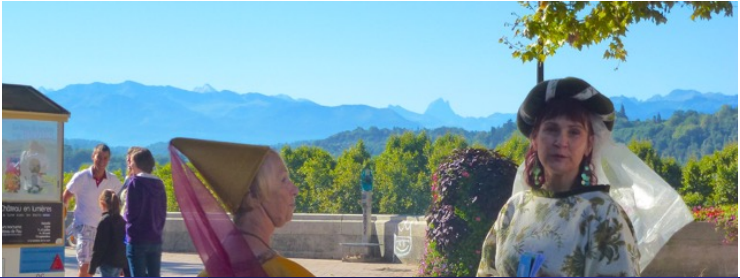
Regularity of electromagnetic fields on Lipschitz domains

Martin Costabel

IRMAR, Université de Rennes 1

Conference in honor of Abderrahmane Bendali

Pau, 12–14 December 2017



Variational spaces, or div-curl system

For $\Omega \subset \mathbb{R}^3$, we consider the standard function spaces:

$$X_N = H(\mathrm{div}, \Omega) \cap H_0(\mathbf{curl}, \Omega) \quad (\text{“Electric energy space”})$$
$$= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \mathrm{div}\, u \in L^2(\Omega), \mathbf{curl}\, u \in L^2(\Omega; \mathbb{C}^3), u \times n = 0 \text{ on } \partial\Omega\}$$
$$X_T = H_0(\mathrm{div}, \Omega) \cap H(\mathbf{curl}, \Omega) \quad (\text{“Magnetic energy space”})$$
$$= \{u \in L^2(\Omega; \mathbb{C}^3) \mid \mathrm{div}\, u \in L^2(\Omega), \mathbf{curl}\, u \in L^2(\Omega; \mathbb{C}^3), u \cdot n = 0 \text{ on } \partial\Omega\}$$

In other words, we consider solutions $u \in L^2(\Omega; \mathbb{C}^3)$ of the div-**curl** system

$$\mathrm{div}\, u = g \in L^2(\Omega); \quad \mathbf{curl}\, u = h \in L^2(\Omega; \mathbb{C}^3)$$

with homogeneous boundary conditions (“PEC”)

$$u \times n = 0 \qquad \text{or} \qquad u \cdot n = 0 \qquad \text{on } \partial\Omega$$

Problem

What is the (Sobolev) regularity of u?

What is known (for bounded domains Ω)

1. **Ω smooth or convex: H^1 regularity**

[Gaffney 1951, Friedrichs 1955, Morrey 1956]: (“Gaffney inequality”)
Ω smooth (C^2)

$$\exists C \ : \ \|u\|_{H^1} \le C\left(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2} + \|\mathbf{curl}\, u\|\right)_{L^2} \quad \forall u \in X_N \cup X_T$$

[Filonov 1997] $C^{3/2+\varepsilon}$, $\varepsilon > 0$
[Saranen 1982, Mitrea-et-al 2005, Dacorogna-et-al 2017] Ω convex

2. **Non-convex corners: No H^1 regularity**

[Co. 1990 ... Co.-Dauge 1999 ... Amrouche-et-al 1998]

$$\partial\Omega \text{ has an edge of opening } \pi/\alpha \quad \Longrightarrow \quad X_N, X_T \not\subset H^\alpha(\Omega)$$

3. **Ω Lipschitz: Inclusions $X_N \hookrightarrow L^2(\Omega)$ and $X_T \hookrightarrow L^2(\Omega)$ are compact**

[Weck 1974] piecewise smooth
[Weber 1980, Picard 1984] Lipschitz

4. **Ω Lipschitz: $H^{\frac{1}{2}}$ regularity**

[Co. 1990, Mitrea 2002] $X_N \cup X_T \subset H^{\frac{1}{2}}(\Omega)$

What is known (for bounded domains Ω)

1. Ω smooth or convex: H^1 regularity

[Gaffney 1951, Friedrichs 1955, Morrey 1956]: ("Gaffney inequality")
Ω smooth (C^2)

$$\exists C \;:\; \|u\|_{H^1} \le C\left(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2} + \|\mathbf{curl}\, u\|\right)_{L^2} \quad \forall u \in X_N \cup X_T$$

[Filonov 1997] $C^{3/2+\varepsilon}$, $\varepsilon > 0$
[Saranen 1982, Mitrea-et-al 2005, Dacorogna-et-al 2017] Ω convex

2. Non-convex corners: No H^1 regularity

[Co. 1990 ... Co.-Dauge 1999 ... Amrouche-et-al 1998]

$$\partial\Omega \text{ has an edge of opening } \pi/\alpha \quad \Longrightarrow \quad X_N, X_T \not\subset H^\alpha(\Omega)$$

