic problems
- Cost: assembly in $O(N^2)$, factorisation/solve in $O(N^3)$ complexity. In practice $N = O(f^2)$
- Fast solvers & HPC are mandatory
  - FMM algorithm & iterative solvers (Greengard and Rokhlin, 90's):
    - ✳ matrix-by-vector product in $O(N \log N)$ operations
    - ✳ issues: low frequency or locally refined meshes, ill-conditioned problems, big number of right hand sides (RHS)…



  - H-matrix direct solvers (Hackbusch 00's):
    - ✳ overall complexity $O(N (\log N)^2)$ complexity in the case of asymptotically smooth kernel (like Laplace's kernel)
    - ✳ absence of solid theoretical bases for the oscillating kernel until recently



Hi-BOX

Supported by a grant from **DGA**, the French Government Defence Procurement and Technology Agency, in the framework of **RAPID** program (dual innovation support scheme), under project No. 132906163.

Aims: a generic software library implementing state-of-the-art fast direct and iterative solvers for existing BEM-based codes

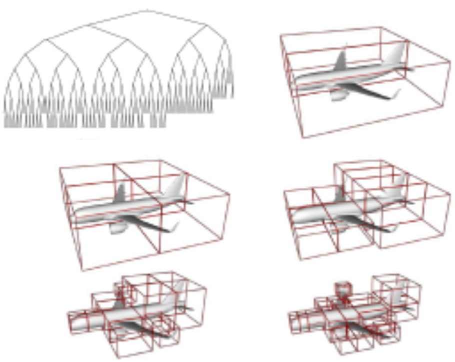
Leader: **IMACS**. Partners: **Airbus Group** and **Inria** (the French Institue for Research in Computer Science and Automation)

Focus on the local approximate rank of the Helmholtz kernel

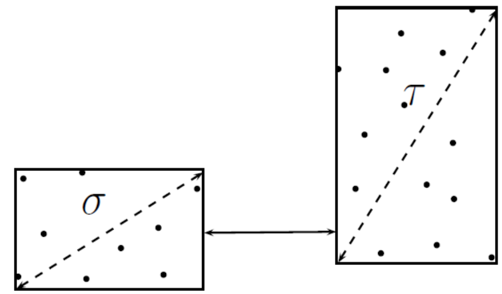
(Kieran Delamotte PhD thesis)

H-matrix

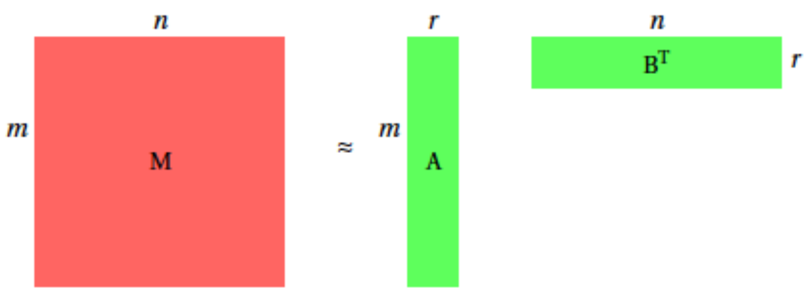
Cluster tree



Admissibility

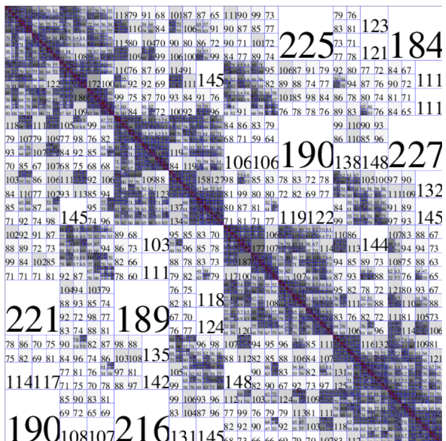


Compression (low rank approximation)



algebraic or analytic method

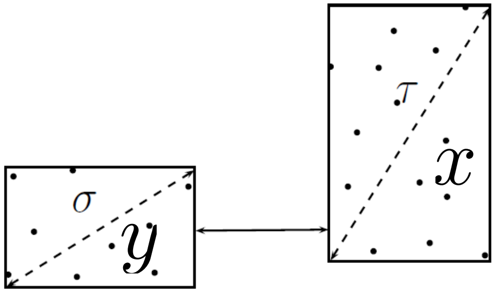
Hierarchical matrix



Compression (analytic method)

$$M_{ij} = \int_\Gamma \int_\Gamma G(x,y)\Phi_j(y)\Phi_i(x)\, d\Gamma(y) d\Gamma(x)$$

$$G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$



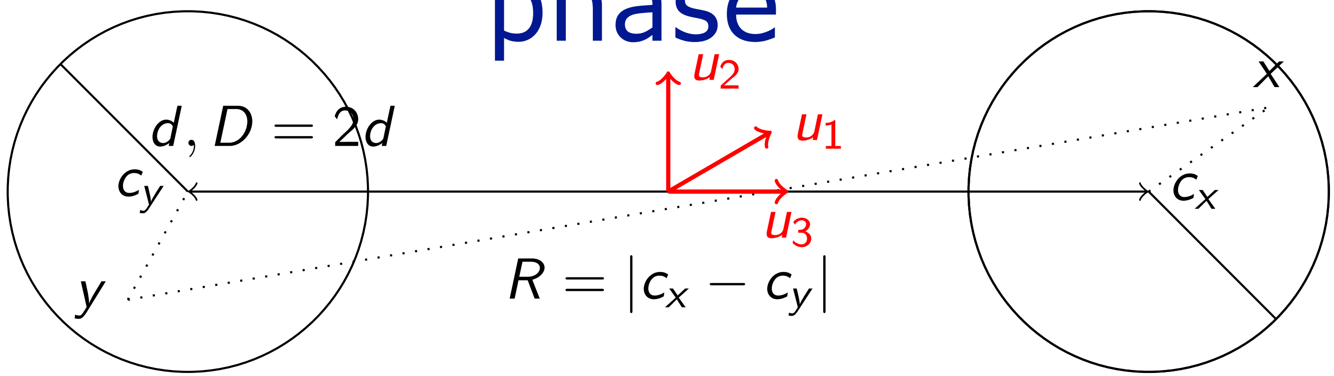
Low rank approximation of the Green's kernel

$$G(x,y) \approx \sum_{q=1}^{r} u_q(x) v_q(y)$$

➔ Low rank approximation of M

Justification for non asymptotically smooth kernels?

Taylor expansion of the phase



- D is the diameter of the clusters,
- R the distance between clusters

$$
\begin{aligned}
x &= \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3, \\
y &= \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3
\end{aligned}
$$

In the (u_1, u_2, u_3) basis the kernel reads as

$$G(x, y) = \frac{e^{ikR}}{|x-y|} e^{ik(\xi_3 - \eta_3)} e^{ik\frac{(\xi_1-\eta_1)^2}{2R}} e^{ik\frac{(\xi_2-\eta_2)^2}{2R}} e^{i\Phi}$$

with $\Phi = \mathcal{O}(k\frac{D^3}{R^2})$

Admissibility condition & rank growth

- <u>Classical static admissibility condition (Hackbusch *et*)</u>

$$R \geq \alpha D$$

  - this works well for 'low frequencies', i.e. if combined with :

$$(kD) \leq \beta$$

  - the phase term is bounded then the kernel has essentially the same rank as the Laplace one
  - *otherwise, we show that rank may grow like* k^2 *in the high frequency regime. The result would lead to a poor asymptotic compression ratio*

Admissibility condition & rank growth

- Fraunhofer admissibility condition (Engquist, Darve, ...)

$$R \geq \alpha \max((kD), 1)D$$

  - 2nd order terms are bounded
  - frequency-independant rank (plane waves);
  - too restrictive : clusters with big radii, *i.e.* (kD) »1, have to be far => small blocks with cst rank: *Divide but NOT Conquer* unless at very high frequencies.

