Eléments finis mixtes de Raviart Thomas de type Petrov-Galerkin

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Conference in honor of Abderrahmane Bendali
Université de Pau et des Pays de l'Adour
mardi 12 décembre 2017

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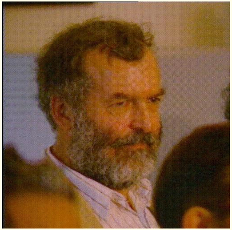
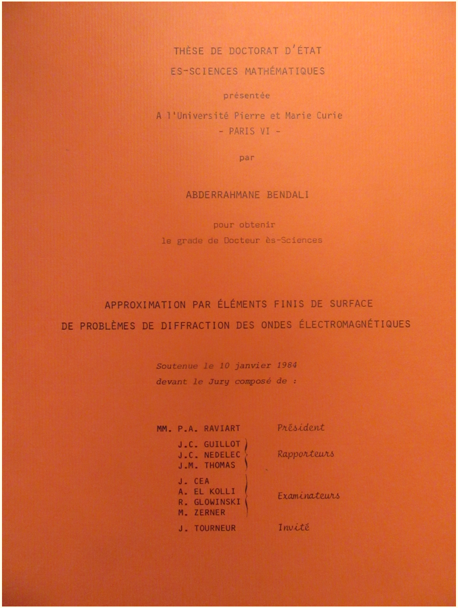
Frères et sœurs en mathématiques...







Thèse d'Etat d'Abderrahmane à l'X, 10 janvier 1984



THÈSE DE DOCTORAT D'ÉTAT

ES-SCIENCES MATHÉMATIQUES

présentée

A l'Université Pierre et Marie Curie

- PARIS VI -

par

ABDERRAHMANE BENDALI

pour obtenir

le grade de Docteur ès-Sciences

APPROXIMATION PAR ÉLÉMENTS FINIS DE SURFACE

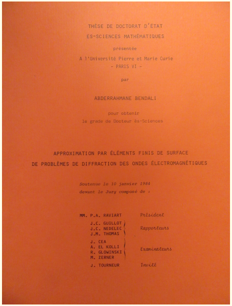
DE PROBLÈMES DE DIFFRACTION DES ONDES ÉLECTROMAGNÉTIQUES

Soutenue le 10 janvier 1984

devant le Jury composé de :

MM. P.A. RAVIART	*Président*
J.C. GUILLOT J.C. NEDELEC J.M. THOMAS	*Rapporteurs*
J. CEA A. EL KOLLI R. GLOWINSKI M. ZERNER	*Examinateurs*
J. TOURNEUR	*Invité*

Je me souviens...



P.-A. Raviart (le jour-même) :

"tu n'as même pas mis de cravatte !"

J.-C. Nédélec (en privé, plus tard) :

"dans les articles, il n'y a pas l'information...
tout est dans la thèse de Bendali !"

Une référence indispensable pour les champs de vecteurs...

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **107**, 537–560 (1985)

A Variational Approach for the Vector Potential Formulation of the Stokes and Navier–Stokes Problems in Three Dimensional Domains

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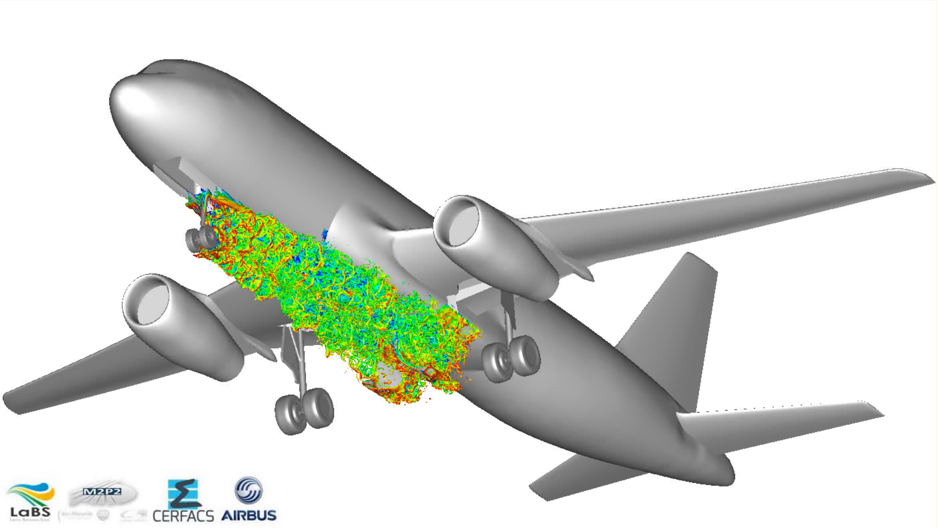
Discrete vector fields (ProLB software, 2016)



Raviart-Thomas finite elements of Petrov-Galerkin type

Outline

- Finite volumes
- Mixed finite elements for the Poisson equation
- Butterfly Petrov-Galerkin finite volume scheme
- A new variant of the same framework
- Construction a possible dual basis vector field
- Sufficient stability conditions
- Conclusion

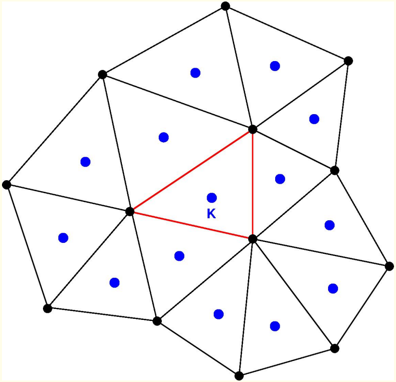
Finite volumes

Motivation: finite volumes for the Navier Stokes equations

$$\frac{\partial W}{\partial t} + \mathrm{div} F(W) + \mathrm{div}\, G(W,\ \nabla W) = 0$$

mean value in the triangle K : $W_K \equiv \frac{1}{|K|} \int_K W(x)\, \mathrm{d}x$

$$\frac{\mathrm{d} W_K}{\mathrm{d}t} + \frac{1}{|K|} \int_{\partial K} F \bullet n \, \mathrm{d}\gamma + \frac{1}{|K|} \int_{\partial K} G \bullet n \, \mathrm{d}\gamma = 0$$