3. Ω Lipschitz: Inclusions $X_N \hookrightarrow L^2(\Omega)$ and $X_T \hookrightarrow L^2(\Omega)$ are compact

[Weck 1974] piecewise smooth
[Weber 1980, Picard 1984] Lipschitz

4. Ω Lipschitz: $H^{\frac{1}{2}}$ regularity

[Co. 1990, Mitrea 2002] $X_N \cup X_T \subset H^{\frac{1}{2}}(\Omega)$

1. **Ω smooth or convex: H^1 regularity**

[Gaffney 1951, Friedrichs 1955, Morrey 1956]: ("Gaffney inequality")
Ω smooth (C^2)

$$\exists C \ : \ \|u\|_{H^1} \leq C\big(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2} + \|\mathbf{curl}\, u\|\big)_{L^2} \quad \forall u \in X_N \cup X_T$$

[Filonov 1997] $C^{3/2+\varepsilon}$, $\varepsilon > 0$
[Saranen 1982, Mitrea-et-al 2005, Dacorogna-et-al 2017] Ω convex

2. **Non-convex corners: No H^1 regularity**

[Co. 1990 ... Co.-Dauge 1999 ... Amrouche-et-al 1998]

$$\partial\Omega \text{ has an edge of opening } \pi/\alpha \qquad \Longrightarrow \qquad X_N, X_T \not\subset H^\alpha(\Omega)$$

3. **Ω Lipschitz: Inclusions $X_N \hookrightarrow L^2(\Omega)$ and $X_T \hookrightarrow L^2(\Omega)$ are compact**

[Weck 1974] piecewise smooth
[Weber 1980, Picard 1984] Lipschitz

4. **Ω Lipschitz: $H^{\frac12}$ regularity**

[Co. 1990, Mitrea 2002] $X_N \cup X_T \subset H^{\frac12}(\Omega)$

What is known (for bounded domains Ω)

1. **Ω smooth or convex: H^1 regularity**

[Gaffney 1951, Friedrichs 1955, Morrey 1956]: ("Gaffney inequality")
Ω smooth (C^2)

$$\exists C \;:\; \|u\|_{H^1} \leq C\big(\|u\|_{L^2} + \|\operatorname{div} u\|_{L^2} + \|\mathbf{curl}\, u\|\big)_{L^2} \quad \forall u \in X_N \cup X_T$$

[Filonov 1997] $C^{3/2+\varepsilon}$, $\varepsilon > 0$
[Saranen 1982, Mitrea-et-al 2005, Dacorogna-et-al 2017] Ω convex

2. **Non-convex corners: No H^1 regularity**

[Co. 1990 ... Co.-Dauge 1999 ... Amrouche-et-al 1998]

$$\partial\Omega \text{ has an edge of opening } \pi/\alpha \quad \Longrightarrow \quad X_N, X_T \not\subset H^{\alpha}(\Omega)$$

3. **Ω Lipschitz: Inclusions $X_N \hookrightarrow L^2(\Omega)$ and $X_T \hookrightarrow L^2(\Omega)$ are compact**

[Weck 1974] piecewise smooth
[Weber 1980, Picard 1984] Lipschitz

4. **Ω Lipschitz: $H^{\frac{1}{2}}$ regularity**

[Co. 1990, Mitrea 2002] $X_N \cup X_T \subset H^{\frac{1}{2}}(\Omega)$

Observation: For any bounded piecewise smooth Lipschitz domain Ω (curved polyhedron), there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$X_N \cup X_T \subset H^{\frac{1}{2}+\varepsilon}(\Omega)$$

This $\varepsilon(\Omega)$ can be arbitrarily small, depending on the angles of the corners and edges.
$\varepsilon > 0$ is useful in numerical analysis.
Example: [Alonso-Valli 1999] Domain decomposition methods

Does there exist such an $\varepsilon(\Omega) > 0$ for every bounded Lipschitz domain Ω?

No

Observation: For any bounded piecewise smooth Lipschitz domain Ω (curved polyhedron), there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$X_N \cup X_T \subset H^{\frac{1}{2}+\varepsilon}(\Omega)$$

This $\varepsilon(\Omega)$ can be arbitrarily small, depending on the angles of the corners and edges.
$\varepsilon > 0$ is useful in numerical analysis.
Example: [Alonso-Valli 1999] Domain decomposition methods

Question

Does there exist such an $\varepsilon(\Omega) > 0$ for every bounded Lipschitz domain Ω ?

No

The question

Observation: For any bounded piecewise smooth Lipschitz domain Ω (curved polyhedron), there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$X_N \cup X_T \subset H^{\frac{1}{2}+\varepsilon}(\Omega)$$

This $\varepsilon(\Omega)$ can be arbitrarily small, depending on the angles of the corners and edges.
$\varepsilon > 0$ is useful in numerical analysis.
Example: [Alonso-Valli 1999] Domain decomposition methods

Question

Does there exist such an $\varepsilon(\Omega) > 0$ for every bounded Lipschitz domain Ω ?