New admissibility condition & rank control

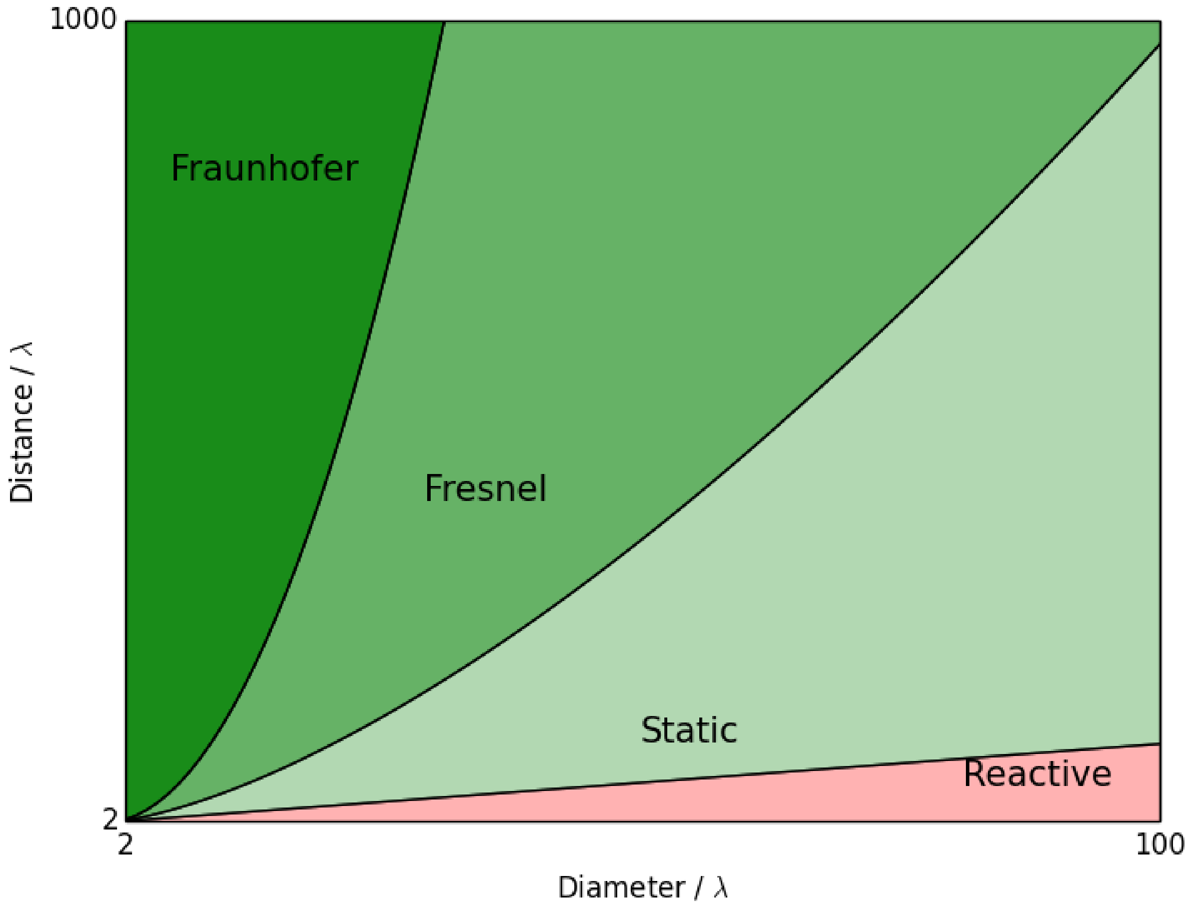
- Fresnel admissibility condition

$$R \geq \alpha \max((kD)^{\frac{1}{2}}, 1)D$$

  - *3rd order terms are bounded*
  - *less restrictive than Fraunhofer condition : bigger blocks with small rank*
  - *dominant phase terms are*

$$e^{ik\frac{(\xi_1-\eta_1)^2}{2R}} \, e^{ik\frac{(\xi_2-\eta_2)^2}{2R}}$$

Zoning



1000
2
2
100
Fraunhofer
Fresnel
Static
Reactive
Distance / λ
Diameter / λ

Fresnel or Fox-Li operator

- *2nd order phase terms : product of 1D operators*

$$e^{ik\frac{(\xi_1-\eta_1)^2}{2R}} = e^{ik\frac{\xi_1^2}{2R}} e^{ik\frac{\xi_1\eta_1}{R}} e^{ik\frac{\eta_1^2}{2R}}$$

- *we study the operator*

$$F_c : \lambda \mapsto [F_c\lambda](x) = \int_{-1}^{1} e^{icxt}\lambda(t)dt \qquad x \in [-1, 1]$$

- *operator studied in laser context by Fox and Li, in signal theory at Bell Labs by Slepian, Pollak, Landau, connected to results by Szegö and Widom*

SVD

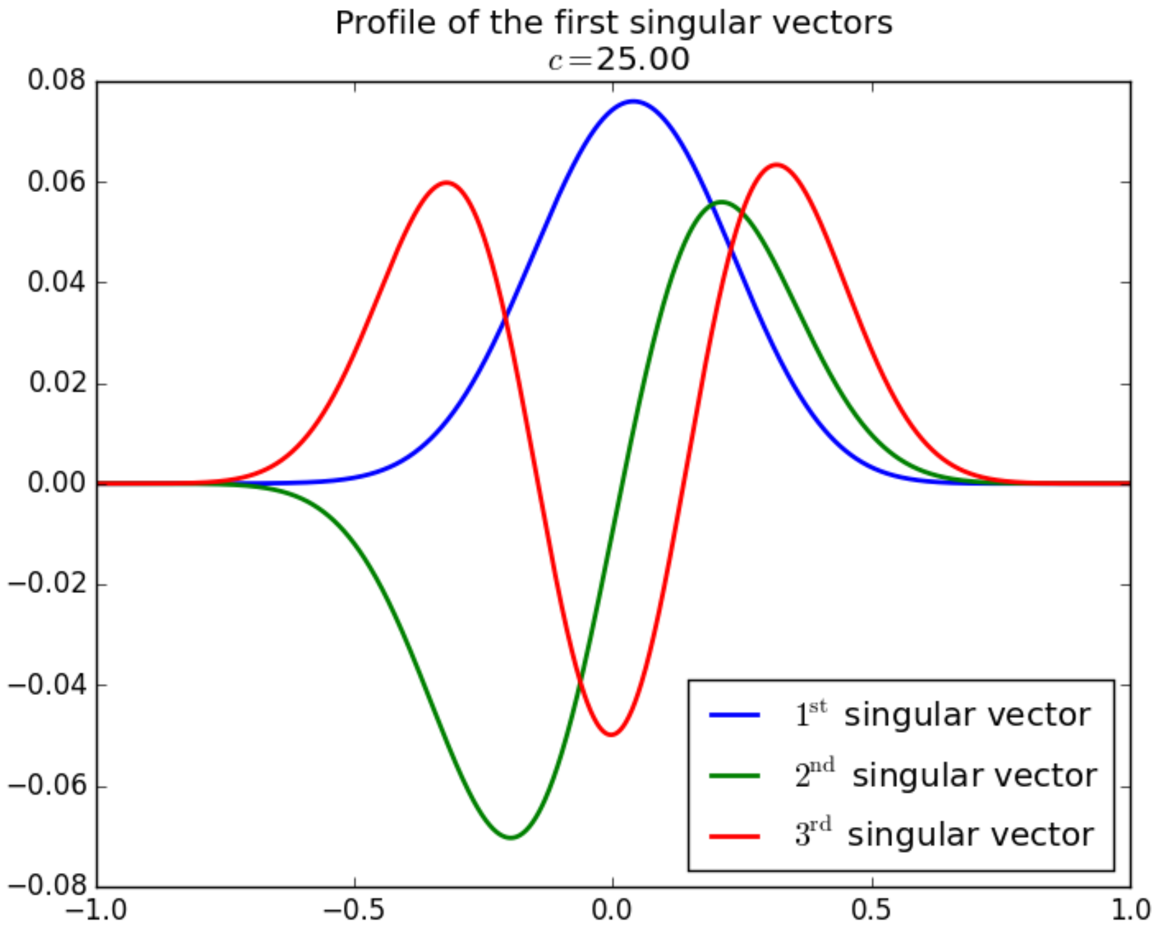
Singular values of F_c are the eigenvalues of Q_c:

$$Q_c = \frac{c}{2\pi} F_c^* F_c \qquad Q_c \phi(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c(x-t))}{x-t} \phi(t)\, dt$$

Eigenfunctions are the Prolate Spheroidal Wave Functions of order zero

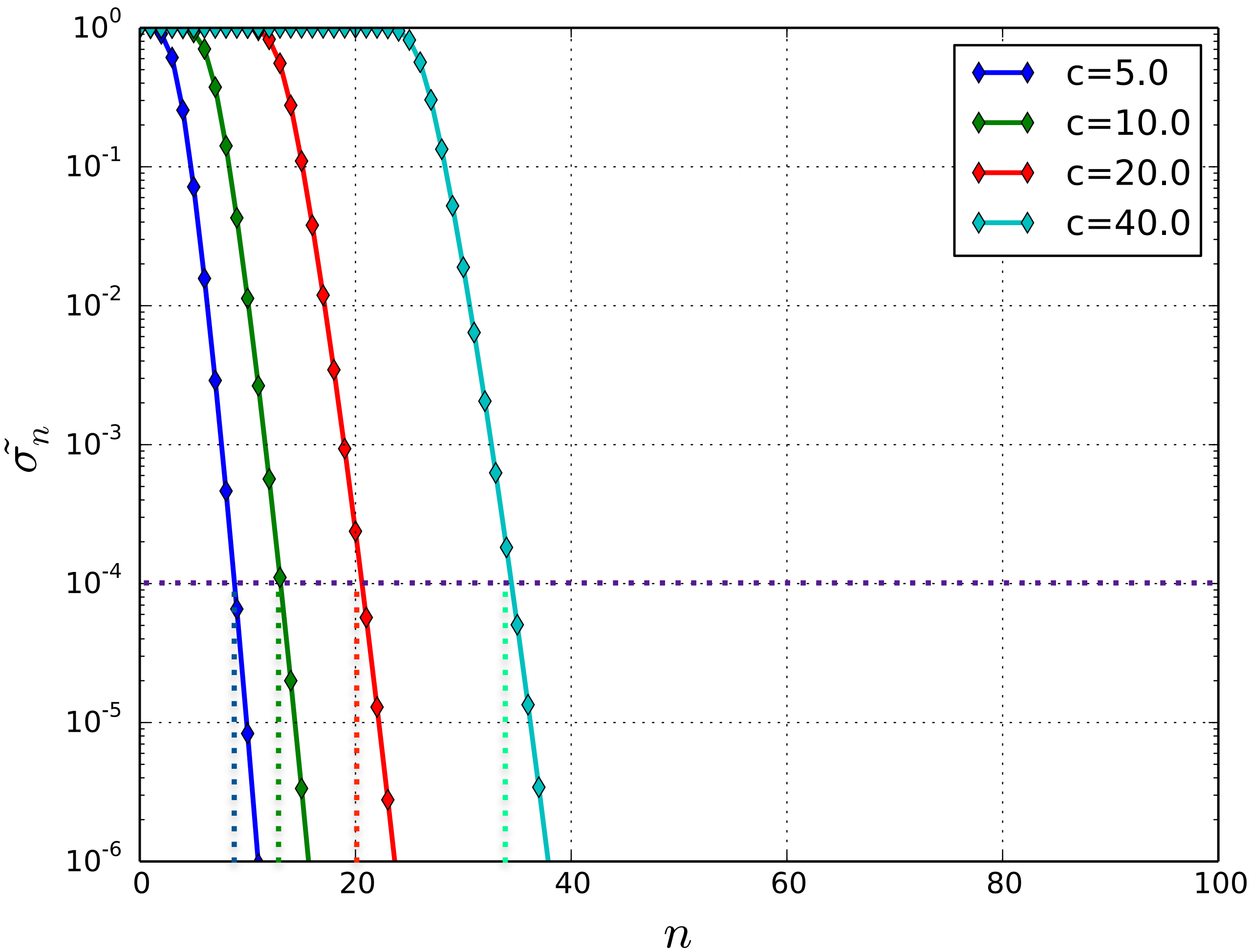
$$(1-x^2)\psi_n''(x) - 2x\psi_n'(x) + (\chi_n - c^2x^2)\psi_n(x) = 0$$