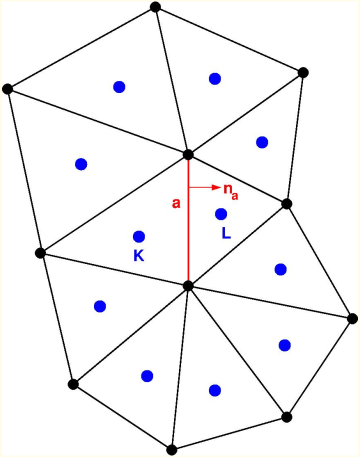
Finite volumes *(ii)*

Model problem

given the mean scalar values u_K in the triangles $K \in \mathcal{T}^2$

how to compute the gradient ∇u

on the edges $a \in \mathcal{T}^1$ of the mesh $\mathcal{T}$?



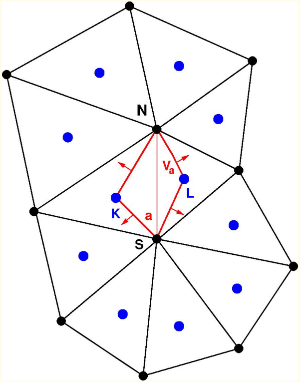
Noh's control volume (1964)

Approach the mean flux on the edge a

by the mean flux on a control volume V_a around the edge:

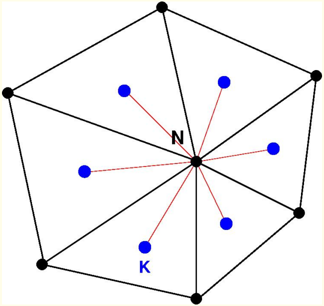
$$\frac{1}{|a|}\int_a \nabla u \,\mathrm{d}\gamma \simeq \frac{1}{|V_a|}\int_{V_a} \nabla u \,\mathrm{d}x = \frac{1}{|V_a|}\int_{\partial V_a} u \bullet n \,\mathrm{d}x$$

The derivation is replaced by a problem of interpolation



Model problem: Laplace equation

Interpolation at the vertices compatible with the linear functions



Nodal value $\overline{u_N}$ with $N \in \mathcal{T}^0$ given from the elements $K \in \mathcal{T}^2$

by $\overline{u_N} = \sum_{\partial K \ni N} \alpha_{NK} \, u_K ,$

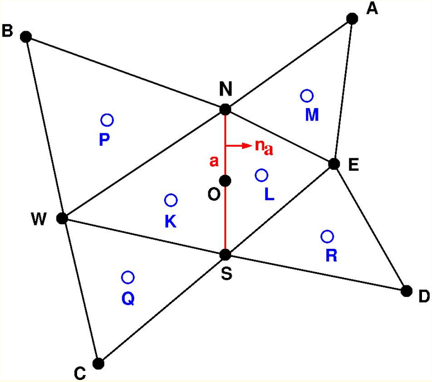
Relations satisfied by the coefficients α_{NK}

$$\sum_{\partial K \ni N} \alpha_{NK} = 1, \quad \sum_{\partial K \ni N} \alpha_{NK} \, x_K = x_N, \quad \sum_{\partial K \ni N} \alpha_{NK} \, y_K = y_N$$

Butterfly stencil for the computation of the gradient

discrete scheme

$$\nabla u(a) = \sum_{K \in \mathcal{T}^2} \alpha_{aK} \, u_K$$



impose that the discrete scheme is exact

for an appropriate linear space of polynomials (FD, 1992)

Mixed finite elements for the Poisson equation

Continuous problem: $u \in \mathrm{L}^2(\Omega)$, $p \in \mathrm{H}(\mathrm{div})$

$p = \nabla u$, $\mathrm{div} p + f = 0$

$$\int_\Omega p \bullet q \, \mathrm{d}x + \int_\Omega u \, \mathrm{div} q \, \mathrm{d}x = 0 \quad \text{for each vector field } q \in \mathrm{H}(\mathrm{div})$$

$$\int_\Omega (\mathrm{div} p + f) \, v \, \mathrm{d}x = 0 \quad \text{for any scalar field } v \in \mathrm{L}^2(\Omega)$$

Discrete problem

$u_\mathcal{T} \in \mathrm{L}^2_\mathcal{T} \equiv P_0$, $p_\mathcal{T} \in \mathrm{H}_\mathcal{T}(\mathrm{div}) \equiv RT$; Raviart-Thomas (1977)

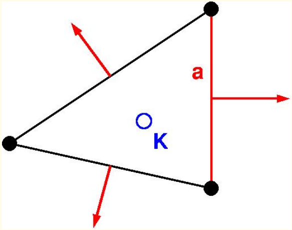
$$p_\mathcal{T} = \sum_{a \in \mathcal{T}^1} p_a \, \varphi_a$$

$$\int_b \varphi_a(x) \bullet n_b \, \mathrm{d}\gamma = \delta_{ab} \, , \; a \in \mathcal{T}^1 \, , \; b \in \mathcal{T}^1$$

$$\varphi_a(x) = \alpha_K + \beta_K \, x \, , \; x \in K \in \mathcal{T}^2$$

$$u_\mathcal{T}(x) = u_K \, , \; x \in K \in \mathcal{T}^2$$

$N_e + N_a$ scalar unknowns



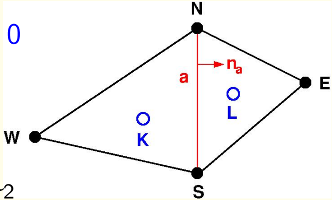
Mixed finite elements for the Poisson equation *(ii)*

$$p_{\mathcal{T}} = \sum_{b \in \mathcal{T}^1} p_b \, \varphi_b \,, \quad u_{\mathcal{T}} \in P_0 \,,$$

$$\int_\Omega p_{\mathcal{T}} \bullet \varphi_a \, \mathrm{d}x + \int_\Omega u_{\mathcal{T}} \, \mathrm{div} \varphi_a \, \mathrm{d}x = 0 \,, \qquad \forall \, a \in \mathcal{T}^1$$

