Transparent boundary conditions for wave propagation in fractal networks

Patrick JOLY

Conference in the honor of Abderrahmane Bendali
Pau, December 2017

Joint work with Maryna Kachanosvska and Adrien Semin

UMR CNRS-ENSTA-INRIA

When I began my PhD in 1980, for me, most important algerian people where footballers, a lot of them were playing in France

Lakhdar Belloumi

Rachid Mekhloufi

Mustapha Dahleb

Rabah Madjer

Salah Assad

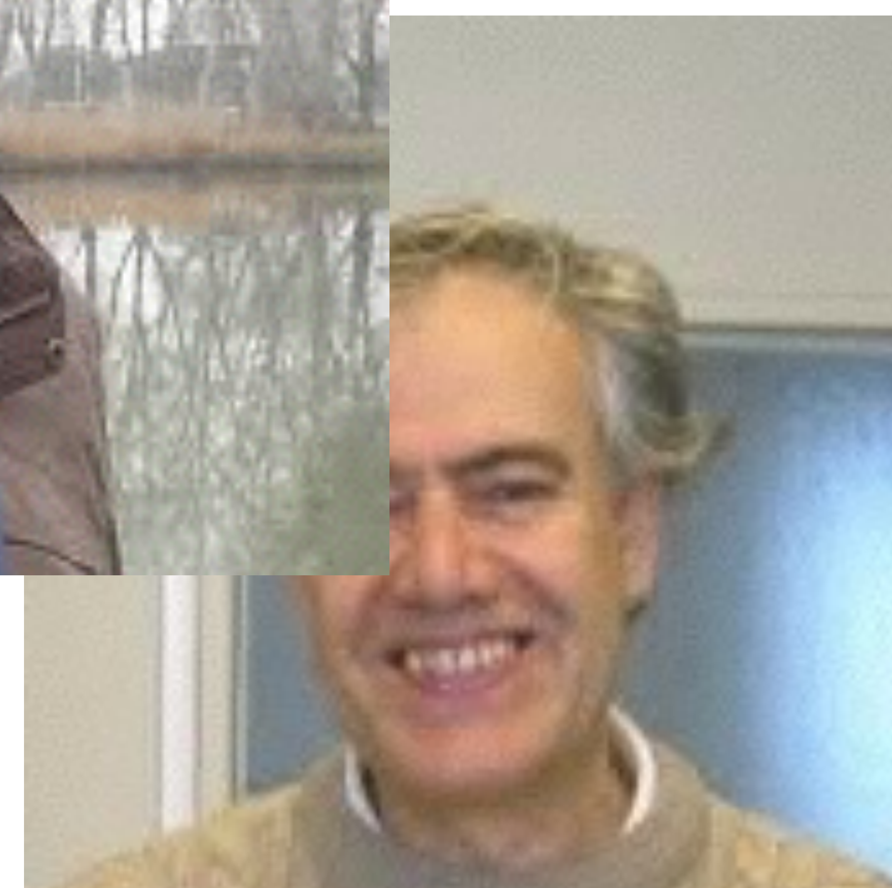
Going every week at Ecole Polytechnique, I realised that Algeria had also a good team of applied mathematicians, many them playing in France

Rachid Touzani

Kamel Hamdache

A. Bendali

Mohamed Amara



Youcef Amirat

The only problem that I ever had with this distingiuished person is when I wanted to send him an e-mail becasuse of the spelling of his first name

Abderramane Bendali

h

Finding the good solution needs several iterations, which is ok for a numerical analyst). The main issue was to find where to put the h, a a paradox for a numerical analyst

