Eigenpairs of the curl operator on axisymmetric tori

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Outline

Outline

Force-free magnetic fields

Let Ω be a bounded three-dimensional domain.

- Problem: Find eigenpairs of the curl operator in Ω

(♦) $$\mathbf{curl}\, \boldsymbol{u} = \kappa \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega$$

with $\kappa \in \mathbb{R}$, $\boldsymbol{u} \neq 0$. Field $\boldsymbol{u}$ called Beltrami field or force-free magnetic field.

- Difficulty:
Find complementary conditions on $\partial\Omega$ so that (♦) is the eigen-equation associated with a (unbounded) self-adjoint operator.
- Theoretical contributions:
PICARD (1976, 1998), YOSHIDA, GIGA (1990), BOULMEZAOUD, MADAY, AMMARI (1999), NICAISE (2013), HIPTMAIR, KOTIUGA, TORDEUX (2012)
- Computational / Numerical contributions:
CANTARELLA (2000), MORSE (2007), RODRÍGUEZ, VENEGAS (2014), LARA, RODRÍGUEZ, VENEGAS (2016), ALONSO RODRÍGUEZ, CAMAÑO, RODRÍGUEZ, VALLI, VENEGAS (2016).

Recent references

The paper

A. ALONSO RODRÍGUEZ, J. CAMAÑO, R. RODRÍGUEZ, A. VALLI, P. VENEGAS, *Finite element approximation of the spectrum of the curl operator in a multiply-connected domain*. UDEC preprint (2016).

summarizes theory from

R. HIPTMAIR, P. R. KOTIUGA, S. TORDEUX, *Self-adjoint curl operators*. Ann. Mat. Pura Appl. (4), 191 (2012), pp. 431–457.

proposes and analyzes a numerical method.

Boundary and circulation conditions

Let $b_1(\mathcal{O})$ be the first Betti number of $\mathcal{O}$.

Simply connected domains

If Ω is simply connected, i.e. its homotopy group is trivial ($\Leftrightarrow b_1(\Omega) = 0$), then the normal (or perfect conductor magnetic) boundary condition

$$(\heartsuit) \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega$$

is sufficient for ($\diamondsuit$)-($\heartsuit$) to have a well-defined real and discrete spectrum.

Admissible circulation conditions in non-simply connected domains

If $b_1(\Omega) =: g > 0$, then ($\diamondsuit$)-($\heartsuit$) has to be completed by g “admissible” circulation conditions ($\mathscr{C}_\ell$ = cycle and $\boldsymbol{t}$ unit tangent vector field to $\mathscr{C}_\ell$)

$$(\spadesuit) \qquad \oint_{\mathscr{C}_\ell} \boldsymbol{u} \cdot \boldsymbol{t} = 0, \quad \ell = 1, \ldots, g$$

to have a well-defined real and discrete spectrum.

Canonical set of cycles

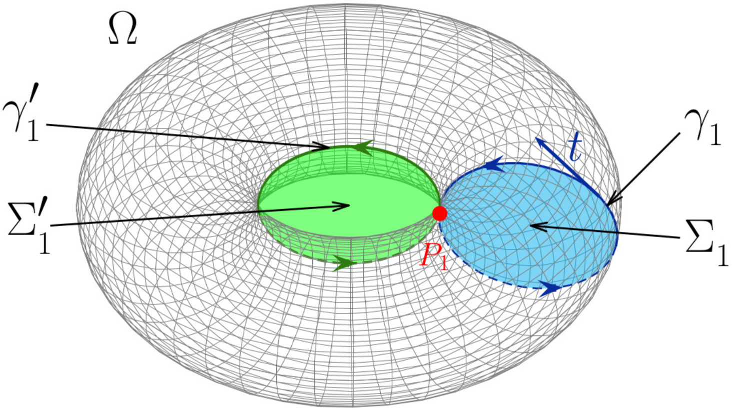
As soon as $g > 0$, the choice of cycles $\mathscr{C}_\ell$ is non-unique!
Denote by $\mathrm{H}_1(\mathcal{O})$ the first homology group of $\mathcal{O}$.

For a large family of domains : (exceptions do exist !)

There exist $2g$ cycles with the following properties

1. $\{\gamma_\ell\}_{\ell=1}^g \cup \{\gamma'_\ell\}_{\ell=1}^g$ generators of $\mathrm{H}_1(\partial\Omega)$ and
$$\gamma_\ell \cap \gamma'_{\ell'} \neq \emptyset \quad \text{iff} \quad \ell = \ell'.$$
2. $\{\gamma_\ell\}_{\ell=1}^g$ generators of $\mathrm{H}_1(\Omega^{\complement})$.
3. $\{\gamma'_\ell\}_{\ell=1}^g$ generators of $\mathrm{H}_1(\Omega)$.
4. The γ_ℓ are the boundaries of cutting surfaces $\Sigma_\ell \subset \Omega$ such that $\Omega \setminus \cup_{\ell=1}^g \Sigma_\ell$ is simply connected.
5. The γ'_ℓ are the boundaries of cutting surfaces $\Sigma'_\ell \subset \Omega^{\complement}$ such that $\Omega^{\complement} \setminus \cup_{\ell=1}^g \Sigma'_\ell$ is simply connected.