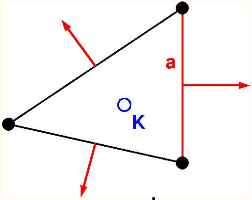
$$\sum_{b \in \mathcal{T}^1} \Big(\int_\Omega \varphi_a \bullet \varphi_b \, \mathrm{d}x \Big) \, p_b + u_K - u_L = 0$$

$$\partial^c a = (K, L) \,, \quad a \in \mathcal{T}^1$$



$$\int_K \mathrm{div} p_{\mathcal{T}} \, \mathrm{d}x + \int_K f \, \mathrm{d}x = 0 \,, \quad \forall \, K \in \mathcal{T}^2$$

$$\sum_{a \in \mathcal{T}^1 \cap \partial K} p_a \, \mathrm{d}\gamma + \int_K f \, \mathrm{d}x = 0 \,, \quad K \in \mathcal{T}^2$$



A finite volume method ?

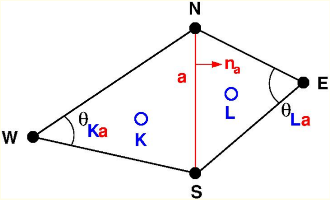
No! The mass-matrix induces a nonlocal gradient operator!

Mass lumping of Baranger, Maitre and Oudin (1996)

$$p_{\mathcal{T}} = \sum_{b \in \mathcal{T}^1} p_b \, \varphi_b \,, \quad u_{\mathcal{T}} \in P_0 ,$$

$$\sum_{b \in \mathcal{T}^1} \Big(\int_\Omega \varphi_a \bullet \varphi_b \, \mathrm{d}x \Big) \, p_b + u_K - u_L = 0 \,, \quad a \in \mathcal{T}^1$$

Replace the mass matrix $\int_\Omega \varphi_a \bullet \varphi_b \, \mathrm{d}x$ by a correct approximation

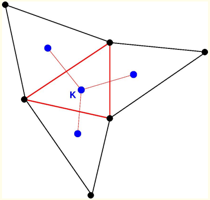


then $\quad p_a = \dfrac{u_L - u_K}{\xi_{Ka} + \xi_{La}}$

with $\quad \xi_{Ka} = \frac{1}{2} \, \mathrm{cotan} \, \theta_{Ka}$

$$\partial^c a = (K, L) \,, \quad a \in \mathcal{T}^1$$

$$\sum_{a \in \mathcal{T}^1 \cap \partial K} p_a \, \mathrm{d}\gamma + \int_K f \, \mathrm{d}x = 0 \,, \quad K \in \mathcal{T}^2$$



VF4 scheme of Herbin (1995)

Petrov-Galerkin finite volumes

Reformulate the mixed finite element method
to enforce the explicitation of the gradient on an edge
in terms of the values in the triangles.

Continuous problem: $u \in \mathrm{L}^2(\Omega)$, $p \in \mathrm{H}(\mathrm{div})$,

$$(p\,,\,q) + (u\,,\,\mathrm{div}q) = 0, \qquad \forall\, q \in \mathrm{H}(\mathrm{div})$$
$$(\mathrm{div}p\,,\,v) = -(f\,,\,v), \qquad \forall\, v \in \mathrm{L}^2(\Omega)$$

Discrete problem: $u_{\mathcal{T}} \in \mathrm{L}^2_{\mathcal{T}} \equiv P_0$, $p_{\mathcal{T}} \in \mathrm{H}_{\mathcal{T}}(\mathrm{div}) \equiv RT$

test functions $v \in \mathrm{L}^2_{\mathcal{T}}$, $q \in \mathrm{H}^\star_{\mathcal{T}}(\mathrm{div})$

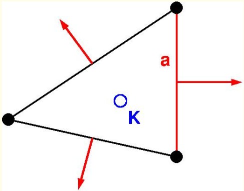
Discrete functional space for test functions $\mathrm{H}^\star_{\mathcal{T}}(\mathrm{div})$

- generated by vector fields $\varphi^\star_a$: $\mathrm{H}^\star_{\mathcal{T}}(\mathrm{div}) = \mathrm{span}\,(\varphi^\star_a\,,\, a \in \mathcal{T}^1)$
- conforming in the space $\mathrm{H}(\mathrm{div})$: $\varphi^\star_a \in \mathrm{H}(\mathrm{div})$
- the family $\{\varphi^\star_a\,,\, a \in \mathcal{T}^1\}$ represent exactly
  the algebraic dual basis of the Raviart-Thomas family
  for the L^2 scalar product: $(\varphi_a\,,\,\varphi^\star_b) = 0\,,\ \ \forall\, a \neq b \in \mathcal{T}^1$.

Petrov-Galerkin finite volumes *(ii)*

$$\left\{ \begin{array}{l} u_{\mathcal{T}} \in L^2_{\mathcal{T}}(\Omega)\,, \quad p_{\mathcal{T}} \in \mathrm{H}_{\mathcal{T}}(\mathrm{div}) \\ (p_{\mathcal{T}}\,,\, q) + (u_{\mathcal{T}}\,,\, \mathrm{div}\, q) \;=\; 0\,, \qquad \forall\, q \in \mathrm{H}^\star_{\mathcal{T}}(\mathrm{div}) \\ (\mathrm{div}\, p_{\mathcal{T}}\,,\, v) + (f\,,\, v) \;=\; 0\,, \qquad \forall\, v \in L^2_{\mathcal{T}}(\Omega)\,. \end{array} \right.$$

A finite volume scheme!