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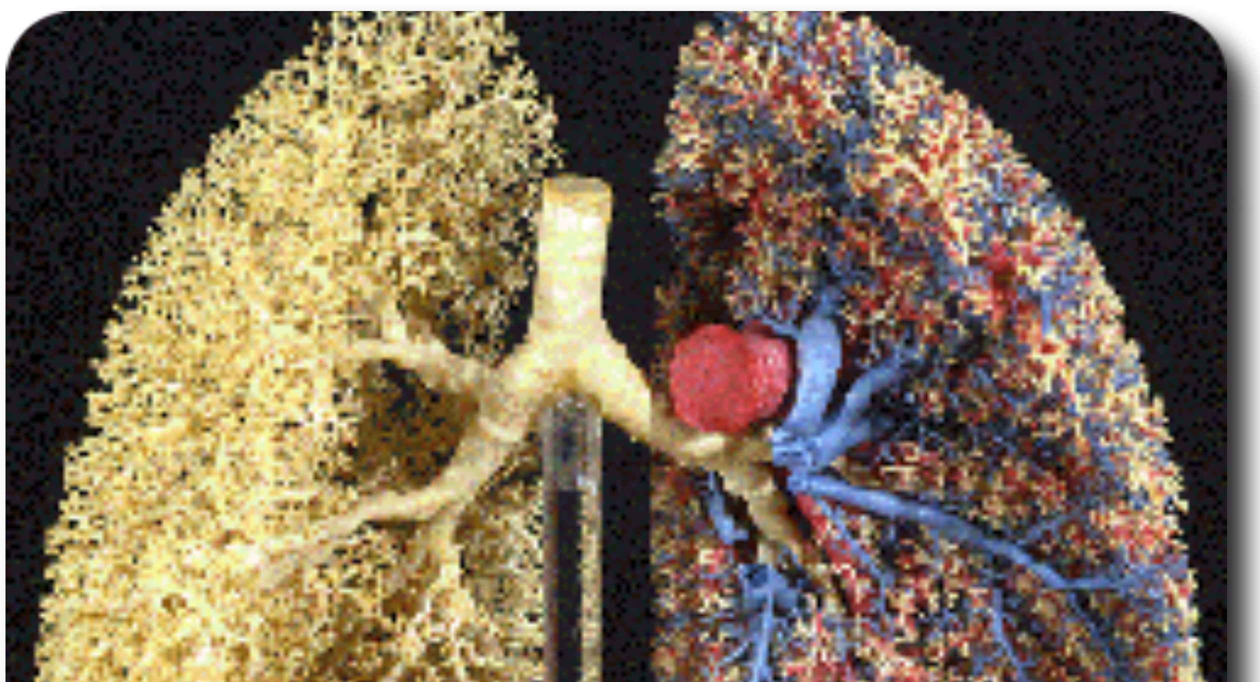
Example : the lung

An application : propagation of acoustic waves in humans lungs (to detect crackles).

- dyadic tree with 23 generations,
- over than 8 million slots,
- the geometry is «almost» self similar.