Borrowed Figure 1 in [ACRVV]



Distinct choices of circulation conditions

Theorem

Let $g \neq 0$. Choose $g_1 \in \{0, \dots, g\}$ and a subset $\mathfrak{L} \subset \{1, \dots, g\}$ of cardinal g_1. Set $\mathfrak{L}' := \{1, \dots, g\} \setminus \mathfrak{L}$. Then the set of cycles

$$\{\gamma_\ell,\ \ell \in \mathfrak{L}\} \cup \{\gamma'_\ell,\ \ell \in \mathfrak{L}'\}$$

defines admissible circulation conditions.

A finite set of realizations for Beltrami fields

For each subset $\mathfrak{L} \subset \{1, \dots, g\}$ (including trivial ones), the problem

$$\begin{cases} (\blacklozenge)\ \ \mathbf{curl}\, \boldsymbol{u} = \kappa \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \\ (\heartsuit)\ \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on} \quad \partial\Omega \\ (\spadesuit)\ \ \oint_{\gamma_\ell} \boldsymbol{u} \cdot \boldsymbol{t} = 0,\ \ \forall \ell \in \mathfrak{L} \quad \oint_{\gamma'_\ell} \boldsymbol{u} \cdot \boldsymbol{t} = 0,\ \ \forall \ell \in \mathfrak{L}' & \end{cases}$$

has a well-defined real and discrete spectrum denoted $\mathfrak{S}(\Omega, \mathfrak{L})$.

Questions

Dependence on $\mathfrak{L}$ of the spectrum $\mathfrak{S}(\Omega, \mathfrak{L})$:

1. Does $\mathfrak{S}(\Omega, \mathfrak{L})$ vary with $\mathfrak{L}$?
2. Does $\mathfrak{S}(\Omega, \mathfrak{L})$ depend on the cardinal $\#\mathfrak{L} := g_1$ only?
3. If $\mathfrak{S}(\Omega, \mathfrak{L}_1)$ and $\mathfrak{S}(\Omega, \mathfrak{L}_2)$ are distinct, do they differ by a finite or infinite set of eigenvalues?
4. Do there exist other ways of defining admissible circulation conditions?

We try to bring answers by addressing axisymmetric domains Ω.

Nota bene:
From now on, we concentrate on non-zero κ's.
$\Longrightarrow$
We drop the gauge condition $\operatorname{div} \boldsymbol{u} = 0$.
Boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ is equivalent to $\boldsymbol{n} \cdot \mathbf{curl}\, \boldsymbol{u} = 0$.
We do not discuss *harmonic fields*.

Outline

Axisymmetry

Denote by $\mathbb{T}$ the 1D torus $\mathbb{R}/(2\pi\mathbb{Z})$.

- $\Omega \subset \mathbb{R}^3$ axisymmetric $\Longrightarrow \exists$ a symmetry axis $\mathfrak{A}$ such that Ω invariant by rotations around this axis.
- $(r, \theta, z) \subset \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}$ cylindrical coordinates associated with $\mathfrak{A}$
- $\omega \subset \mathbb{R}_+ \times \mathbb{R}$ *meridian domain* of Ω, i.e.

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^3, \quad (r, z) \in \omega \quad \text{and} \quad \theta \in \mathbb{T}\}.$$

Assume for simplicity that $\overline{\Omega} \cap \mathfrak{A}$ is empty.

A canonical set of cycles

The number g is the number of connected components $\partial_\ell\omega$ of $\partial\omega$
Choose $\theta_\ell \in \mathbb{T}$ and $(r_\ell, z_\ell) \in \partial_\ell\omega$, for $\ell = 1, \dots, g$.

1. Set $\gamma_\ell = \{\boldsymbol{x} \in \mathbb{R}^3, \quad (r, z) \in \partial_\ell\omega, \quad \theta = \theta_\ell\}$
2. Set $\gamma'_\ell = \{\boldsymbol{x} \in \mathbb{R}^3, \quad (r, z) = (r_\ell, z_\ell), \quad \theta \in \mathbb{T}\}$

This set is a canonical set of cycles for Ω.