Singular vectors of F_c

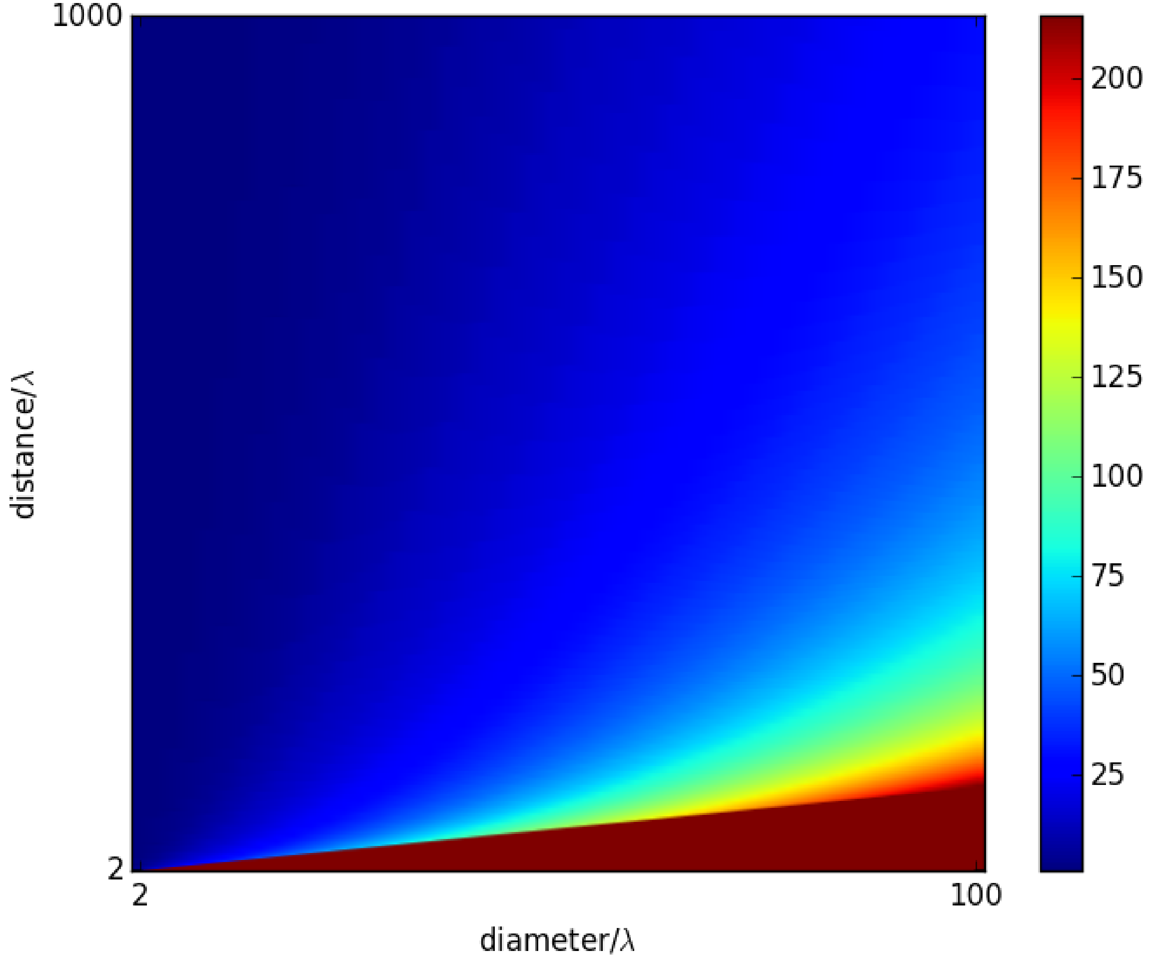


Prolate spheroidal wave functions of order zero

Singular values & ε-rank of F_c



ε-rank of F_c



Approximate rank estimation

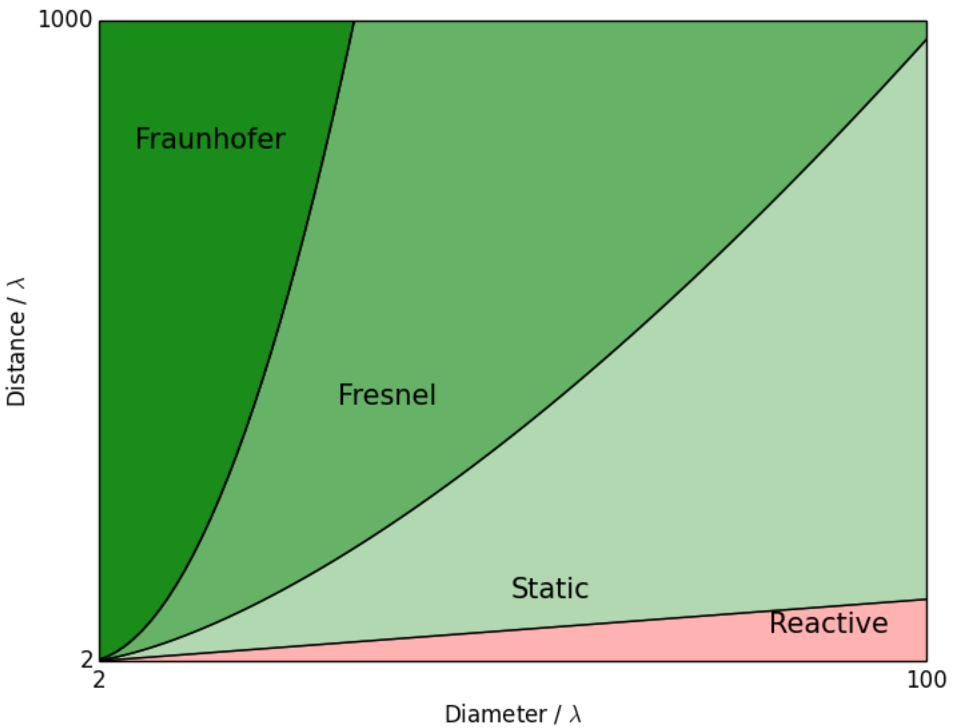
Theorem[Landau-Widom ('80)]
[Conjectured by Slepian ('65)]

$$N(c,\alpha) = \#\{n \geq 1 : \; \mu_n > \alpha\}$$
$$N(c,\alpha) = \frac{2c}{\pi} + \frac{1}{\pi^2}\log\left(\frac{1-\alpha}{\alpha}\right)\log(c) + o(\log(c))$$

That provides the *ε-rank of the Fresnel kernel:*

$$\varepsilon - \mathrm{rank}(F_c) \simeq \frac{2c}{\pi} + \frac{2}{\pi^2}|\log(\varepsilon)|\log(c)$$

Admissibility - rank



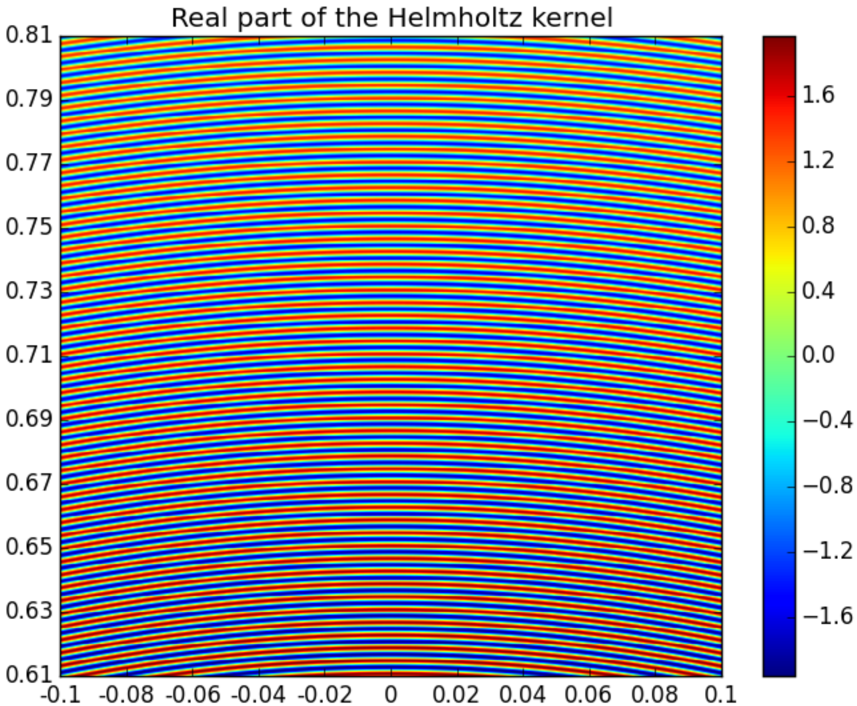
$$c = \frac{(kd)^2}{kR}$$

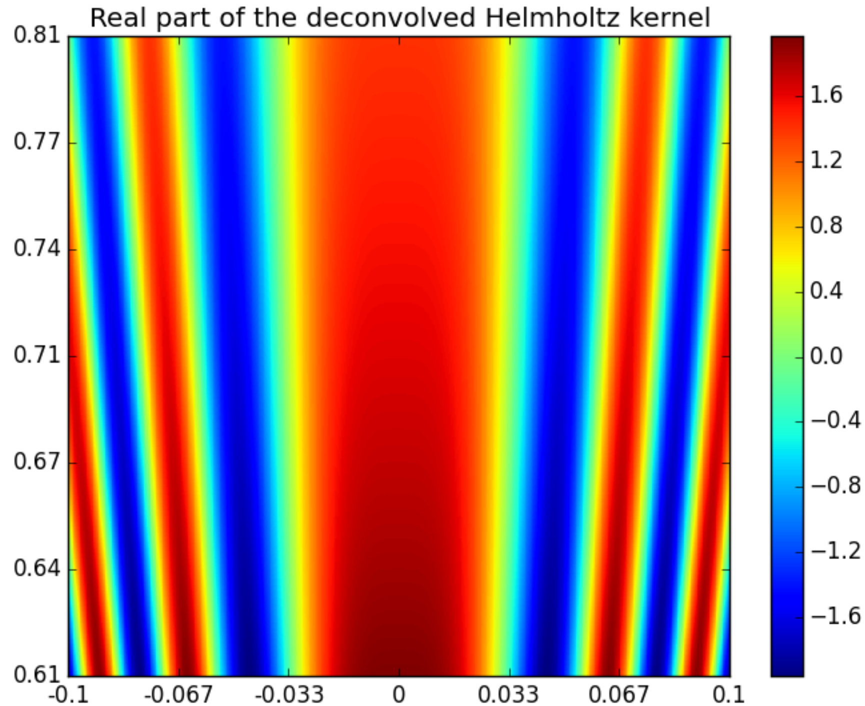
Hackbusch	$R \sim d$	$c \sim kd$
Fresnel	$R \sim k^{\frac{1}{2}} d^{\frac{3}{2}}$	$c \sim (kd)^{\frac{1}{2}}$
Fraunhofer	$R \sim kd^2$	$c \sim 1$