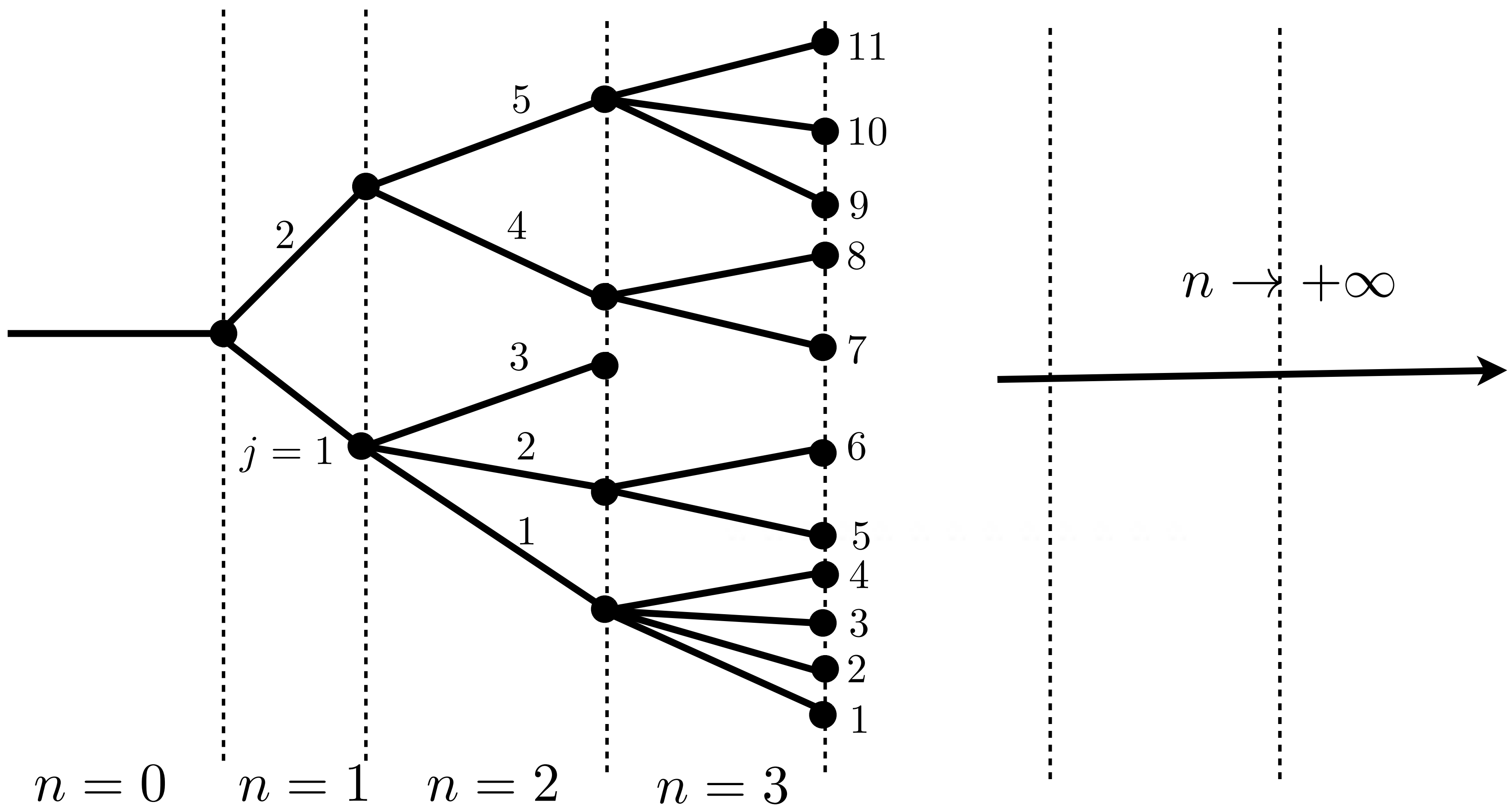
Molding obtained by Ewald R. Wiebel, University of Berne, Switzerland

Can be modeled mathematically as an infinite quasi-self similar tree

B. Maury, D. Salort, C. Vannier, Trace theorems for trees, applications for the human lung, *Network and Heterogeneous Media 1 (3), 469-500 (2009)*

Wave propagation in a tree

Goal : study the propagation of acoustic waves in a network of thin slots and particularly in infinite trees (seen as a limit case of a very large number of slots)



By a tree, we mean a graph with the additional notion of branches and successive generations

→ natural numbering of edges with two indices $i \equiv (n, j)$

Wave propagation in a tree

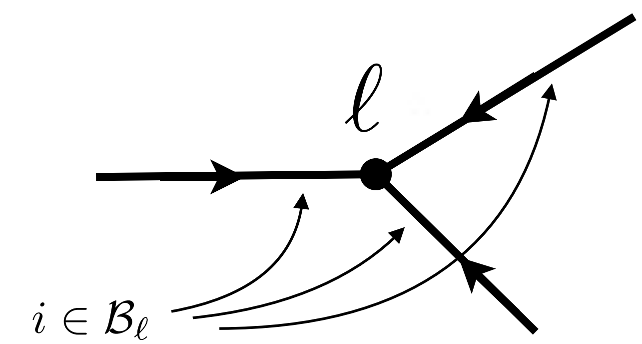
Notation : s denotes a generalized abcissa along the tree