The curl equation

Let (u_r, u_θ, u_z) be the cylindrical components of $\boldsymbol{u} = (u_1, u_2, u_3)$:

$$u_r = \cos\theta\; u_1 + \sin\theta\; u_2, \quad u_\theta = -\sin\theta\; u_1 + \cos\theta\; u_2, \quad u_z = u_3.$$

The eigen-equation $\mathbf{curl}\,\boldsymbol{u} = \kappa\boldsymbol{u}$ becomes

$$\begin{cases} \frac{1}{r}\partial_\theta u_z - \partial_z u_\theta & = \kappa u_r \\ \partial_z u_r - \partial_r u_z & = \kappa u_\theta \\ \partial_r u_\theta + \frac{1}{r} u_\theta - \frac{1}{r}\partial_\theta u_r & = \kappa u_z \end{cases} \quad \text{for} \quad (r, z, \theta) \in \omega \times \mathbb{T}$$

We can diagonalize this system by angular Fourier transformation

$$u \mapsto u^m(r,z) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-\mathrm{i}m\theta} u(r,\theta,z)\,d\theta, \quad m \in \mathbb{Z}$$

and obtain

$$(\blacklozenge^m) \quad \begin{cases} \frac{1}{r}\mathrm{i}m\, u_z^m - \partial_z u_\theta^m & = \kappa u_r^m \\ \partial_z u_r^m - \partial_r u_z^m & = \kappa u_\theta^m \\ \partial_r u_\theta^m + \frac{1}{r} u_\theta^m - \frac{1}{r}\mathrm{i}m\, u_r^m & = \kappa u_z^m, \end{cases} \quad \text{for} \quad (r,z) \in \omega, \quad \forall m \in \mathbb{Z}$$

Boundary and circulation conditions

The normal $\boldsymbol{n}$ to $\partial\Omega$ has no angular component: $\boldsymbol{n} = n_r \boldsymbol{e}_r + n_z \boldsymbol{e}_z$

1. The boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ is transformed into

$$(\heartsuit^m) \qquad n_r\, u_r^m + n_z\, u_z^m = 0 \quad \text{for} \quad (r, z) \in \partial\omega, \quad \forall m \in \mathbb{Z}$$

2. A unit tangent field to the cycle $\gamma_\ell = \partial_\ell\omega$ is given by $\boldsymbol{t} = n_r \boldsymbol{e}_z - n_z \boldsymbol{e}_r$
The circulation condition $\oint_{\gamma_\ell} \boldsymbol{u} \cdot \boldsymbol{t} = 0$ is transformed into

$$(\spadesuit_\ell^m) \qquad \oint_{\partial_\ell\omega} n_r\, u_z^m - n_z\, u_r^m = 0 \qquad \forall m \in \mathbb{Z}$$

3. For $\gamma_\ell' = (r_\ell, z_\ell)\mathbb{T}$, we find $\boldsymbol{t} = \boldsymbol{e}_\theta$ and the condition $\oint_{\gamma_\ell'} \boldsymbol{u} \cdot \boldsymbol{t} = 0$ is

$$(\spadesuit_\ell'^m) \qquad \oint_{\mathbb{T}} u_\theta(r_\ell, \theta, z_\ell)\, d\theta = 0, \quad \text{with} \quad u_\theta = \sum_{m\in\mathbb{Z}} e^{\mathrm{i}m\theta} u_\theta^m$$

Hence

Lemma

The condition $\oint_{\gamma_\ell'} \boldsymbol{u} \cdot \boldsymbol{t} = 0$ is equivalent to condition $(\spadesuit_\ell'^0) : u_\theta^0(r_\ell, z_\ell) = 0$

Elimination

Recall equations ($\blacklozenge^m$.1) and ($\blacklozenge^m$.3)

$$\frac{1}{r}\mathrm{i}m\,u_z^m - \partial_z u_\theta^m = \kappa u_r^m \quad \text{and} \quad \frac{1}{r}\partial_r(ru_\theta^m) - \frac{1}{r}\mathrm{i}m\,u_r^m = \kappa u_z^m$$

This can be viewed as the system

$$\begin{cases} \frac{1}{r}\mathrm{i}m\,u_z^m - \kappa u_r^m = \partial_z u_\theta^m \\ \kappa u_z^m + \frac{1}{r}\mathrm{i}m\,u_r^m = \frac{1}{r}\partial_r(ru_\theta^m) \end{cases}$$

whose solution provides

Elimination formulas

($\clubsuit^m$)
$$\begin{cases} u_z^m = d_m[\kappa]\Big(\frac{1}{r^2}\mathrm{i}m\,\partial_z(\Phi) + \frac{1}{r}\kappa\,\partial_r(\Phi)\Big) \\ u_r^m = d_m[\kappa]\Big(-\frac{1}{r}\kappa\,\partial_z(\Phi) + \frac{1}{r^2}\mathrm{i}m\,\partial_r(\Phi)\Big) \end{cases}$$

where we have set

$$d_m[\kappa] = \frac{1}{\kappa^2 - \frac{m^2}{r^2}} \quad \text{and} \quad \Phi = ru_\theta^m.$$

Outline

Diagonalization

For each $m \in \mathbb{Z}$, investigate problem (♦^{m})-(♥^{m}) possibly completed by a mixture of circulation conditions (♠^{m}) and (♠$'^m$).

Remark for non-zero m

For $m \neq 0$, conditions (♠$'^m$) are automatically satisfied.

- We expect that the problem (♦^{m})-(♥^{m}) will be self-adjoint by itself...
- Are conditions (♠^{m}) redundant?