HCA2 method

- This provides an alternative to ACA algorithm
- HCA2: interpolation based technique (functional equivalent of ACA)
- Fresnel admissibility condition
- *Oriented boxes*
- *Number of interpolation points* chosen according to Landau-Widom theorem in (u_1,u_2) directions and like O(log **ε**) along u_3
- block rank may grow like $k^{1/2}$ or k depending on the cross section

Hybrid Cross Approximation HCA2





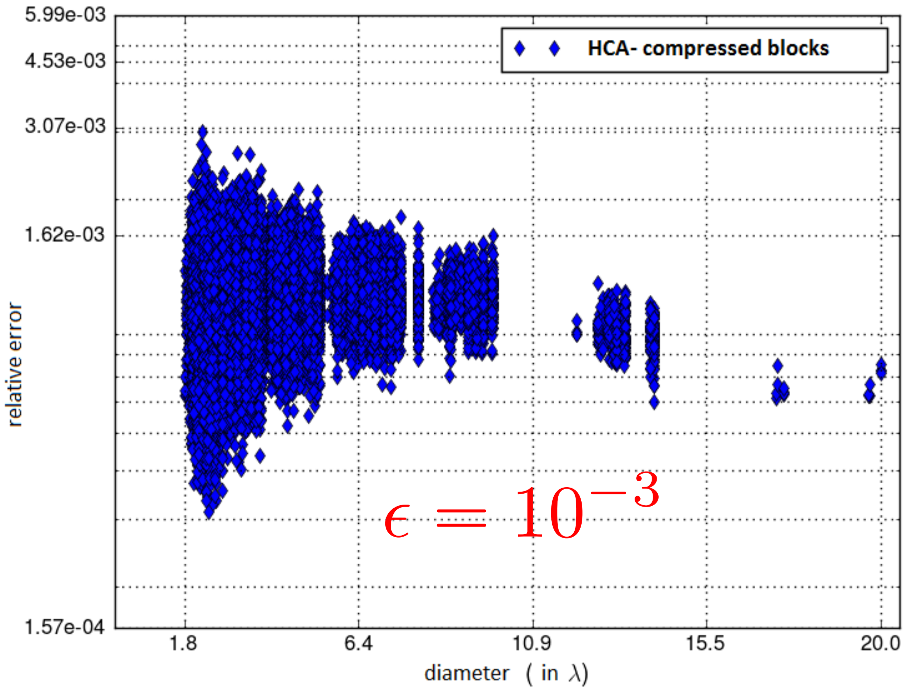
$$G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

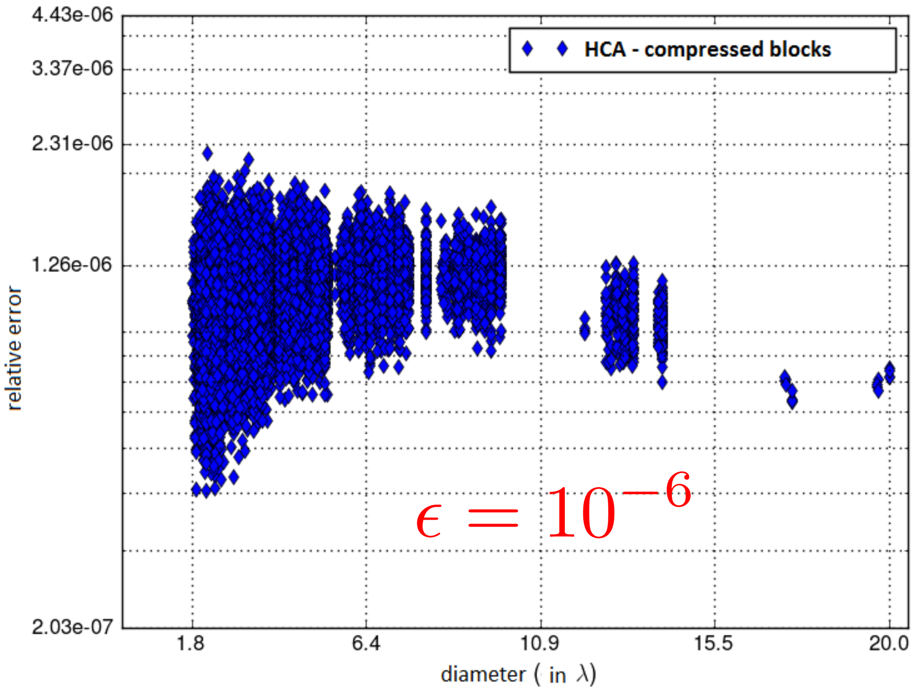
$$G_{\hat{u}_3}(h) = \frac{e^{ik(|x-y|-\hat{u}_3\cdot(x-y))}}{4\pi|x-y|}$$

- Fresnel admissibility condition
- Interpolation nodes chosen according to Landau-Widom theorem