$\mu : \mathcal{T} \longrightarrow \mathbb{R}^+$ $\qquad \mu(s) = \mu_i$ $\qquad$ along the edge $\mathrm{n}^o\ i$

$$\boxed{\mu\, \partial_t^2 u - \partial_s(\,\mu\, \partial_s u) = 0}$$

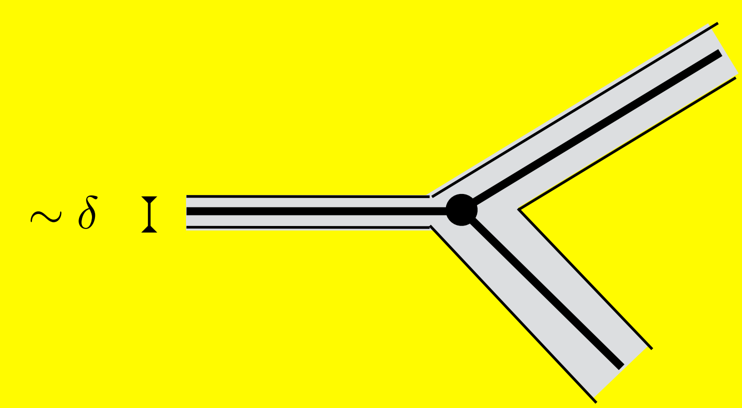
1D wave equations + Continuity + Kirchoff conditions

$$u_i = u_j\,, \quad \forall\ (i,j) \in \mathcal{B}_\ell \qquad \sum_{i \in \mathcal{B}_\ell} \mu_i\, \partial_{s_i} u_i = 0$$



This model is justified by an asymptotic analysis of the 3D acoustic wave equation in a thin network $(\delta \to 0)$ with homogeneous Neumann boundary conditions