The road map (visiting possibly the points in a different order):

1. We choose $m \in \mathbb{Z}$
2. We set $\Phi(r, z) = r\, u_\theta^m(r, z)$, for $(r, z) \in \omega$
3. Using elimination formulas (♣^{m}) we transform problem (♦^{m})-(♥^{m}) into a problem set on Φ.
4. We complete by circulations conditions (♠) read on Φ.

Elimination in boundary and circulation conditions

Denote $\boxed{\partial_{\boldsymbol{n}} = n_r\partial_r + n_z\partial_z \quad \text{and} \quad \partial_{\boldsymbol{t}} = n_r\partial_z - n_z\partial_r \quad \text{on} \quad \partial\omega}$

1. Elimination formulas inserted in Boundary Cond. $n_r\, u_r^m + n_z\, u_z^m = 0 \quad \Longrightarrow$

   ($\heartsuit^m$) $\quad \mathrm{i}m\,\partial_{\boldsymbol{n}}\Phi - r\,\kappa\,\partial_{\boldsymbol{t}}\Phi = 0 \quad \text{on} \quad \partial\omega$ *(oblique derivative)*

2. Elimination formulas in Circulation Cond. $\oint_{\partial_\ell\omega}(n_r\, u_z^m - n_z\, u_r^m) = 0 \quad \Longrightarrow$

   ($\spadesuit_\ell^m$) $$\oint_{\partial_\ell\omega} d[\kappa]\Big(\tfrac{1}{r^2}\mathrm{i}m\,\partial_{\boldsymbol{t}}\Phi + \tfrac{1}{r}\kappa\,\partial_{\boldsymbol{n}}\Phi\Big) = 0$$

3. If $\boxed{m\neq 0}$, ($\heartsuit^m$) $\Longrightarrow \frac{1}{r}\kappa\,\partial_{\boldsymbol{n}}\Phi = -\mathrm{i}\frac{1}{m}\kappa^2\,\partial_{\boldsymbol{t}}\Phi$ on $\partial\omega$. Therefore

$$\oint_{\partial_\ell\omega} d[\kappa]\Big(\tfrac{1}{r^2}\mathrm{i}m\,\partial_{\boldsymbol{t}}\Phi + \tfrac{1}{r}\kappa\,\partial_{\boldsymbol{n}}\Phi\Big) = \oint_{\partial_\ell\omega} d[\kappa]\Big(\tfrac{1}{r^2}\mathrm{i}m\,\partial_{\boldsymbol{t}}\Phi - \mathrm{i}\tfrac{1}{m}\kappa^2\,\partial_{\boldsymbol{t}}\Phi\Big)$$
$$= \mathrm{i}\frac{1}{m}\oint_{\partial_\ell\omega} d[\kappa]\Big(\tfrac{m^2}{r^2}\,\partial_{\boldsymbol{t}}\Phi - \kappa^2\,\partial_{\boldsymbol{t}}\Phi\Big)$$
$$= -\mathrm{i}\frac{1}{m}\oint_{\partial_\ell\omega}\partial_{\boldsymbol{t}}\Phi = 0$$

Boundary and circulation conditions when $m \neq 0$

We put together results concerning

- boundary conditions ($\heartsuit^m$)
- circulation conditions ($\spadesuit_\ell^m$) and ($\spadesuit_\ell'^m$)

Proposition

Assume that $\overline{\omega}$ is disjoint from the rotation axis $\mathfrak{A}$.
Let $\boldsymbol{u}$ satisfy the boundary condition ($\heartsuit$) : $\boldsymbol{u} \cdot \boldsymbol{n} = 0$.
Let $m \neq 0$ and set

$$\Phi = r u_\theta^m \quad \text{and} \quad \boldsymbol{u}^m = e^{im\theta}(u_r^m \boldsymbol{e}_r + u_\theta^m \boldsymbol{e}_\theta + u_z^m \boldsymbol{e}_z)$$

Then

- Φ satisfy the complex oblique derivative condition

  ($\heartsuit^m$) $$\mathrm{i} m\, \partial_{\boldsymbol{n}} \Phi - r\, \kappa\, \partial_{\boldsymbol{t}} \Phi = 0 \quad \text{on} \quad \partial\omega$$

- $\boldsymbol{u}^m$ satisfies all circulation conditions ($\spadesuit_\ell^m$) and ($\spadesuit_\ell'^m$) $\forall \ell$.

Outline

$m = 0$: Boundary and circulation conditions

Boundary conditions

The boundary condition (♥^{0}) is

$$\partial_t \Phi = 0 \quad \text{on} \quad \partial\omega$$

Therefore Φ is constant on each connected component $\partial_\ell\omega$ of $\partial\omega$.

Circulation conditions

Assume boundary conditions (♥^{0}).