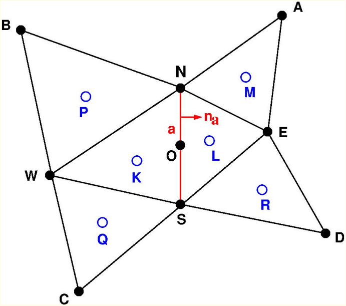


$$p_{\mathcal{T}} \;=\; \sum_{b \in \mathcal{T}^1} p_b\, \varphi_b\,, \quad u_{\mathcal{T}} \in P_0, \qquad \partial^c a = (K, L)\,, \quad a \in \mathcal{T}^1$$

$$(\varphi_a\,,\, \varphi^\star_a)\; p_a = -(u_{\mathcal{T}}\,,\, \mathrm{div}\, \varphi^\star_a)\;, \quad a \in \mathcal{T}^1$$

$$\sum_{a \in \mathcal{T}^1 \cap\, \partial K} p_a\, \mathrm{d}\gamma + (f\,,\, 1)_K = 0\,, \quad K \in \mathcal{T}^2$$

Butterfly Petrov-Galerkin finite volume scheme



Support $\mathcal{V}(\mathrm{SN})$ of the dual Raviart-Thomas basis function $\varphi^\star_{\mathrm{SN}}$

then

$$(\varphi_a \,,\, \varphi^\star_a)\; p_a = -(u_{\mathcal{T}} \,,\, \mathrm{div}\, \varphi^\star_a)$$
$$= \text{linear combination of } \; u_K,\, u_L,\, u_M,\, u_P,\, u_Q,\, u_R \;, \quad a \in \mathcal{T}^1$$

Presented at the 3th conference

"Finite Volumes for Complex Applications" (Porquerolles, 2002)

Butterfly Petrov-Galerkin finite volume scheme *(ii)*

Numerical tests with

Sophie Borel (DEA Orsay, 2002)
Christophe Le Potier (CEA Saclay)
Mahdi Tekitek (DEA Orsay, 2003).

Various schemes for boundary conditions

Recovering exact low degree polynomial solutions

Experimental convergence obtained for two-dimensional test cases

No complete mathematical understanding of the convergence

Presented by Mahdi Tekitek at the 4th conference
"Finite Volumes for Complex Applications" (Marrakech, 2005)

Working with Isabelle and Charles (Pau, June 2014)

A new variant of the same framework

$\Omega \subset \mathbb{R}^2$, bounded, convex, $\partial\Omega$ polyhedral

Right hand side $f \in \mathrm{L}^2(\Omega)$

Dirichlet problem $\qquad -\Delta u = f \;\text{ in}\,\Omega, \qquad u = 0 \;\text{ on}\,\partial\Omega$

Mixed variational formulation of the continuous problem:

$$u \in \mathrm{L}^2(\Omega), \qquad p \in \mathrm{H}(\mathrm{div})$$

$$\begin{aligned} (p\,,\,q) + (u\,,\,\mathrm{div} q) &= 0, & \forall\, q &\in \mathrm{H}(\mathrm{div}) \\ (\mathrm{div} p\,,\,v) &= -(f\,,\,v), & \forall\, v &\in \mathrm{L}^2(\Omega) \end{aligned}$$

Discrete problem: $\qquad u_{\mathcal{T}} \in \mathrm{L}^2_{\mathcal{T}} \equiv P_0, \;\; p_{\mathcal{T}} \in \mathrm{H}_{\mathcal{T}}(\mathrm{div}) \equiv RT$

test functions $\qquad v \in \mathrm{L}^2_{\mathcal{T}}, \;\; q \in \mathrm{H}^\star_{\mathcal{T}}(\mathrm{div})$

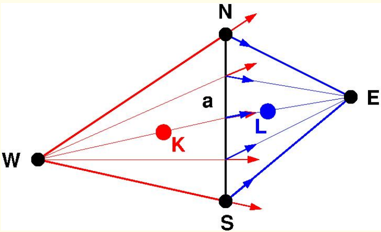
$$\mathrm{H}^\star_{\mathcal{T}}(\mathrm{div}) = <\varphi^\star_a\,,\, a \in \mathcal{T}^1>$$

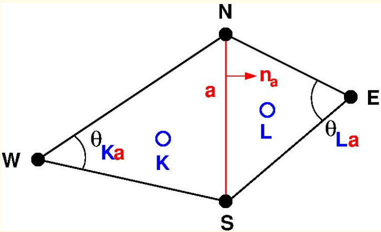
$$\varphi^\star_a \in \mathrm{H}(\mathrm{div})$$

$$(\varphi_a\,,\,\varphi^\star_b) = 0\,, \quad \forall\, a \neq b \in \mathcal{T}^1.$$

with Isabelle Greff and Charles Pierre (Pau, 2013-2017).

A new variant of the same framework *(ii)*





Raviart-Thomas basis function φ_a for the edge $a = (S, N)$

$$\varphi_a(x) = \begin{cases} \frac{1}{2|K|}(x - W) = \frac{1}{4|K|}\nabla|x - W|^2, \; x \in K \\ -\frac{1}{2|L|}(x - E) = -\frac{1}{4|L|}\nabla|x - E|^2, \; x \in L \\ 0 \quad \text{elsewhere}. \end{cases}$$

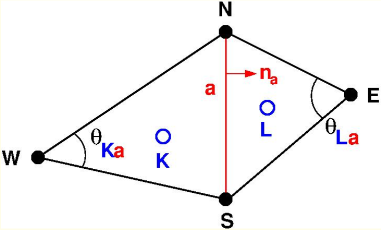
Dual Raviart-Thomas basis function $\varphi_a^\star$

hypothesis: $\mathrm{supp}(\varphi_a^\star) \subset \mathrm{supp}(\varphi_a)$

$$\text{then } p = \sum_{a \in \mathcal{T}^1} p_a \, \varphi_a \quad \text{and} \quad p_a = \frac{u_L - u_K}{(\varphi_a^\star, \varphi_a)}$$

$\varphi_a^\star \in \mathrm{H}(\mathrm{div})$ then $\varphi_a^\star \bullet n = 0$ on the four edges NWSEN.