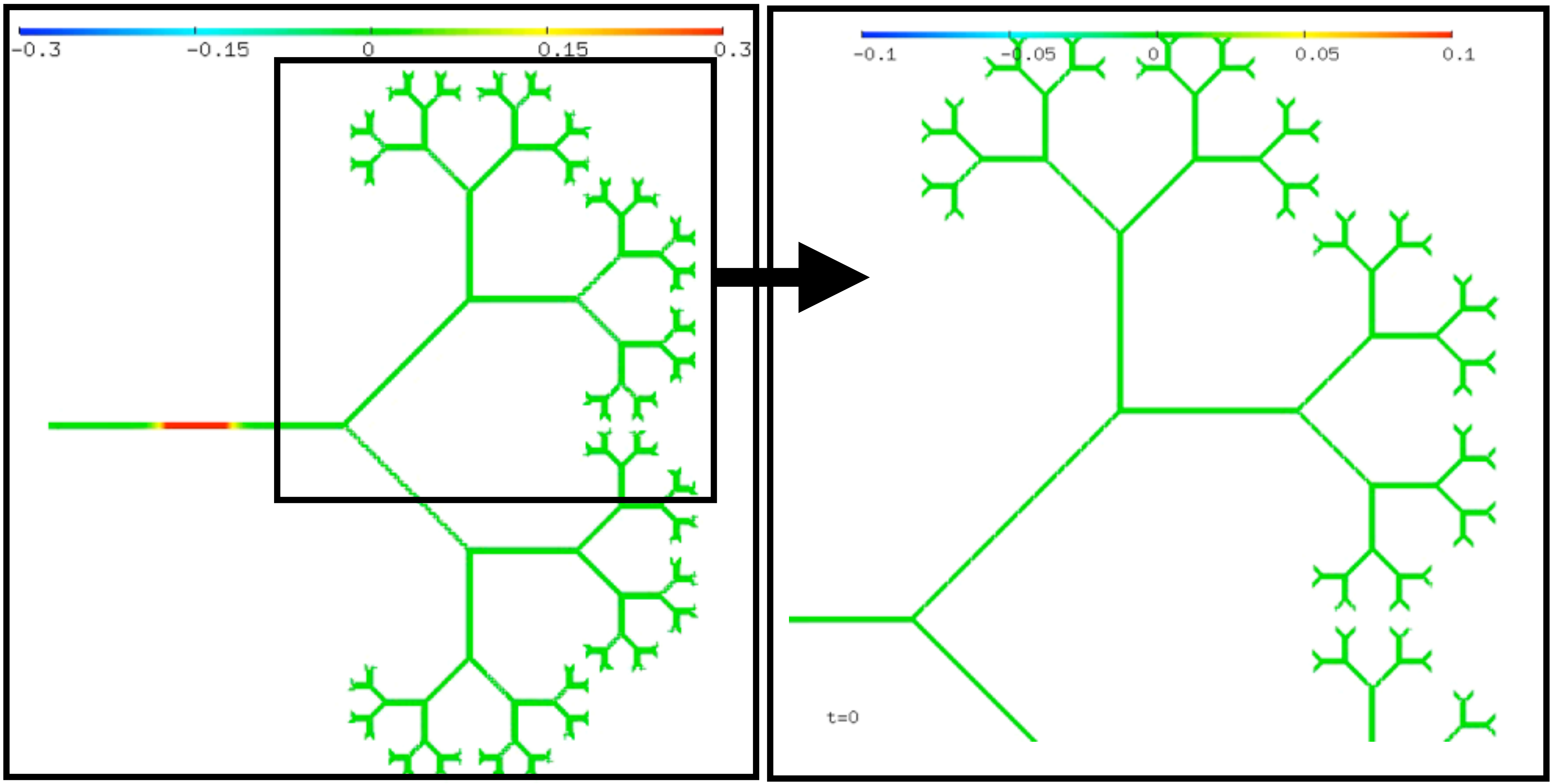
The transverse cross section of the thin slot Γ_i^δ is $\mu_i\, \delta^2$.



A complex phenomenon : reflections up to infinity

Solution computer with many generations and brute force (one week of computation)