- Take $m = 0$ in (♠$_\ell^m$):

(♠$_\ell^0$)
$$\oint_{\partial_\ell\omega} \partial_n \Phi \, \frac{1}{r} \, \mathrm{d}\sigma = 0$$

- Condition (♠$_\ell'^{0}$) is $\Phi(r_\ell, z_\ell) = 0$ for a $(r_\ell, z_\ell) \in \partial_\ell\omega$. With (♥^{0}) :

(♠$_\ell'^{0}$)
$$\Phi\big|_{\partial_\ell\omega} = 0$$

$m = 0$: Variational spaces

$$H^1_\heartsuit(\omega) := \{\Phi \in H^1(\omega), \quad \partial_t \Phi = 0 \quad \text{on} \quad \partial\omega\}$$

For $\Phi \in H^1_\heartsuit(\omega)$ let

$$\mathfrak{B} : \Phi \longmapsto (b_\ell)_{\ell=1}^g \quad \text{with} \quad b_\ell = \oint_{\partial_\ell \omega} \partial_{\boldsymbol{n}} \Phi \, \frac{1}{r} \, \mathrm{d}\sigma$$

$$\mathfrak{C} : \Phi \longmapsto (c_\ell)_{\ell=1}^g \quad \text{with} \quad c_\ell = \Phi\big|_{\partial_\ell \omega}$$

and denote for $\Phi, \Psi \in H^1_\heartsuit(\omega)$

$$\langle \mathfrak{B}\Phi, \mathfrak{C}\Psi \rangle := \sum_{\ell=1}^g b_\ell(\Phi)\, c_\ell(\Psi)$$

A family of variational spaces

Let us choose a subspace $\mathfrak{E}$ of $\mathbb{R}^g$ (possibly trivial).

Define the variational space $V = V[\mathfrak{E}]$ as

$$V[\mathfrak{E}] := \{\Phi \in H^1_\heartsuit(\omega), \quad \mathfrak{C}\Phi \in \mathfrak{E}\}.$$

$m = 0$: Self-adjoint realizations

Elimination formulas are

$$(♣^0) \qquad u_z^m = \frac{1}{r}\frac{1}{\kappa}\,\partial_r\Phi \quad \text{and} \quad u_r^m = -\frac{1}{r}\frac{1}{\kappa}\,\partial_z\Phi$$

Equation (♦.2) is $\partial_z u_r^m - \partial_r u_z^m = \kappa u_\theta^m$. It yields

$$(*) \qquad -\partial_z \frac{1}{r}\,\partial_z\Phi - \partial_r\frac{1}{r}\,\partial_r\Phi = \kappa^2\frac{1}{r}\Phi$$

Theorem

The variational eigen-problem: Find $\kappa \in \mathbb{R}$ s.t. $\exists$ non-zero $\Phi \in V[\mathfrak{E}]$

$$(♦_{[0]}) \qquad \forall\Psi \in V[\mathfrak{E}], \quad \int_\omega \nabla\Phi\,\nabla\Psi\,\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z = \kappa^2\int_\omega \Phi\,\Psi\,\frac{1}{r}\,\mathrm{d}r\,\mathrm{d}z$$

has a discrete set of solutions denoted by $\mathfrak{S}(\omega, \mathfrak{E})$. The eigenvectors Φ satisfy the eigen-equation $(*)$ and the natural condition

$$\langle \mathfrak{B}\Phi, \mathfrak{C}\Psi\rangle = 0 \quad \forall\Psi \in V[\mathfrak{E}] \qquad \text{i.e.} \qquad \mathfrak{B}\Phi \in \mathfrak{E}^\perp$$

Outline

$m = 0$: Example of a circular torus

The meridian domain ω is a disk, with center $(r_0, 0)$ and radius $R < r_0$.
The boundary $\partial\omega$ has one connected component.

So $g = 1$ and we have only two distinct choices for the space $\mathfrak{C}$

- If $\mathfrak{C} = \{0\}$, $V = H^1_0(\omega)$,
  $\Rightarrow$ Circulation along cycle γ'_1 is 0, $\boxed{g_1 = 0}$ Defines problem $\boxed{\text{P0}}$
- If $\mathfrak{C} = \mathbb{R}$, $V = \{\Phi \in H^1(\omega),\ \partial_t\Phi = 0 \text{ on } \partial\omega\}$,
  $\Rightarrow$ Circulation along cycle γ_1 is 0, $\boxed{g_1 = 1}$ Defines problem $\boxed{\text{P1}}$

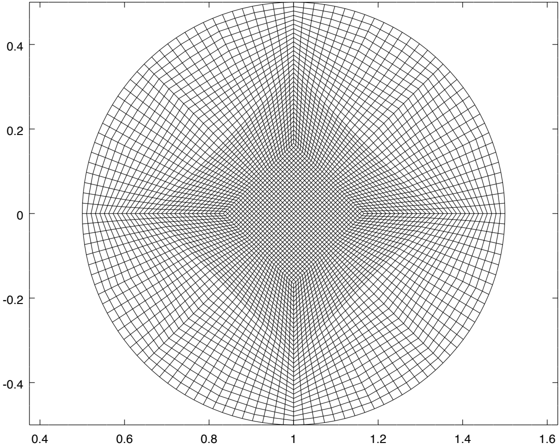
P0 As standard for a Dirichlet problem: On $H^1_0(\omega)$

$$\int_\omega \nabla\Phi\,\nabla\Psi\,\tfrac{1}{r}\,\mathrm{d}r\,\mathrm{d}z = \kappa^2 \int_\omega \Phi\,\Psi\,\tfrac{1}{r}\,\mathrm{d}r\,\mathrm{d}z$$