Short Answer

No

Theorem [Co. Nov 2017, arXiv : 1711.07179]

There exists a bounded domain $\Omega \subset \mathbb{R}^3$ such that

1. $\partial\Omega \in C^1$
2. $X_N, X_T \not\subset H^{\frac{1}{2}+\varepsilon}(\Omega)$ for any $\varepsilon > 0$

In this domain, homog. Dirichlet implies Neumann:

$$H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H^1_0(\Omega) = H_0^{\frac{3}{2}+\varepsilon}(\Omega)$$

Analogous properties are true for L^p Sobolev spaces, $1 \leq p < \infty$.

The construction of Ω follows ideas from [Filonov 1997], who constructs a $C^{\frac{3}{2}}$ domain with similar properties for $\varepsilon = \frac{1}{2}$.

Reduction to the (scalar) Laplacian; Ω bounded Lipschitz

General principle

Main Maxwell singularities are gradients of Laplace singularities
("Electrostatic or magnetostatic fields")

Conversely, all non-H^1 elements of $X_N \cup X_T$ come from gradients:

$$H(\mathrm{div},\Omega) \cap H(\mathbf{curl},\Omega) = H^1(\Omega;\mathbb{C}^3) + \nabla H^1(\Delta,\Omega)$$

(Proof: $\mathbf{curl}\, u = h \in L^2 \Rightarrow \exists w \in H^1 : \mathbf{curl}\, w = h$.)
With boundary conditions: Birman-Solomyak decomposition

$X_N = H^1 \cap X_N + \nabla H^1_0(\Delta)$ — true for any Lipschitz domain

$X_T = H^1 \cap X_T + \nabla H^1(\Delta) \cap X_T$ — true for any piecewise smooth domain

[Filonov 1997] True for $C^{\frac{3}{2}+\epsilon}$ domains, but $C^{\frac{3}{2}}$ counterexample for X_T

Reduction to the (scalar) Laplacian; Ω bounded Lipschitz

General principle

Main Maxwell singularities are gradients of Laplace singularities
("Electrostatic or magnetostatic fields")

$$\Delta v = g \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega; \quad u = \nabla v; \quad \Longrightarrow \begin{cases} \mathbf{curl}\, u = 0;\ \operatorname{div} u = g \\ v \in H^1, g \in L^2 \Rightarrow u \in X_N \\ v \in H^{s+1}(\Omega) \Leftrightarrow u \in H^s(\Omega;\mathbb{C}^3) \end{cases}$$

Conversely, all non-H^1 elements of $X_N \cup X_T$ come from gradients:

$$H(\operatorname{div},\Omega) \cap H(\mathbf{curl},\Omega) = H^1(\Omega;\mathbb{C}^3) + \nabla H^1(\Delta,\Omega)$$

(Proof: $\operatorname{curl} u = h \in L^2 \Rightarrow \exists w \in H^1 : \operatorname{curl} w = h$.)
With boundary conditions: Birman-Solomyak decomposition

$X_N = H^1 \cap X_N + \nabla H^1_0(\Delta)$ — true for any Lipschitz domain

$X_T = H^1 \cap X_T + \nabla H^1(\Delta) \cap X_T$ — true for any piecewise smooth domain

[Filonov 1997] True for $C^{\frac{3}{2}+\epsilon}$ domains, but $C^{\frac{3}{2}}$ counterexample for X_T

Reduction to the (scalar) Laplacian; Ω bounded Lipschitz

General principle

Main Maxwell singularities are gradients of Laplace singularities
("Electrostatic or magnetostatic fields")

$$\Delta v = g \text{ in } \Omega;\ \partial_n v = 0 \text{ on } \partial\Omega; \quad u = \nabla v; \quad \Longrightarrow \begin{cases} \mathbf{curl}\, u = 0;\ \operatorname{div} u = g \\ v \in H^1, g \in L^2 \Rightarrow u \in X_T \\ v \in H^{s+1}(\Omega) \Leftrightarrow u \in H^s(\Omega;\mathbb{C}^3) \end{cases}$$

Conversely, all non-H^1 elements of $X_N \cup X_T$ come from gradients:

$$H(\operatorname{div},\Omega) \cap H(\mathbf{curl},\Omega) = H^1(\Omega;\mathbb{C}^3) + \nabla H^1(\Delta,\Omega)$$

(Proof: $\mathbf{curl}\, u = h \in L^2 \Rightarrow \exists w \in H^1 : \mathbf{curl}\, w = h$.)
With boundary conditions: Birman-Solomyak decomposition