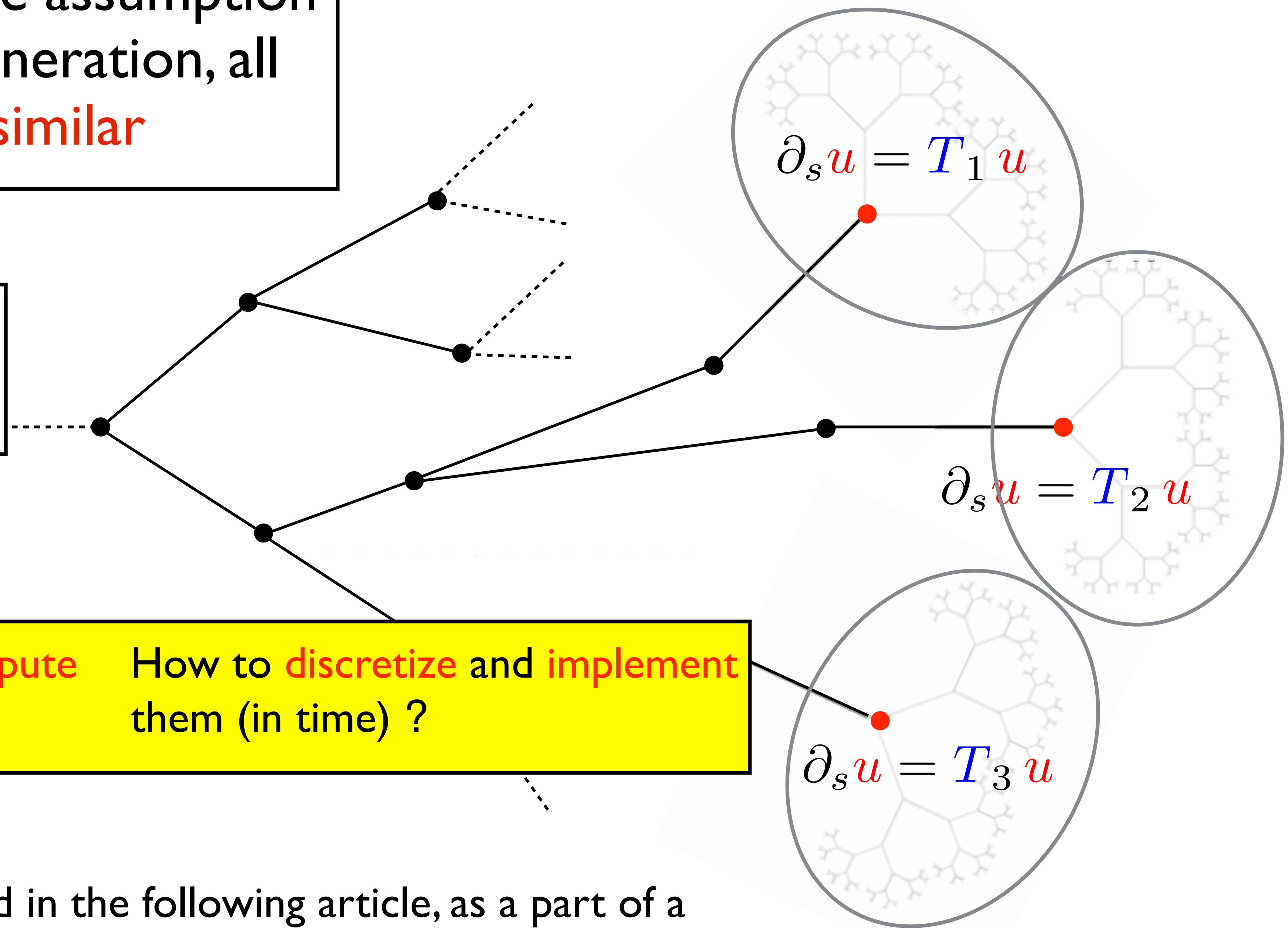
Zoom with amplified color scale



Major difficulty : Treat numerically the fact that the tree is infinite

can be done under the assumption that after a certain generation, all the subtrees are self-similar

Transparent DtN conditions at nodes •



Main questions

How to characterize and compute the operators T_m ?

How to discretize and implement them (in time) ?

Similar questions were adressed in the following article, as a part of a series of works devoted to elliptic boundary value problems in fractal domains

Y. Achdou, C, Sabot, N. Tchou, Diffusion and propagation problems in some ramified domains with a fractal boundary, *ESAIM : M2AN 40(4), 623-652 (2006)*