P1 Penalize the tangential derivative: On $H^1(\omega)$, with large Λ,

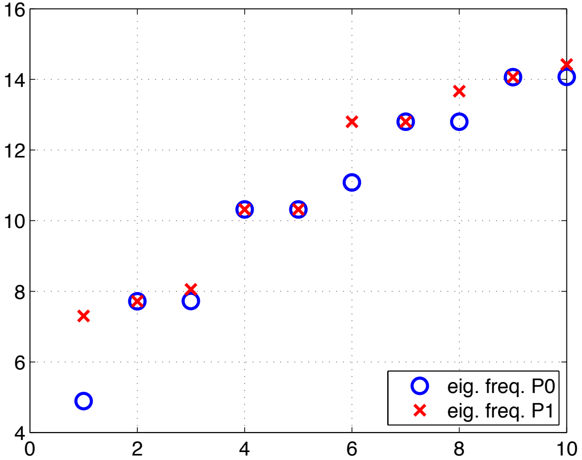
$$\int_\omega \nabla\Phi\,\nabla\Psi\,\tfrac{1}{r}\,\mathrm{d}r\,\mathrm{d}z + \Lambda \int_{\partial\omega} \partial_t\Phi\,\partial_t\Psi\,\mathrm{d}\sigma = \kappa^2 \int_\omega \Phi\,\Psi\,\tfrac{1}{r}\,\mathrm{d}r\,\mathrm{d}z$$

Circular torus: radius $R = 0.5$ and center $r_0 = 1$



The mesh for Q1 elements

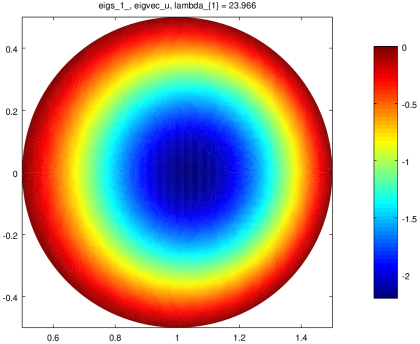
Circular torus: radius $R = 0.5$ and center $r_0 = 1$



Positive κ_j versus rank j for problems P0 and P1

$0 < \kappa_1 <= \kappa_2 <= \dots$ P1 has one null ev: $\kappa_0 = 0$

($m = 0$) Circular torus $R = 0.5$, $r_0 = 1$. E-vector Φ

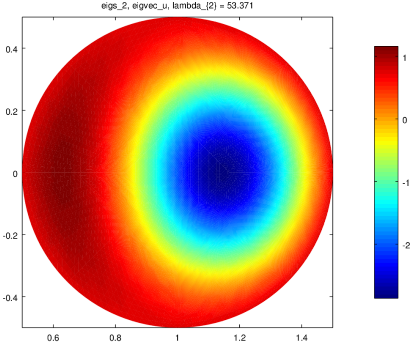


P0 eigvect. 1

$\kappa_1 = 4.8943$

Compare with

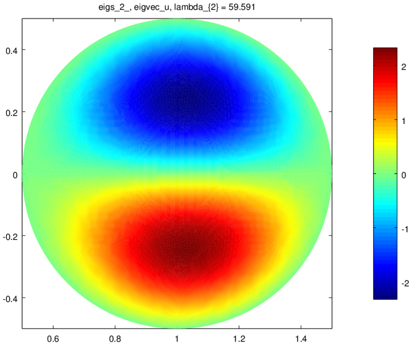
$\kappa_1 = 4.89561$ (extrapolated, [ACRVV])