$X_N = H^1 \cap X_N + \nabla H^1_0(\Delta)$ — true for any Lipschitz domain

$X_T = H^1 \cap X_T + \nabla H^1(\Delta) \cap X_T$ — true for any piecewise smooth domain

[Filonov 1997] True for $C^{\frac{3}{2}+\varepsilon}$ domains, but $C^{\frac{3}{2}}$ counterexample for X_T

Reduction to the (scalar) Laplacian; Ω bounded Lipschitz

General principle

Main Maxwell singularities are gradients of Laplace singularities
("Electrostatic or magnetostatic fields")

$$\Delta v = g \text{ in } \Omega;\ \partial_n v = 0 \text{ on } \partial\Omega; \quad u = \nabla v; \quad \Longrightarrow \begin{cases} \mathbf{curl}\, u = 0;\ \operatorname{div} u = g \\ v \in H^1, g \in L^2 \Rightarrow u \in X_T \\ v \in H^{s+1}(\Omega) \Leftrightarrow u \in H^s(\Omega;\mathbb{C}^3) \end{cases}$$

Conversely, all non-H^1 elements of $X_N \cup X_T$ come from gradients:

$$H(\operatorname{div},\Omega) \cap H(\mathbf{curl},\Omega) = H^1(\Omega;\mathbb{C}^3) + \nabla H^1(\Delta,\Omega)$$

(Proof: $\mathbf{curl}\, u = h \in L^2 \Rightarrow \exists\, w \in H^1 : \mathbf{curl}\, w = h$.)

With boundary conditions: Birman-Solomyak decomposition

$$X_N = H^1 \cap X_N + \nabla H^1_0(\Delta) \qquad \text{true for any Lipschitz domain}$$

$$X_T = H^1 \cap X_T + \nabla H^1(\Delta) \cap X_T \qquad \text{true for any piecewise smooth domain}$$

[Filonov 1997] True for $C^{\frac{3}{2}+\epsilon}$ domains, but $C^{\frac{3}{2}}$ counterexample for X_T

General principle

Main Maxwell singularities are gradients of Laplace singularities
("Electrostatic or magnetostatic fields")

$$\Delta v = g \text{ in } \Omega;\ \partial_n v = 0 \text{ on } \partial\Omega; \quad u = \nabla v; \quad \Longrightarrow \begin{cases} \mathbf{curl}\, u = 0;\ \operatorname{div} u = g \\ v \in H^1, g \in L^2 \Rightarrow u \in X_T \\ v \in H^{s+1}(\Omega) \Leftrightarrow u \in H^s(\Omega;\mathbb{C}^3) \end{cases}$$

Conversely, all non-H^1 elements of $X_N \cup X_T$ come from gradients:

$$H(\operatorname{div},\Omega) \cap H(\mathbf{curl},\Omega) = H^1(\Omega;\mathbb{C}^3) + \nabla H^1(\Delta,\Omega)$$

(Proof: $\mathbf{curl}\, u = h \in L^2 \Rightarrow \exists\, w \in H^1 : \mathbf{curl}\, w = h$.)

With boundary conditions: Birman-Solomyak decomposition

$X_N = H^1 \cap X_N + \nabla H^1_0(\Delta)$ — true for any Lipschitz domain

$X_T = H^1 \cap X_T + \nabla H^1(\Delta) \cap X_T$ — true for any piecewise smooth domain

[Filonov 1997] True for $C^{\frac{3}{2}+\varepsilon}$ domains, but $C^{\frac{3}{2}}$ counterexample for X_T

The construction

1. We construct a 2π-periodic continuous function f via Fourier series:

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x)$$

2. Setting $F(x) = 1 + \int_0^x f\,dx$, we define $\Omega \subset \mathbb{R}^2$ in polar coordinates (r, θ)

$$\Omega = \{(r, \theta) \mid r < F(\theta)\}$$

3. In higher dimensions, we define Ω in cylindrical coordinates (r, θ, z)

$$\Omega = \{(r, \theta, z) \mid r^2 < (1 - |z|^2)\, F(\theta)^2\}$$

It is clear that Ω is a C^1 domain.
For f, the coefficients will be chosen as follows:

$$a_k = q^{-k};\ b_k = 2^{q^k}, \quad q \in \mathbb{N} \text{ large enough } (q = 66 \text{ will do})$$

(In Filonov's counterexample, $b_k = q^{2k}$)

The construction

1. We construct a 2π-periodic continuous function f via Fourier series:

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x)$$

2. Setting $F(x) = 1 + \int_0^x f\,dx$, we define $\Omega \subset \mathbb{R}^2$ in polar coordinates (r, θ)

$$\Omega = \{(r, \theta) \mid r < F(\theta)\}$$

3. In higher dimensions, we define Ω in cylindrical coordinates (r, θ, z)

$$\Omega = \{(r, \theta, z) \mid r^2 < (1 - |z|^2)\,F(\theta)^2\}$$

It is clear that Ω is a C^1 domain.