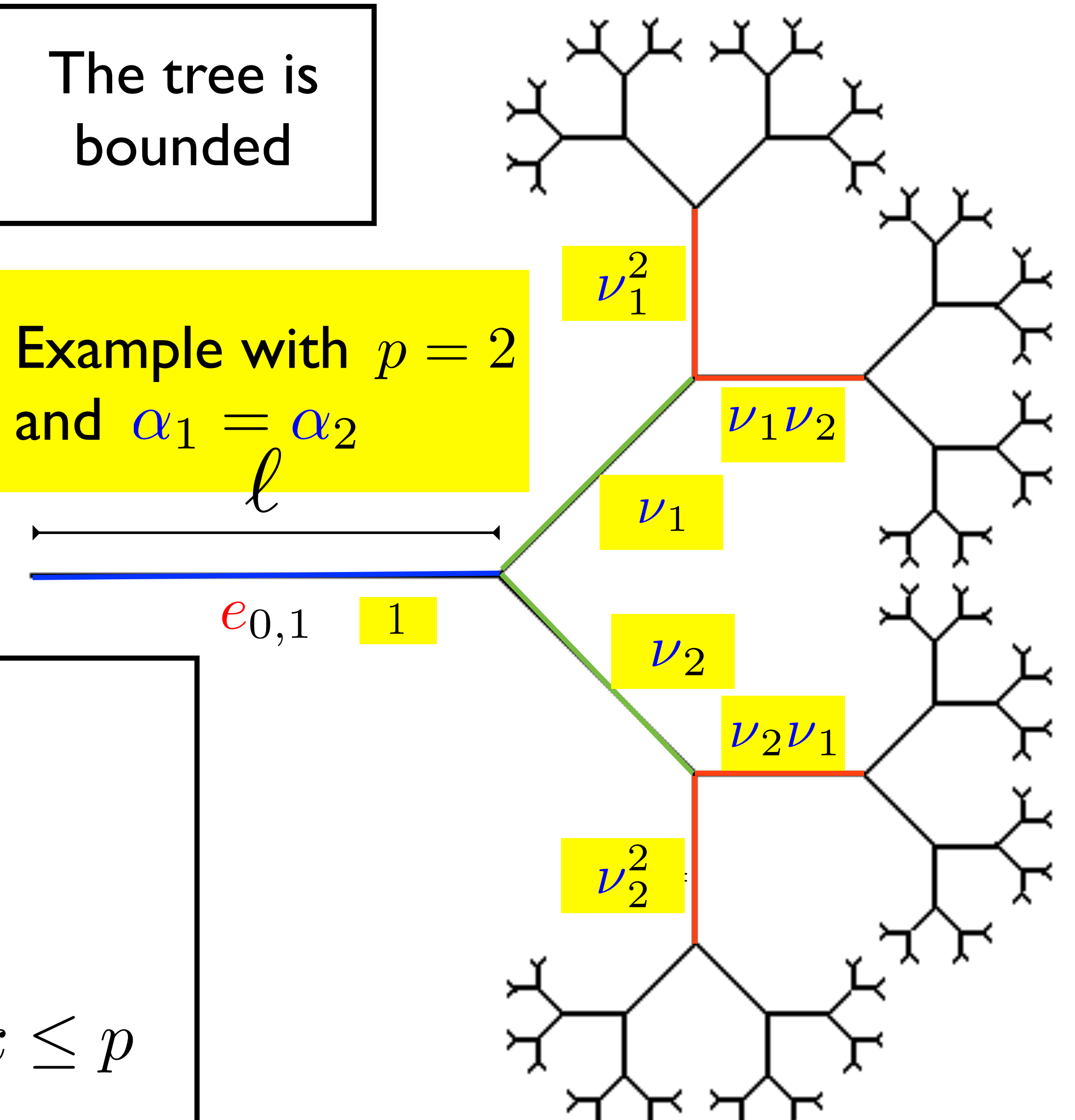
A self-similar p-adyc tree

Σ : a root segment of length ℓ

p strictly contractant direct similitudes s_j of ratio $\alpha_j < 1$.

The tree is bounded

Example with $p = 2$ and $\alpha_1 = \alpha_2$



The branches of the tree are

$$e_{0,1} \quad = \quad \Sigma$$

$$e_{1,k} = s_k(e_{0,1}), \quad 1 \leq k \leq p$$

$$e_{n+1,p(j-1)+k} = s_k(e_{n,j}), \quad 1 \leq j \leq p^n, \; 1 \leq k \leq p$$

Self-similarity of the coefficients : $\exists\ (\nu_1, \cdots, \nu_p) > 0$ such that

$$\mu\big(s_k(e_{n,j})\big) = \nu_k \cdot \mu(e_{n,j})$$

Example : the human lung $\quad p = 2, \quad \alpha_1 = \alpha_2 \simeq 0.85, \quad \nu_k = \alpha_k^2$