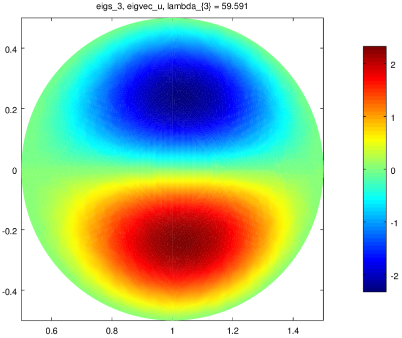


P1 eigvect. 1

$\kappa_1 = 7.3024$

($m = 0$) Circular torus $R = 0.5$, $r_0 = 1$. E-vector Φ





P0 eigvect. 2

$\kappa_2 = 7.7165$

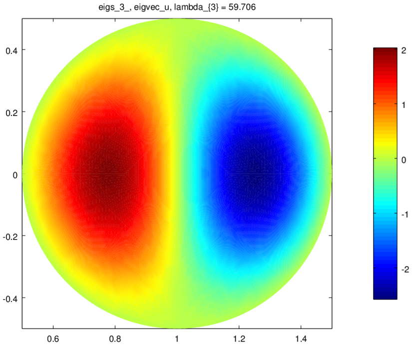
P1 eigvect. 2

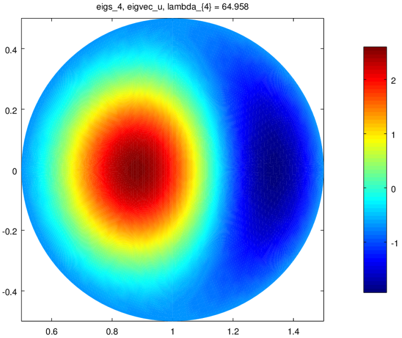
$\kappa_2 = 7.7165$

Odd

8-digit coincidence

($m = 0$) Circular torus $R = 0.5$, $r_0 = 1$. E-vector Φ





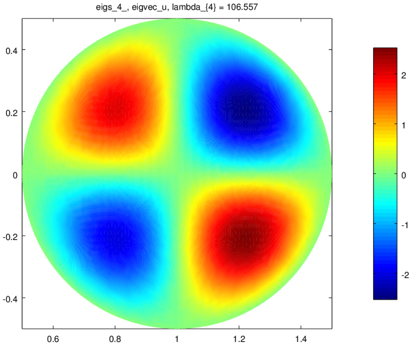
P0 eigvect. 3

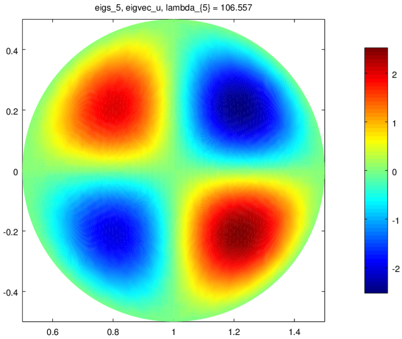
$\kappa_3 = 7.7239$

P1 eigvect. 3

$\kappa_3 = 8.0558$

($m = 0$) Circular torus $R = 0.5$, $r_0 = 1$. E-vector Φ





P0 eigvect. 4
$\kappa_4 = 10.3162$

P1 eigvect. 4
$\kappa_4 = 10.3162$

Odd

Outline

$m \neq 0$: Eigenproblem

Non-linear with respect to κ.

Note that $\kappa^2\tau(\kappa) = \dfrac{1}{r^2 - \frac{m^2}{\kappa^2}}$ and set $\boxed{\rho := \dfrac{m}{\kappa}}$

Introduce the forms $a[\rho]$ and $b[\rho]$

$$a[\rho](\Phi,\Psi) = \int_\omega (r^2-\rho^2)^{-1}\Big\{\nabla\Phi\cdot\nabla\Psi + i\rho(r^2-\rho^2)^{-1}\big(\partial_z\Phi\,\Psi - \Phi\,\partial_z\Psi\big)\Big\}\, r\,\mathrm{d}r\mathrm{d}z$$

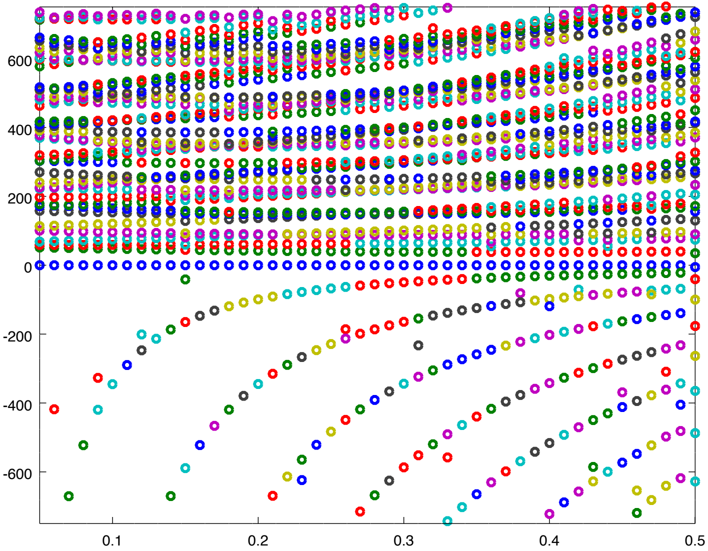
$$b[\rho](\Phi,\Psi) = i\,\frac{1}{2\rho}\int_{\partial\omega}(r^2-\rho^2)^{-1}\Big\{\partial_{\boldsymbol{t}}\Phi\;\Psi - \Phi\;\partial_{\boldsymbol{t}}\Psi\Big\}\, r^2\mathrm{d}\sigma$$

$[\rho]$ - eigenproblem: Find non-zero $\Phi \in H^1(\omega)$ and $\lambda \in \mathbb{R}_+$ s.t.

$$(\blacklozenge_{[\rho]}) \qquad \forall\Psi \in H^1(\omega), \quad a[\rho](\Phi,\Psi) + b[\rho](\Phi,\Psi) = \lambda\int_\omega \Phi\Psi\,\frac{1}{r}\,\mathrm{d}r\mathrm{d}z\,.$$

Set $\kappa = \sqrt{\lambda}$ and look for ρ such that $\kappa\rho =: m \in \mathbb{N}^*$.