For f, the coefficients will be chosen as follows:

$$a_k = q^{-k};\ b_k = 2^{q^k}, \quad q \in \mathbb{N} \text{ large enough } (q = 66 \text{ will do})$$

(In Filonov's counterexample, $b_k = q^{2k}$)

The construction

1. We construct a 2π-periodic continuous function f via Fourier series:

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x)$$

2. Setting $F(x) = 1 + \int_0^x f\,dx$, we define $\Omega \subset \mathbb{R}^2$ in polar coordinates (r, θ)

$$\Omega = \{(r, \theta) \mid r < F(\theta)\}$$

3. In higher dimensions, we define Ω in cylindrical coordinates (r, θ, z)

$$\Omega = \{(r, \theta, z) \mid r^2 < (1 - |z|^2)\, F(\theta)^2\}$$

It is clear that Ω is a C^1 domain.
For f, the coefficients will be chosen as follows:

$$a_k = q^{-k};\ b_k = 2^{q^k}, \quad q \in \mathbb{N} \text{ large enough} \quad (q = 66 \text{ will do})$$

(In Filonov's counterexample, $b_k = q^{2k}$)

Properties of the function f

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x) \quad a_k = q^{-k};\ b_k = 2^{q^k};\ q \text{ large enough}$$

Lemma (Strongly nowhere Hölder continuous)

$$\forall x \in [0,2\pi], \forall \varepsilon > 0 : \int_0^{2\pi} \frac{|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}} dy = +\infty$$

Corollary 1 (Not a Sobolev multiplier)

If $a, b \in H^\varepsilon([0,2\pi])$, $\varepsilon > 0$ and if $af = b$, then $a = b = 0$.

Proof of the corollary: Writing the H^ε (Slobodeckij) seminorm:

$$|b|_\varepsilon = \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|b(y)-b(x)|^2}{|y-x|^{1+2\varepsilon}} dy\, dx\Big)^{\frac12}$$

$$\geq \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|a(x)|^2\,|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}} dy\, dx\Big)^{\frac12} - \|f\|_{L^\infty}|a|_\varepsilon$$

If $a \in H^{\frac12+\varepsilon}([0,2\pi])$, we find $|b|_\varepsilon = +\infty$, unless $a(x) = 0$ for a.e. $x \in [0,2\pi]$. □

Properties of the function f

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(b_k x) \qquad a_k = q^{-k};\ b_k = 2^{q^k};\ q \text{ large enough}$$

Lemma (Strongly nowhere Hölder continuous)

$$\forall x \in [0,2\pi], \forall \varepsilon > 0 : \int_0^{2\pi} \frac{|f(y) - f(x)|^2}{|y-x|^{1+2\varepsilon}} dy = +\infty$$

Corollary 1 (Not a Sobolev multiplier)

If $a, b \in H^\varepsilon([0,2\pi])$, $\varepsilon > 0$ and if $af = b$, then $a = b = 0$.

Proof of the corollary: Writing the H^ε (Slobodeckij) seminorm:

$$|b|_\varepsilon = \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|b(y) - b(x)|^2}{|y-x|^{1+2\varepsilon}} dy\, dx\Big)^{\frac12}$$

$$\geq \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|a(x)|^2\,|f(y) - f(x)|^2}{|y-x|^{1+2\varepsilon}} dy\, dx\Big)^{\frac12} - \|f\|_{L^\infty}|a|_\varepsilon$$

If $a \in H^{\frac12+\varepsilon}([0,2\pi])$, we find $|b|_\varepsilon = +\infty$, unless $a(x) = 0$ for a.e. $x \in [0,2\pi]$. □

Properties of the function f

$$f(x)=\sum_{k=1}^{\infty} a_k \sin(b_k x) \qquad a_k=q^{-k};\ b_k=2^{q^k};\ q \text{ large enough}$$

Lemma (Strongly nowhere Hölder continuous)

$$\forall x\in[0,2\pi], \forall\varepsilon>0:\ \int_0^{2\pi} \frac{|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}}\,dy=+\infty$$

Corollary 1 (Not a Sobolev multiplier)

If $a,b\in H^{\varepsilon}([0,2\pi])$, $\varepsilon>0$ and if $af=b$, then $a=b=0$.