A new variant of the same framework *(iii)*



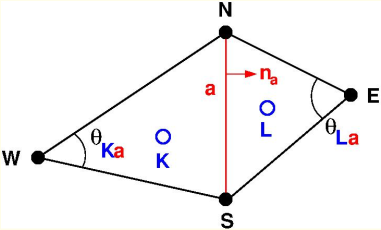
$$0 = (\varphi_a^\star\,,\,\varphi_{\mathrm{NW}}) = \frac{1}{4\,|K|}\int_K \varphi_a^\star\,\nabla|x-S|^2\,\mathrm{d}x$$

$$0 = \int_{\partial K}(\varphi_a^\star \bullet n)\,\,|x-S|^2\,\mathrm{d}\gamma - \int_K \mathrm{div}\,\varphi_a^\star\,\,|x-S|^2\,\mathrm{d}x$$

$$= \int_a (\varphi_a^\star \bullet n)\,\,|x-S|^2\,\mathrm{d}\gamma - \int_K \mathrm{div}\,\varphi_a^\star\,\,|x-S|^2\,\mathrm{d}x$$

Impose that these two integral are both equal to zero

A new variant of the same framework *(iv)*



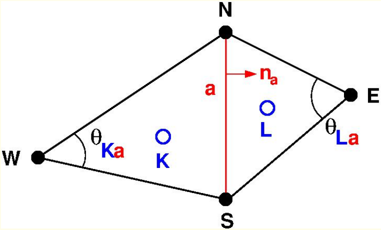
Boundary term : $\displaystyle \int_a (\varphi_a^\star \bullet n)\ |x - S|^2 \, \mathrm{d}\gamma = 0$

Flux of $\varphi_a^\star$ on the edge a: $\displaystyle \varphi_a^\star \bullet n_a \equiv \frac{1}{|a|} g(s)$

with a universal function g defined on $[0, 1]$ such that

- $g(s) = g(1 - s)\,, \quad \forall\, s \in [0, 1]$
- $\displaystyle \int_0^1 g(s)\, \mathrm{d}s = 1,$ scaling $\displaystyle \int_a (\varphi_a^\star \bullet n)\ \mathrm{d}\gamma = 1$
- $\displaystyle \int_0^1 s^2\, g(s)\, \mathrm{d}s = 0.$

A new variant of the same framework *(v)*



Two-dimensional term : $$\int_K \mathrm{div}\, \varphi_a^\star \;\; |x - S|^2 \, \mathrm{d}x = 0$$

We impose that the previous relation is true

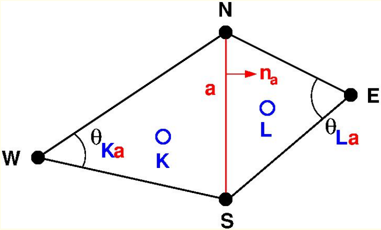
for the three vertices of the triangle K

Introduce $\delta_K \equiv \mathrm{div}\, \varphi_a^\star$ in the triangle K

We search δ_K such that $$\int_K \delta_K \;\; |x - A|^2 \, \mathrm{d}x = 0$$

for each vertex of the triangle K

Recovering the VF4 scheme



$$(\varphi_a^\star\,,\,\varphi_a) = \frac{1}{4\,|K|}\int_K \varphi_a^\star\,\nabla|x-W|^2\,\mathrm{d}x - \frac{1}{4\,|L|}\int_L \varphi_a^\star\,\nabla|x-E|^2\,\mathrm{d}x$$

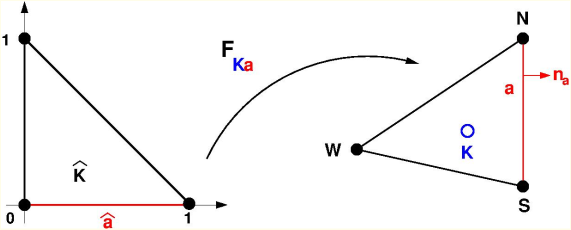
$$= \frac{1}{4\,|K|}\int_{\partial K}(\varphi_a^\star \bullet n)\,|x-W|^2\,\mathrm{d}\gamma - \frac{1}{4\,|L|}\int_{\partial L}(\varphi_a^\star \bullet n)\,|x-E|^2\,\mathrm{d}\gamma$$

$$= \frac{1}{4\,|K|}\int_a(\varphi_a^\star \bullet n)\,|x-W|^2\,\mathrm{d}\gamma + \frac{1}{4\,|L|}\int_a(\varphi_a^\star \bullet n)\,|x-E|^2\,\mathrm{d}\gamma$$

$$= \frac{1}{2}\left(\mathrm{cotan}\,\theta_{Ka} + \mathrm{cotan}\,\theta_{La}\right)$$ after some elementary geometry

be careful ! $0 < \theta_* \le \theta \le \theta^* < \frac{\pi}{2}\,, \quad \forall\,\theta \in \mathcal{T}^{-1}$

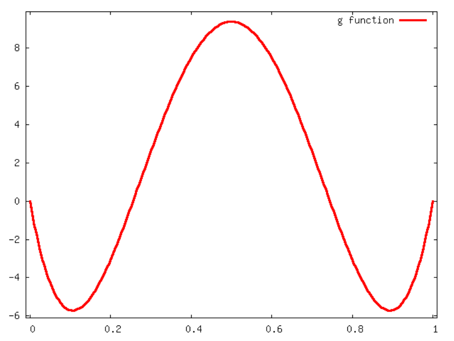
Construction a possible dual basis vector field