- Control of the precision & of the growth of the block rank

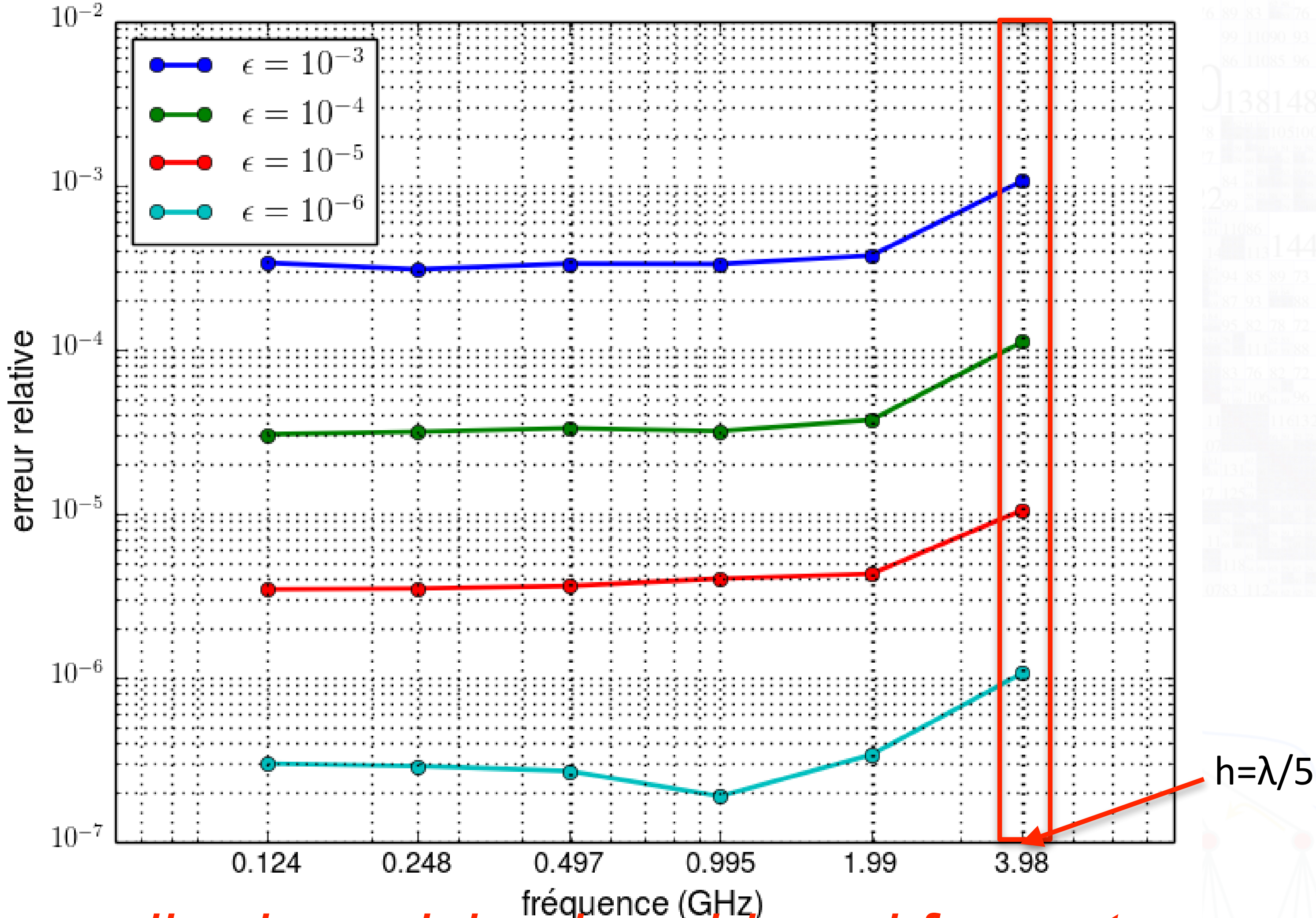
Block relative L^2 error





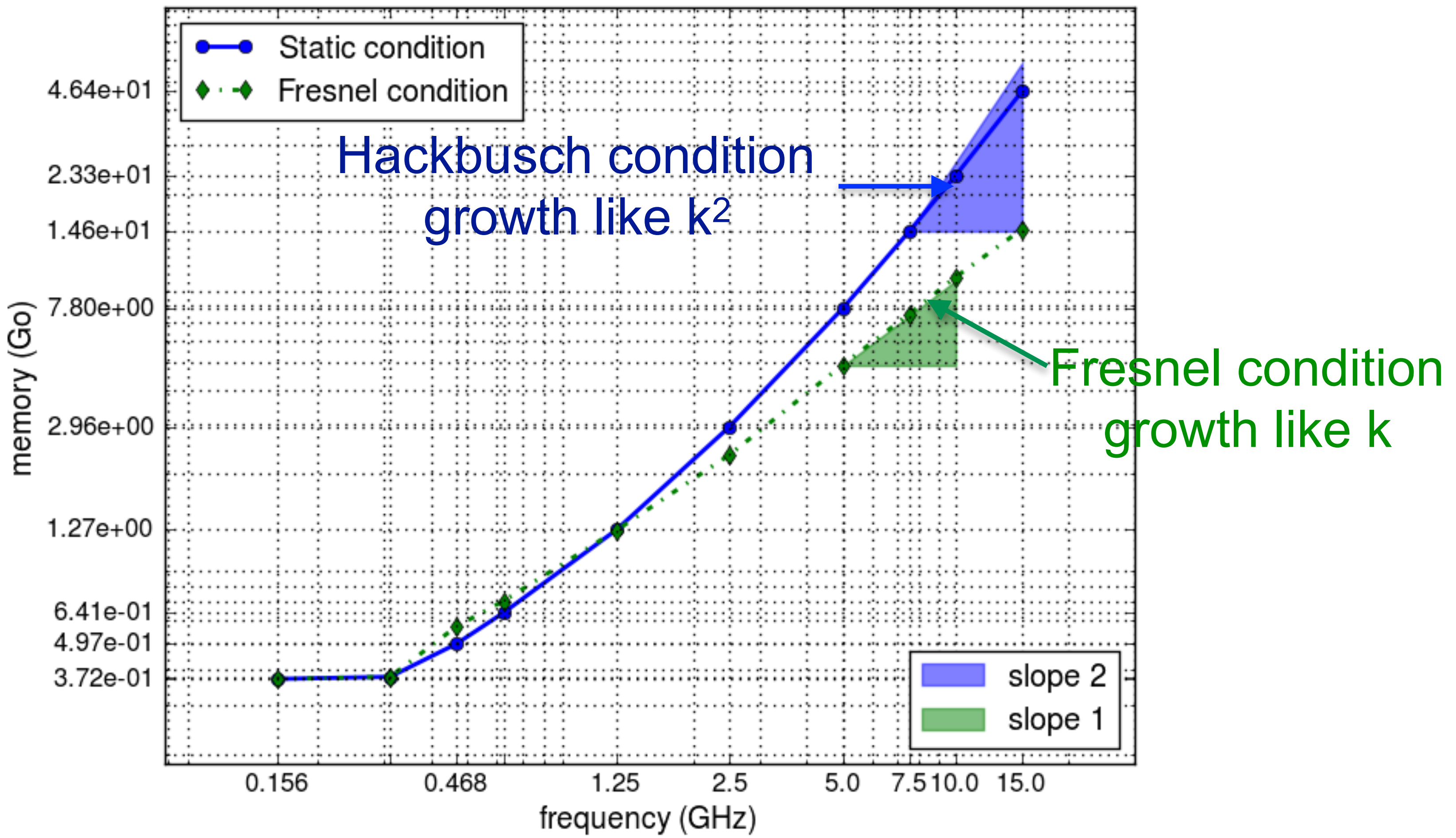
prescribed precision is achieved blockwise