Proof of the corollary: Writing the H^{ε} (Slobodeckij) seminorm:

$$\begin{aligned}|b|_{\varepsilon} &= \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|b(y)-b(x)|^2}{|y-x|^{1+2\varepsilon}}\,dy\,dx\Big)^{\frac12}\\ &\geq \Big(\int_0^{2\pi}\int_0^{2\pi} \frac{|a(x)|^2\,|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}}\,dy\,dx\Big)^{\frac12} - \|f\|_{L^\infty}|a|_{\varepsilon}\end{aligned}$$

If $a\in H^{\frac12+\varepsilon}([0,2\pi])$, we find $|b|_{\varepsilon}=+\infty$, unless $a(x)=0$ for a.e. $x\in[0,2\pi]$. $\square$

Corollary 2 (Dirichlet implies Neumann)

$$H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H^1_0(\Omega) = H_0^{\frac{3}{2}+\varepsilon}(\Omega)$$

Corollary 3 (Laplace non-regularity)

1. (Dirichlet)
   The solution $v \in H^1_0(\Omega)$ of $\Delta v = 1$ in Ω satisfies

   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$

   The same is true for all Dirichlet eigenfunctions in Ω.

2. (Neumann)
   There exists $g \in C(\overline{\Omega})$ such that the solutions $v \in H^1(\Omega)$ of $\Delta v = g$ in Ω, $\partial_n v = 0$ on $\partial\Omega$ satisfy

   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$

Corollary 4

Corollary 2 (Dirichlet implies Neumann)

$$H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H_0^1(\Omega) = H_0^{\frac{3}{2}+\varepsilon}(\Omega)$$

Corollary 3 (Laplace non-regularity)

1. (Dirichlet)
   The solution $v \in H_0^1(\Omega)$ of $\Delta v = 1$ in Ω satisfies

   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$

   The same is true for all Dirichlet eigenfunctions in Ω.

2. (Neumann)
   There exists $g \in C(\overline{\Omega})$ such that the solutions $v \in H^1(\Omega)$ of $\Delta v = g$ in Ω, $\partial_n v = 0$ on $\partial\Omega$ satisfy

   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$

Corollary 4

Properties of the domain Ω

Corollary 2 (Dirichlet implies Neumann)

$$H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H^1_0(\Omega) = H_0^{\frac{3}{2}+\varepsilon}(\Omega)$$

Corollary 3 (Laplace non-regularity)

1. (Dirichlet)
   The solution $v \in H^1_0(\Omega)$ of $\Delta v = 1$ in Ω satisfies
   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$
   The same is true for all Dirichlet eigenfunctions in Ω.
2. (Neumann)
   There exists $g \in C(\overline{\Omega})$ such that the solutions $v \in H^1(\Omega)$ of $\Delta v = g$ in Ω, $\partial_n v = 0$ on $\partial\Omega$ satisfy
   $$v \notin H^{\frac{3}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$$

Corollary 4 The Theorem: $X_N, X_T \not\subset H^{\frac{1}{2}+\varepsilon}(\Omega) \quad \forall \varepsilon > 0$

Some proofs: Proof of the Lemma

Recall $f = \sum_{k=1}^{\infty} f_k$, $f_k(x) = a_k \sin b_k x$, $a_k = q^{-k}$, $b_k = 2^{q^k}$.

Define $I_m = [\frac{1}{b_m}, \frac{2}{b_m}]$. Then $\displaystyle\int_0^{2\pi} \frac{|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}} dy \ge \sum_{m=1}^{\infty} \int_{I_m} \frac{|f(x+h)-f(x)|^2}{|h|^{1+2\varepsilon}} dh.$

We show that each term in the sum is $\ge c$ for some $c = c(\varepsilon) > 0$.

$$\Big(\int_{I_m} \frac{|f(x+h)-f(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} \ge \Big(\int_{I_m} \frac{|f_m(x+h)-f_m(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12}$$

$$- \sum_{k\le m-1} \Big(\int_{I_m} \frac{|f_k(x+h)-f_k(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} - \sum_{k\ge m+1} \Big(\int_{I_m} \frac{|f_k(x+h)-f_k(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12}$$

- Resonance frequency $k = m$: Change of scale $t = b_m h$:

$$\Big(\int_{I_m} \frac{|f_m(x+h)-f_m(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} = a_m b_m^{\varepsilon} \Big(\int_1^2 \frac{|\sin(b_m x + t) - \sin(b_m x)|^2}{t^{1+2\varepsilon}} dt\Big)^{\frac12} \ge 5\gamma a_m b_m^{\varepsilon}$$

with $\boxed{\gamma = \frac15 \min_z \int_1^2 |\sin(z+t) - \sin(z)| t^{-2} dt > 0}$.

- Small frequencies $k \le m-1$: $|f_k(x+h) - f_k(x)| \le a_k b_k |h| \le 2 a_k b_k \frac{1}{b_m}$
- Large frequencies $k \ge m+1$: $|f_k(x+h) - f_k(x)| \le 2 a_k$.