Dual basis function $\varphi^{\star}_{K,j}(x) = \dfrac{1}{\det(\mathrm{d}F_{K,a})}\, \mathrm{d}F_{K,a} \bullet \widehat{\nabla} \zeta_K$,

ζ_K solution of a Neumann problem in the reference element $\widehat{K}$

Construction a possible dual basis vector field *(ii)*



Given $g\colon\ [0,\,1] \longrightarrow \mathbb{R}$ such that

$$g(s) = g(1-s)\,,\quad \int_0^1 g(s)\,\mathrm{d}s = 1,\quad \int_0^1 s^2\,g(s)\,\mathrm{d}s = 0.$$

We impose moreover $g(0) = g(1) = 0$

example $g(s) = 30\,s\,(s-1)\,(4 - 21\,s\,(1-s))$

Construction a possible dual basis vector field *(iii)*

Neumann problem in $\widehat{K} = \{(\widehat{x}, \widehat{y}) \,, \widehat{x} \geq 0 \,, \widehat{y} \geq 0 \,, \widehat{x} + \widehat{y} \leq 1\}$

Datum $\widetilde{g} \in \mathrm{H}^{1/2}(\partial\widehat{K})$ on $\partial\widehat{K}$:

$\widetilde{g} = g$ on $\widehat{a} = [0,\, 1] \times \{0\}$, $\quad \widetilde{g} = 0$ elsewhere

Affine function $F_{K,a}$ such that

$\widehat{K} \ni \widehat{x} \longmapsto x = F_{K,a}(\widehat{x}) \in K$ is one to one

Right hand side $\quad \widetilde{\delta_K}(\widehat{x}) = 2|K|\delta_K(x)$.

Solution ζ_K of the Neumann problem

$$-\Delta\zeta_K = \widetilde{\delta_K} \text{ in } \widehat{K}, \qquad \frac{\partial\zeta_K}{\partial n} = \widetilde{g} \text{ on } \partial\widehat{K}$$

$$\text{since} \quad \int_{\widehat{K}} \widetilde{\delta_K}\,\mathrm{d}x = \int_{\partial K} \widetilde{g}\,\mathrm{d}\gamma = 1\,.$$

Solution $\zeta_K \in \mathrm{H}^2(\widehat{K})$

and $\|\zeta_K\|_{2,\hat{K}} \leq C_{\widehat{K}} \left(\|\widetilde{\delta_K}\|_{0,\widehat{K}} + \|\widetilde{g}\|_{1/2,\partial\widehat{K}} \right)$

Satisfy the sufficient stability conditions

Discrete stability

Hypothesis: the interpolation operator and the family $\mathcal{U}$ of meshes

$$\mathrm{H}_{\mathcal{T}}(\mathrm{div}) \ni q \equiv \sum_{a \in \mathcal{T}^1} q_a \, \varphi_a \longmapsto \Pi \, q \equiv \sum_{a \in \mathcal{T}^1} q_a \, \varphi_a^{\star} \in \mathrm{H}_{\mathcal{T}}^{\star}(\mathrm{div})$$

satisfy the conditions

$$\begin{aligned}
&A \, \| q \|_0^2 \;\leq\; (q \, , \Pi \, q) \, , && \forall q \in H_{\mathcal{T}}(\mathrm{div}) \, , \, \forall \mathcal{T} \in \mathcal{U} \\
&\| \Pi \, q \|_0 \;\leq\; B \, \| q \|_0 \, , && \forall q \in H_{\mathcal{T}}(\mathrm{div}) \, , \, \forall \mathcal{T} \in \mathcal{U} \\
&(\mathrm{div} \, q \, , \mathrm{div} \, \Pi \, q) \;\geq\; C \, \| \mathrm{div} \, q \|_0^2 \, , && \forall q \in H_{\mathcal{T}}(\mathrm{div}) \, , \, \forall \mathcal{T} \in \mathcal{U} \\
&\| \mathrm{div} \, \Pi \, q \|_0 \;\leq\; D \, \| \mathrm{div} \, q \|_0 \, , && \forall q \in H_{\mathcal{T}}(\mathrm{div}) \, , \, \forall \mathcal{T} \in \mathcal{U} .
\end{aligned}$$

Then we have the following uniform discrete inf-sup stability condition for the Petrov Galerkin mixed formulation :

$$\inf_{\xi \in V_{\mathcal{T}} , \, \|\xi\|_V = 1} \; \sup_{\eta \in V_{\mathcal{T}}^{\star} , \, \|\eta\|_V \leq 1} \; \gamma(\xi, \eta) \;\geq\; \beta$$

Error estimate : there exists a constant $C > 0$ such that

$$\| \, u - u_{\mathcal{T}} \, \|_0 + \| \, p - p_{\mathcal{T}} \, \|_{\mathrm{div}} \;\leq\; C \; h_{\mathcal{T}} \; \| \, f \, \|_1 .$$

Satisfy the sufficient stability conditions (coefficient C)

Divergence estimate

normalization condition $\int_a \varphi_a^\star \bullet \, \mathrm{d}\gamma = 1$

then $\int_{\partial K} \Pi p \bullet n \, \mathrm{d}s = \int_{\partial K} p \bullet n \, \mathrm{d}s$

and $\int_K \mathrm{div}\, \Pi p \, \mathrm{d}x = \int_K \mathrm{div}\, p \, \mathrm{d}x.$

$(\mathrm{div}\, \Pi p, \mathrm{div}\, p)_{0,K} = \mathrm{div}\, p \int_K \mathrm{div}\, \Pi p \, \mathrm{d}x$

because $\mathrm{div}\, p$ is a constant on K.