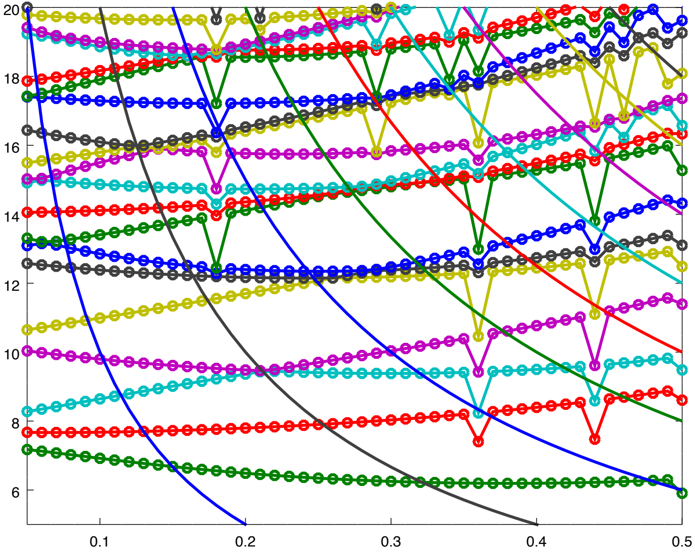
($m \neq 0$) Circular torus $R = 0.5$, $r_0 = 1$: κ versus ρ



For $\rho \in [0.05, 0.5]$ the 40 eigenvalues with least module.

Presence of negative evs

($m \neq 0$) Circular torus $R = 0.5$, $r_0 = 1$: κ versus ρ



For $\rho \in [0.05, 0.5]$ sort the positive evs.

Presence of spurious evs

Plot $\rho \mapsto \frac{m}{\rho}$ $m = 1, \ldots$

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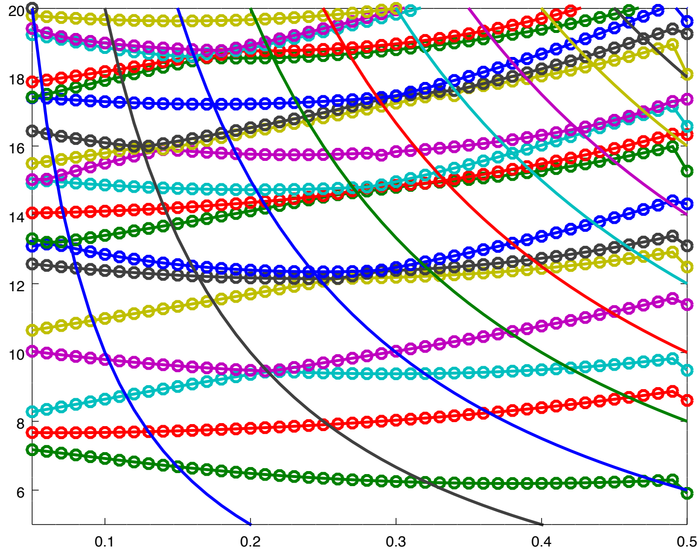

For $\rho \in [0.05, 0.5]$ sort the positive evs and eliminate spurious evs

Plot $\rho \mapsto \frac{m}{\rho}$ $m = 1, 2, \ldots$

Intersection points give the κ's and m's

Outline

Circular torus $R = 0.5$, $r_0 = 1$: Comparison of results

Compare

- our results obtained by 2D parametric FEM computations ($[\rho]$-method)
- those of [ACRVV] obtained by 3D computations incorporating circulation conditions

	# DOF	κ_1	κ_2	κ_3	κ_4	κ_5
Finest mesh	247239	4.9151	6.2717	6.2724	6.3256	6.3256
Extrapolated		4.8956	6.2306	6.2276	6.2798	6.2811
2D	5185	4.8955	6.225*	6.225*	6.275*	6.275*
Angular freq.		$m = 0$	$m = 2$		$m = 3$	

The next (visible) κ is $\kappa \simeq 6.682$ with $m = 1$

New questions

1. We are very curious of what happens in the parametric region

$$\rho \in [r_{\min}, r_{\max}]$$

where $r_{\min}$ and $r_{\max}$ are the minimum and maximal values of r for $(r, z) \in \overline{\omega}$.
We do not yet know if computations are possible to solve problem $(\blacklozenge_{[\rho]})$

2. Question of the dependence on circulation conditions on the spectrum for axisymmetric domains:
   (a) No influence on non-axisymmetric modes.
   (b) In presence of horizontal symmetry, no influence on odd axisymmetric modes.
   (c) For all remaining ones (infinitely many) the spectrum does depend on circulation conditions.

3. Performances of the p-version of FEM (debugging of library necessary)

Still under investigation.

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Thank you for your attention