Frequency stable error



prescribed precision is achieved for matvec

Influence of admissibility condition on the memory growth

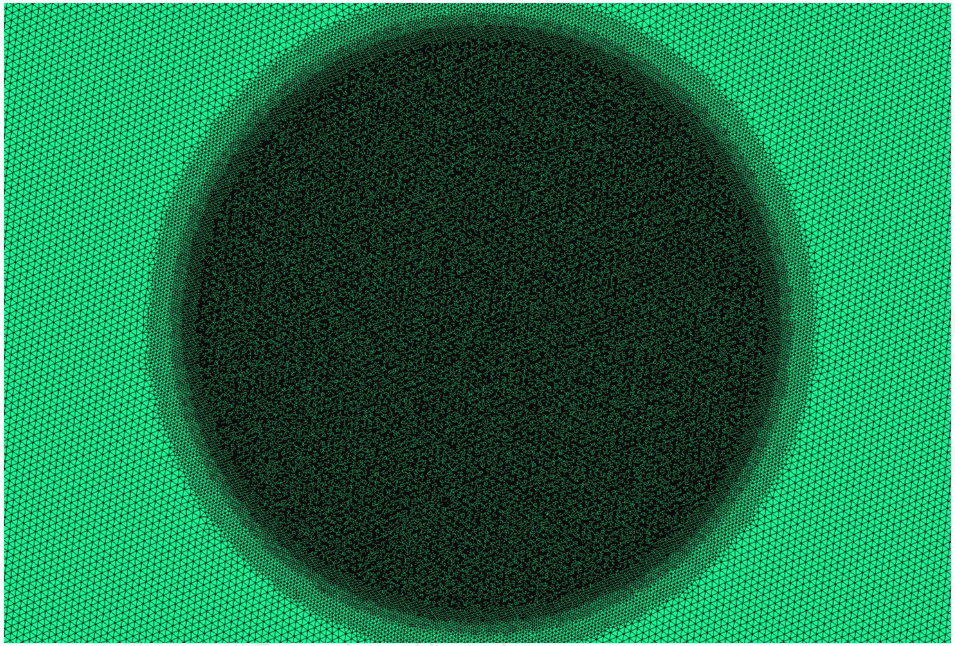


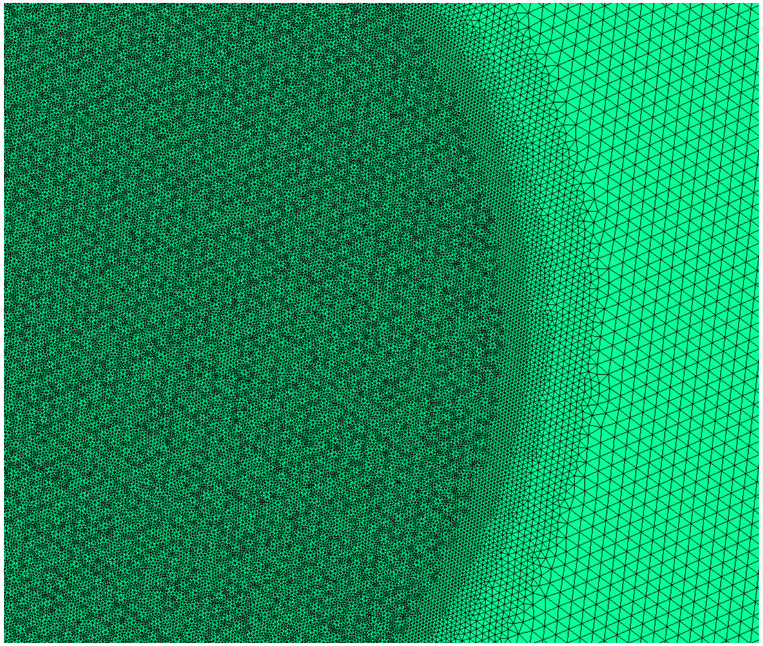
Conclusion & perspectives

- Fresnel admissibility: frequency-dependent condition
- block rank growth control
- HCA2 method for oscillating kernel: robust, precise and efficient
- Better clustering for wave problems exploiting rank estimates
- Improving parallelism:
  - Distributing computation tasks
  - Memory management.
- Hybridisation of solvers for the biggest cases

Example of H-matrix — FMM hybridisation

Locally refined meshes: plate refined in its centre

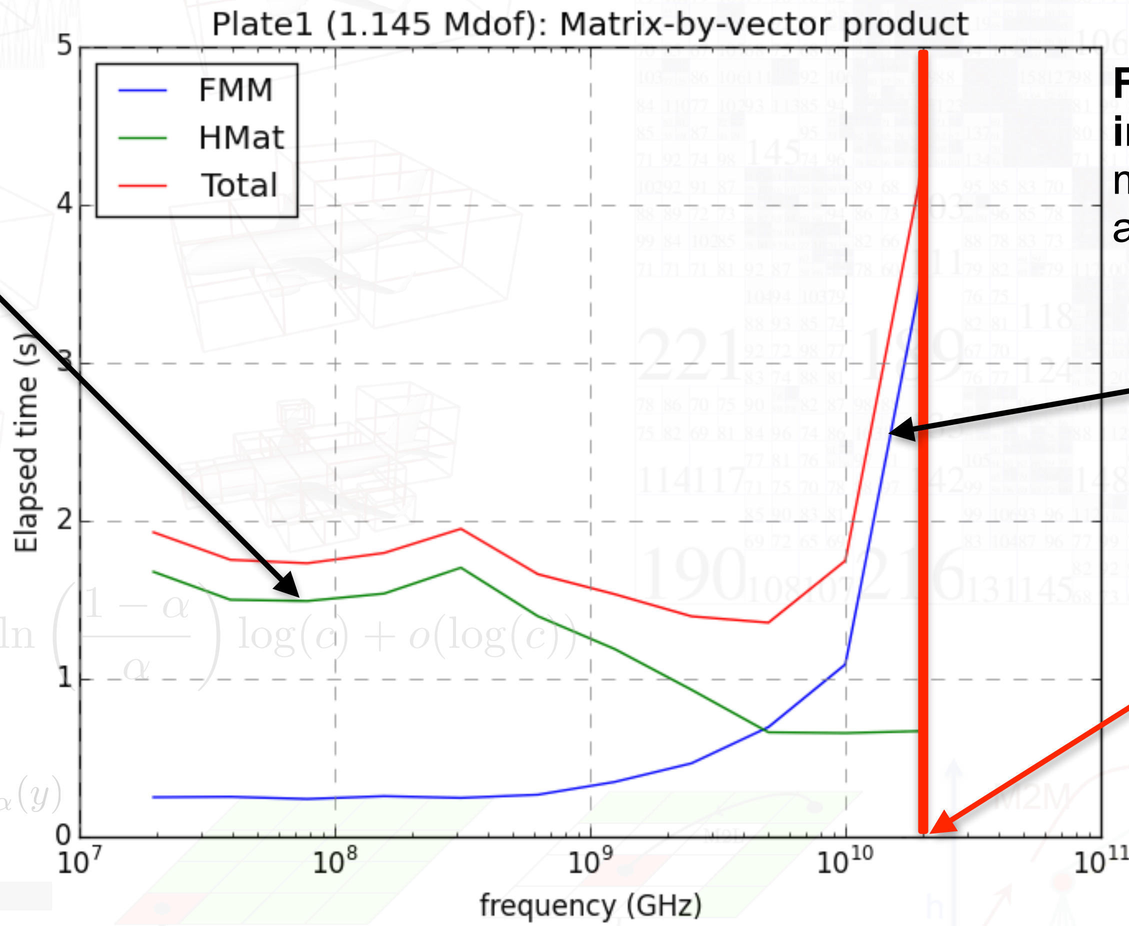




non refined plate: 235 kdof, h_{max} = 2.7mm
2 different refinements:
plate1: 1.1 Mdof — h_{max}/h_{min} ~ 13
plate2: 1.7 Mdof — h_{max}/h_{min} ~ 22

Plates: elapsed time
(6core proc)