Thus, $(\mathrm{div}\, \Pi p, \mathrm{div}\, p)_{0,K} = \| \mathrm{div}\, p \|_{0,K}^2$

and $C = 1$ in the inequality

$$(\mathrm{div}\, q \,,\, \mathrm{div}\, \Pi\, q) \;\geq\; C \, \|\mathrm{div}\, q\|_0^2 \,, \quad \forall q \in H_{\mathcal{T}}(\mathrm{div}) \,,\, \forall\, \mathcal{T} \in \mathcal{U}$$

Satisfy the sufficient stability conditions (coefficient D)

Estimation of $\| \operatorname{div} \Pi p \|_{0,K}$ for $K \in \mathcal{T}^2$

$$p = \sum_{1=1}^{3} p_{K,i} \; \varphi_{K,i} \text{ in } K \quad \text{and} \quad \operatorname{div} p = \sum_{1=1}^{3} p_{K,i}$$

thus $\| \operatorname{div} p \|_{0,K}^2 = \dfrac{1}{|K|} \Big(\sum_{1=1}^{3} p_{K,i} \Big)^2$

$$\Pi p = \sum_{1=1}^{3} p_{K,i} \; \varphi^{\star}_{K,i} \text{ in } K \quad \text{and} \quad \operatorname{div} \Pi p = \delta_K(x) \sum_{1=1}^{3} p_{K,i}$$

$$\| \operatorname{div} \Pi p \|_{0,K}^2 = \int_K \delta_K^2 \, \Big(\sum_{1=1}^{3} p_{K,i} \Big)^2 \, \mathrm{d}x$$

$$\| \operatorname{div} \Pi p \|_{0,K}^2 = |K| \, \Big(\int_K \delta_K^2 \, \mathrm{d}x \Big) \; \| \operatorname{div} p \|_{0,K}^2$$

Satisfy the sufficient stability conditions [D, *(ii)*]

Estimation of $I \equiv |K| \Big(\int_K \delta_K^2 \, \mathrm{d}x \Big)$

with δ_K satisfying the condition $\int_K \delta_K(x) \, \mathrm{d}x = 1$ and $\int_K \delta_K \, |x - A|^2 \, \mathrm{d}x = 0$ for each vertex A of the triangle K.

After a formal calculus, $I \leq \dfrac{8 \,.\, 3^5 \,.\, 23}{5} \Big(\dfrac{1}{\tan \theta_*} \Big)^4 \equiv \nu$

and $\| \mathrm{div} \, \Pi \, q \|_0 \leq D \, \| \mathrm{div} \, q \|_0$ with $D = \sqrt{\nu}$.

Satisfy the sufficient stability conditions (coefficient A)

Minoration of $(q\,,\,\Pi q)$

$$(q\,,\,\Pi q) \;=\; \sum_{a,\,b\,\in\mathcal{T}^1} q_a\, q_b\, (\varphi_a\,,\,\varphi_b^\star)$$
$$=\sum_{a\in\mathcal{T}^1} q_a^2\,(\varphi_a\,,\,\varphi_a^\star) \;=\; \sum_{K\in\mathcal{T}^2} (q\,,\,\Pi q)_{0\,K}$$

with

$$(q\,,\,\Pi q)_{0\,K} \;\equiv\; \sum_{1\leq i\leq 3} q_{K,j}^2\;(\varphi_{K,j}\,,\,\varphi_{K,j}^\star)$$
$$=\;\frac{1}{2}\sum_{1\leq i\leq 3} \mathrm{cotan}\,\theta_j\;q_j^2 \qquad \geq\; \frac{1}{2}\,\mathrm{cotan}\,\theta^\star\;\sum_{1\leq i\leq 3} q_j^2$$

$$(q\,,\,\Pi q)_{0\,K} \;\geq\; \frac{1}{2}\,\mathrm{cotan}\,\theta^\star\;\sum_{1\leq i\leq 3} q_j^2$$

Satisfy the sufficient stability conditions [A, *(ii)*]

We majorate $\| q \|^2_{0\,K}$ by some constant multiplied by $\sum_{1 \leq i \leq 3} q_j^2$

$$\| q \|^2_{0\,K} = \sum_{1 \leq i, j \leq 3} q_i \, q_j \, (\varphi_{K,j} \, , \, \varphi_{K,j}) \leq \frac{5}{4 \ \tan \ \theta_\star} \sum_{1 \leq j \leq 3} q_j^2$$

then

$$(q \, , \, \Pi q)_{0\,K} \geq \frac{1}{2} \mathrm{cotan} \, \theta^\star \sum_{1 \leq i \leq 3} q_j^2$$

$$\geq \frac{1}{2} \mathrm{cotan} \, \theta^\star \ \frac{4}{5} \ \tan \ \theta_\star \ \| q \|^2_{0\,K}$$

and $\quad (q \, , \, \Pi \, q) \geq A \, \|q\|^2_0 \quad$ with $\quad A = \dfrac{2}{5} \mathrm{cotan} \, \theta^\star \ \tan \ \theta_\star$.

Satisfy the sufficient stability conditions (coefficient B)

Majoration of the L^2 norm $\|\Pi q\|_0$

$$\| \Pi q \|_0^2 = \sum_{K \in \mathcal{T}^2} \| \Pi q \|_{0\,K}^2 \leq 3 \sum_{1 \leq j \leq 3} q_{K,j}^2 \; \|\varphi^\star_{K,j}\|_{0\,K}^2$$

Dual basis function $\quad \varphi^\star_{K,j}(x) = \dfrac{1}{\det(\mathrm{d}F_{K,a})} \mathrm{d}F_{K,a} \bullet \widehat{\nabla} \zeta_K \,,$