It follows (easy exercise): $\sum_{k\le m-1} \cdots \le \frac{2}{q-1} a_m b_m^{\varepsilon}$ and $\sum_{k\ge m+1} \cdots \le \frac{2}{q-1} a_m b_m^{\varepsilon}$.

Recall $f = \sum_{k=1}^{\infty} f_k$, $f_k(x) = a_k \sin b_k x$, $a_k = q^{-k}$, $b_k = 2^{q^k}$.

Define $I_m = [\frac{1}{b_m}, \frac{2}{b_m}]$. Then $\displaystyle\int_0^{2\pi} \frac{|f(y)-f(x)|^2}{|y-x|^{1+2\varepsilon}} dy \ge \sum_{m=1}^{\infty} \int_{I_m} \frac{|f(x+h)-f(x)|^2}{|h|^{1+2\varepsilon}} dh.$

We show that each term in the sum is $\ge c$ for some $c = c(\varepsilon) > 0$.

$$\Big(\int_{I_m} \frac{|f(x+h)-f(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} \ge \Big(\int_{I_m} \frac{|f_m(x+h)-f_m(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12}$$
$$- \sum_{k\le m-1} \Big(\int_{I_m} \frac{|f_k(x+h)-f_k(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} - \sum_{k\ge m+1} \Big(\int_{I_m} \frac{|f_k(x+h)-f_k(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12}$$

- Resonance frequency $k = m$: Change of scale $t = b_m h$:

$$\Big(\int_{I_m} \frac{|f_m(x+h)-f_m(x)|^2}{|h|^{1+2\varepsilon}} dh\Big)^{\frac12} = a_m b_m^{\varepsilon} \Big(\int_1^2 \frac{|\sin(b_m x + t) - \sin(b_m x)|^2}{t^{1+2\varepsilon}} dt\Big)^{\frac12} \ge 5\gamma a_m b_m^{\varepsilon}$$

with $\boxed{\gamma = \frac15 \min_z \int_1^2 |\sin(z+t) - \sin(z)| t^{-2} dt > 0}$.

- Small frequencies $k \le m-1$: $|f_k(x+h) - f_k(x)| \le a_k b_k |h| \le 2 a_k b_k \frac{1}{b_m}$
- Large frequencies $k \ge m+1$: $|f_k(x+h) - f_k(x)| \le 2a_k$.

It follows (easy exercise): $\sum_{k\le m-1} \cdots \le \frac{2}{q-1} a_m b_m^{\varepsilon}$ and $\sum_{k\ge m+1} \cdots \le \frac{2}{q-1} a_m b_m^{\varepsilon}$.

Some proofs: End of proof of the Lemma

If $\frac{1}{q-1} < \gamma$, we find

$$\Big(\int_{I_m} \frac{|f(x+h)-f(x)|^2}{|h|^{1+2\varepsilon}}\,dh\Big)^{\frac{1}{2}} \geq a_m b_m^{\varepsilon}(5\gamma-2\gamma-2\gamma) = \gamma\, a_m b_m^{\varepsilon}$$

It remains to show that

$$\sum_{m=1}^{\infty} a_m^2 b_m^{2\varepsilon} = +\infty$$

But $a_m b_m^{\varepsilon} = \varepsilon(2^{\varepsilon q^m})/(\varepsilon q^m) \geq \varepsilon \log 2 = c(\varepsilon) > 0$. □

In 2D, $\partial\Omega$ is given by the curve $r = F(\theta)$ or $x = \binom{F\cos\theta}{F\sin\theta}$.
The tangential vector is

$$t = n^{\perp} = (F^2 + f^2)^{-\frac{1}{2}} \begin{pmatrix} -F\sin\theta + f\cos\theta \\ F\cos\theta + f\sin\theta \end{pmatrix}$$

Hence $n \times \nabla v = 0 \Longleftrightarrow af = b$ if we define

$$a = \partial_1 v\cos\theta + \partial_2 v\sin\theta\,, \quad b = (\partial_1 v\sin\theta - \partial_2 v\cos\theta)F$$

Let now $v \in H^{\frac{3}{2}+\varepsilon}(\Omega)$.
Then $\nabla v\big|_{\partial\Omega} \in H^{\varepsilon}(\partial\Omega;\mathbb{C}^2)$, and also $a, b \in H^{\varepsilon}(\partial\Omega)$.
Thus, if the tangential derivative $n \times \nabla v$ vanishes on $\partial\Omega$, by Corollary 1 we find $a = b = 0$, hence $\nabla v\big|_{\partial\Omega} = 0$. □