Hmat for near field interactions: more important for low frequencies

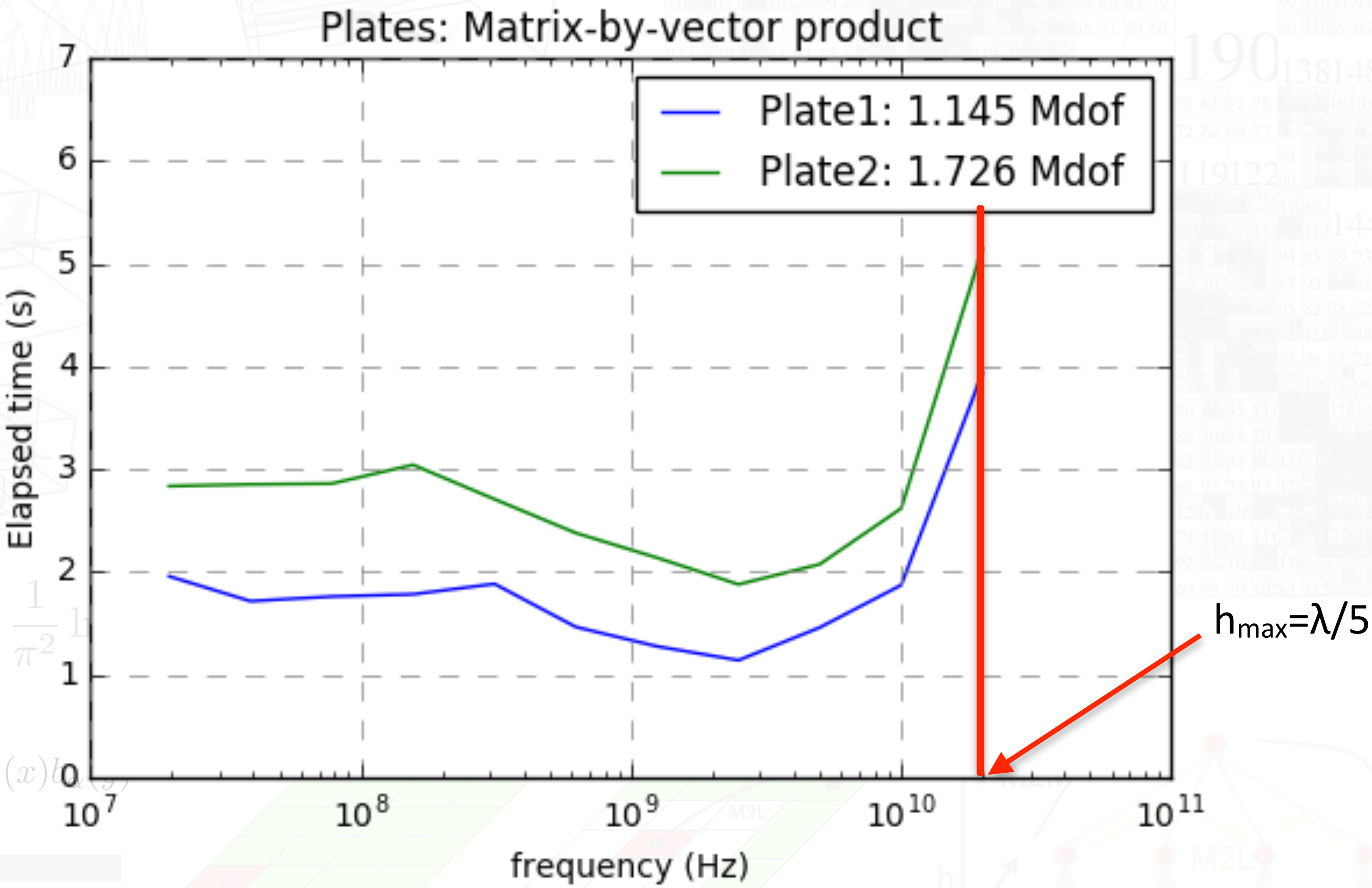


FMM for far field interactions: more multipoles and levels as frequency grows

$h_{max}=\lambda/5$

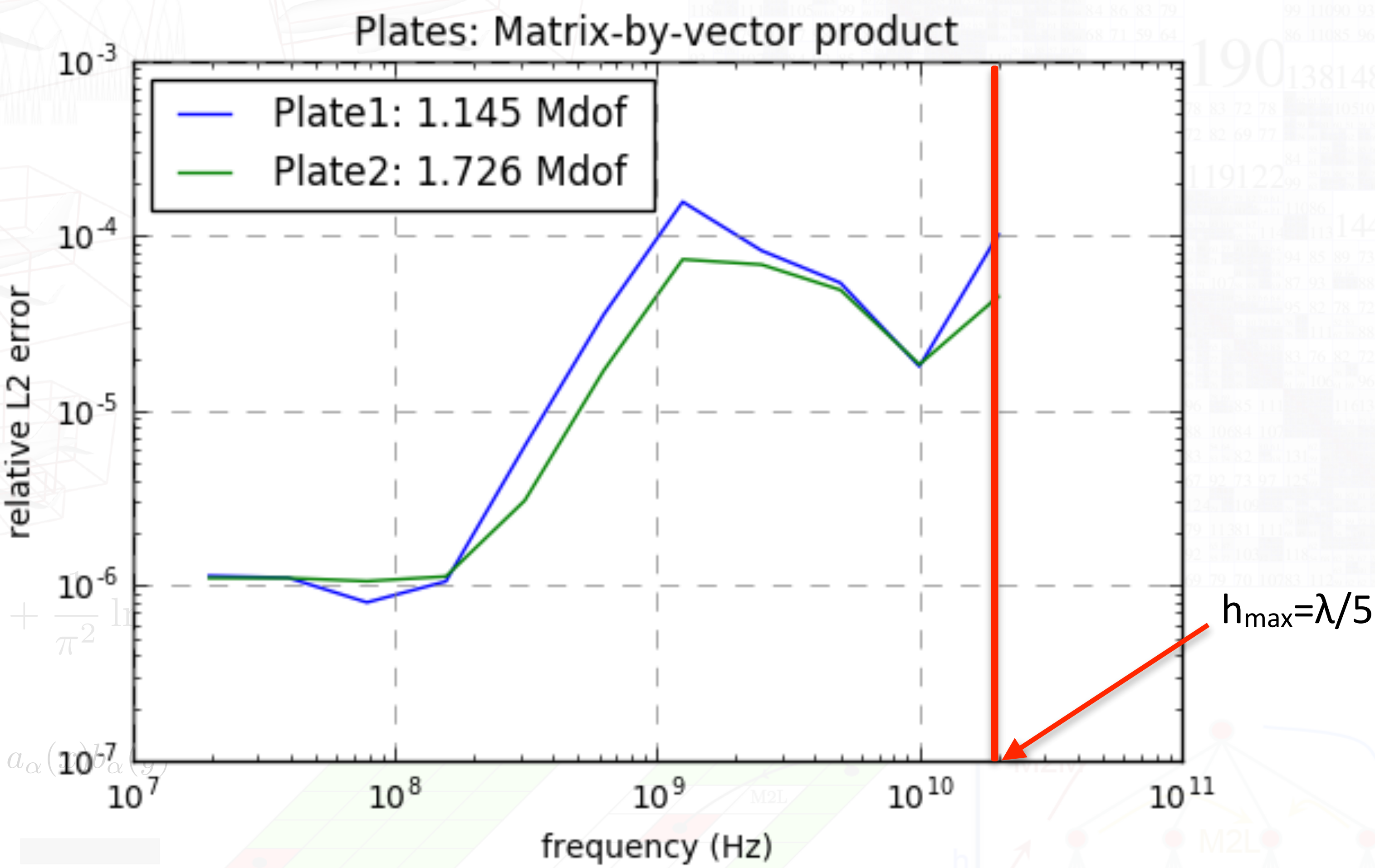
Intel®Core™ i7-3930K processor with 64Go RAM

Plates: elapsed time
(6core proc)



Intel®Core™ i7-3930K processor with 64Go RAM

Plates: precision



H-matrix preconditioner

Approximate factorisation

- assemble the H-matrix with the desired precision
- factorise the matrix with a degraded accuracy
- use this factorised matrix to provide an approximate inverse => preconditoner
- example with a 160640 dof :
  - accurate factorisation (ε_f = 1.E-4) :
    - IIAx-bII/IIbII = 1.2E-4
  - degraded factorisation (ε_f = 5.E-2) + iterative solver :
    - IIAx-bII/IIbII = 2.1E-5 after 5 iterations

Hi-BOX

- Advanced linear algebra algorithm & HPC:
  - ✳ H-matrix, FMM & hybridisation
  - ✳ handling of ooc, multithreading and distribution
  - ✳ block GMRES iterative solver
  - ✳ parallel computing using runtime system
- Library already used by several aeronautic and defence industrials
- Benchmarks showed a significant gain compared to existing solutions