Neumann problem $\quad -\Delta \zeta_K = \widetilde{\delta_K}$ in $\widehat{K}$, $\quad \dfrac{\partial \zeta_k}{\partial n} = \widetilde{g}$ on $\partial \widehat{K}$

solution $\zeta_K \in \mathrm{H}^2(\widehat{K})$ such that $\|\zeta_K\|_{2,\hat{K}} \leq C_{\widehat{K}} \Big(\|\widetilde{\delta_K}\|_{0,\widehat{K}} + \|\widetilde{g}\|_{1/2, \partial \widehat{K}} \Big)$

$$\|\widetilde{\delta_K}\|_{0,\hat{K}}^2 = \int_{\widehat{K}} \widetilde{\delta_K}^2 \, \mathrm{d}\widehat{x} = \int_K (2|K|\delta_K)^2 \frac{\mathrm{d}\hat{x}}{\mathrm{d}x} \, \mathrm{d}x = 2|K| \int_K \delta_K^2 \mathrm{d}x \leq 2\,\nu$$

$$\|\widehat{\nabla} \zeta_K \|_{0,\hat{K}} \leq C_{\widehat{K}} \left(\sqrt{2\,\nu} + \|\widetilde{g}\|_{1/2,\, \partial \widehat{K}} \right)$$

$$\|\varphi^\star_{K,j}\|_{0\,K}^2 \leq \left(\frac{1}{2\,|K|}\right)^2 \left(\frac{8\,|K|}{\sin\,\theta_\star}\right) \|\widehat{\nabla} \zeta_K\|_{0,\hat{K}}^2 \,(2\,|K|)$$

$$\|\varphi^\star_{K,j}\|_{0\,K} \leq \mu_* \equiv \frac{2}{\sqrt{\sin\,\theta_\star}} \, C_{\widehat{K}} \left(\sqrt{2\,\nu} + \|\widetilde{g}\|_{1/2,\, \partial \widehat{K}} \right)$$

Satisfy the sufficient stability conditions [B, *(ii)*]

$$\| \Pi q \|_{0\,K}^2 \leq 3 \sum_{1\leq j\leq 3} q_{K,j}^2 \, \|\varphi_{K,j}^\star\|_{0\,K}^2 \leq 3\,\mu_*^2 \sum_{1\leq j\leq 3} q_{K,j}^2$$

We minorate $\| q \|_{0\,K}^2$ by some constant multiplied by $\sum_{1\leq i\leq 3} q_j^2$

Local Raviart-Thomas mass matrix (Gram)

$$G_K \equiv \left(\varphi_{K,i} \, , \, \varphi_{K,j} \right)_{1\leq i,j\leq 3} \quad \text{for} \quad K \in \mathcal{T}^2$$

Smallest eigenvalue of the Gram matrix

$$\lambda_* \geq \frac{\tan^2\theta_*}{48}$$

Final estimate

$$\| \Pi q \|_{0\,K}^2 \leq 3\,\mu_* \sum_{1\leq j\leq 3} q_{K,j}^2 \leq 3\,\mu_*^2 \frac{1}{\lambda_*} \| q \|_{0\,K}^2$$

and $\|\Pi\, q\|_0 \leq B\,\|q\|_0$ with $B = \dfrac{12\,\mu_*}{\tan\theta_*}$. ouf !

Presented by Charles Pierre at the 8th conference "Finite Volumes for Complex Applications" (Lille, June 2017)

Conclusion

- Reformulation of finite volumes for Poisson equation
  with mixed finite elements of Petrov-Galerkin type
- Explicitation of dual test functions
  of the low degree Raviart-Thomas finite element
- Specific constraints for the dual test functions enforce stability
- The convergence is established with the
  usual methods of mixed finite elements
- Analysis for the Laplace equation is also *a priori* valid in 3D
- Extension to equations with tensorial coefficients...
- The stability of the "butterfly" scheme is still open...

Thank you for your attention !

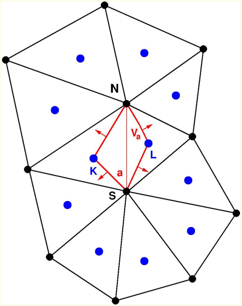
Finite volumes for elliptic problems

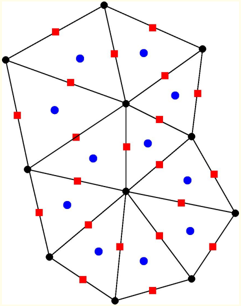
Noh scheme

Convergence theorem of Coudière, Vila and Villedieu (1999)

The affine compatibility of the nodal interpolation

is an essential hyothesis

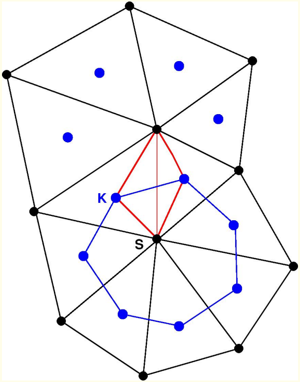




Other method: "footbridge" of Ghidaglia and Pascal (1999)

Finite volumes for elliptic problems *(ii)*

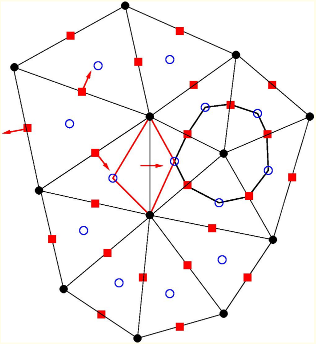
Radical solution: Hermeline (2000), Domelevo and Omnes (2005)
modify the set of degrees of freedom!



Degrees of freedom: nodal values $\overline{u_S}$ for $S \in \mathcal{T}^0$
and volumic values u_K for $K \in \mathcal{T}^2$

Finite volumes for elliptic problems *(iii)*

Petrov-Galerkin mixed formulation of Thomas and Trujillo (1994)



$$
\begin{aligned}
&p = \nabla u, \\
&\mathrm{div}\, p + f = 0, \\
&u \in \mathrm{H}^1_0, \\
&p \in \mathrm{H}(\mathrm{div})
\end{aligned}
$$

$$\int_\Omega (p - \nabla u) \bullet q \,\mathrm{d}x = 0$$

for each vector field q constant in the Noh's cell V_a, $a \in \mathcal{T}^1$

$$\int_\Omega (\mathrm{div}\, p + f)\, v \,\mathrm{d}x = 0$$

for any scalar field v constant in each "Inria cell" around the vertices ($E \in \mathcal{T}^0$)