Again in 2D, let $v \in H_0^1(\Omega)$ solve $\Delta v = 1$ in Ω.
If $v \in H^{\frac{3}{2}+\varepsilon}(\Omega)$, then by Corollary 2 we have $\partial_n v = 0$, and Green's formula leads to a contradiction:

$$0 = \int_{\partial\Omega} \partial_n v \, ds = \int_\Omega \Delta v \, dx = \int_\Omega 1 \, dx \qquad \square$$

A similar argument shows that no Dirichlet eigenfunction on Ω can have $H^{\frac{3}{2}+\varepsilon}$ regularity.
For the Neumann problem, one gets a contradiction if $\Delta v = g$ with a non-zero harmonic polynomial g satisfying $\int_\Omega g = 0$: In 2D, $\partial_n v = 0$ implies that $n \times \nabla v = 0$, hence $v = \text{const}$ on $\partial\Omega$, w.l.o.g. $v = 0$ on $\partial\Omega$:

$$0 = \int_{\partial\Omega} (\partial_n v\, g - v\, \partial_n g)\, ds = \int_\Omega (\Delta v\, g - v \Delta g)\, dx = \int_\Omega g^2 \, dx \qquad \square$$

Again in 2D, let $v \in H_0^1(\Omega)$ solve $\Delta v = 1$ in Ω.
If $v \in H^{\frac{3}{2}+\varepsilon}(\Omega)$, then by Corollary 2 we have $\partial_n v = 0$, and Green's formula leads to a contradiction:

$$0 = \int_{\partial\Omega} \partial_n v \, ds = \int_{\Omega} \Delta v \, dx = \int_{\Omega} 1 \, dx \qquad \square$$

A similar argument shows that no Dirichlet eigenfunction on Ω can have $H^{\frac{3}{2}+\varepsilon}$ regularity.

For the Neumann problem, one gets a contradiction if $\Delta v = g$ with a non-zero harmonic polynomial g satisfying $\int_\Omega g = 0$: In 2D, $\partial_n v = 0$ implies that $n \times \nabla v = 0$, hence $v = \text{const}$ on $\partial\Omega$, w.l.o.g. $v = 0$ on $\partial\Omega$:

$$0 = \int_{\partial\Omega} (\partial_n v \, g - v \, \partial_n g) \, ds = \int_{\Omega} (\Delta v \, g - v \Delta g) \, dx = \int_{\Omega} g^2 \, dx \qquad \square$$

Again in 2D, let $v \in H_0^1(\Omega)$ solve $\Delta v = 1$ in Ω.
If $v \in H^{\frac{3}{2}+\varepsilon}(\Omega)$, then by Corollary 2 we have $\partial_n v = 0$, and Green's formula leads to a contradiction:

$$0 = \int_{\partial\Omega} \partial_n v \, ds = \int_\Omega \Delta v \, dx = \int_\Omega 1 \, dx \qquad \square$$

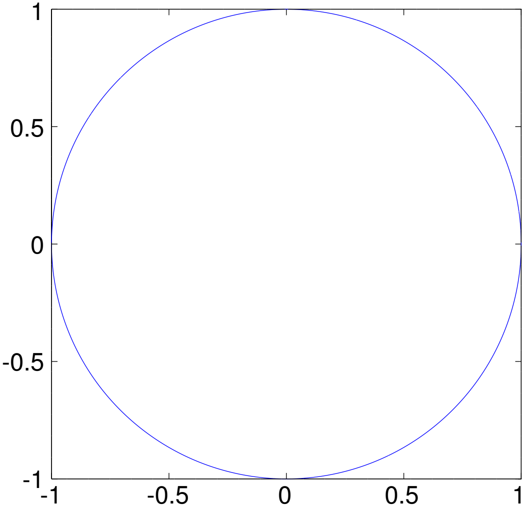
A similar argument shows that no Dirichlet eigenfunction on Ω can have $H^{\frac{3}{2}+\varepsilon}$ regularity.
For the Neumann problem, one gets a contradiction if $\Delta v = g$ with a non-zero harmonic polynomial g satisfying $\int_\Omega g = 0$: In 2D, $\partial_n v = 0$ implies that $n \times \nabla v = 0$, hence $v = \mathrm{const}$ on $\partial\Omega$, w.l.o.g. $v = 0$ on $\partial\Omega$:

$$0 = \int_{\partial\Omega} (\partial_n v \, g - v \, \partial_n g) \, ds = \int_\Omega (\Delta v \, g - v \Delta g) \, dx = \int_\Omega g^2 \, dx \qquad \square$$

Picture of the domain Ω in 2D **(Finally !)**

Here is a graph of the boundary curve $r = F(\theta)$, for $q = 66$:

Here is a graph of the boundary curve $r = F(\theta)$, for $q = 66$:



A somewhat more meaningful picture...

A somewhat more meaningful picture...

Thank you for your attention!