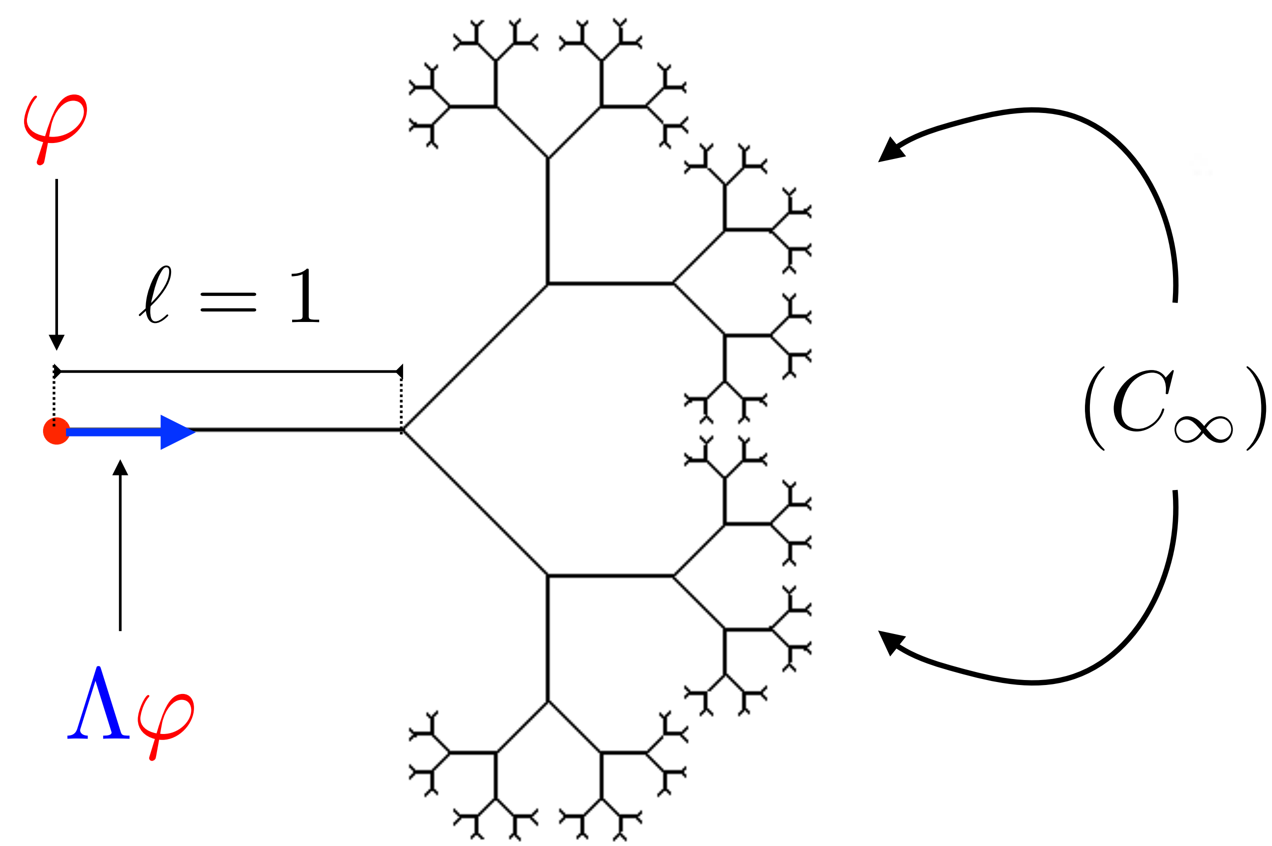
The reference DtN operator

$$\mu\,\partial_t^2 u_\varphi - \partial_s(\,\mu\,\partial_s u_\varphi) = 0 \quad \text{along } \mathcal{T}, \quad u_\varphi(\mathbf{0}, t) = \varphi(t)$$

$+$ some condition (C_∞) at infinity (to be made precise)

The Dirichlet-to-Neumann operator : $\Lambda\varphi(t) := \partial_s u_\varphi(\mathbf{0}, t)$

Reference tree



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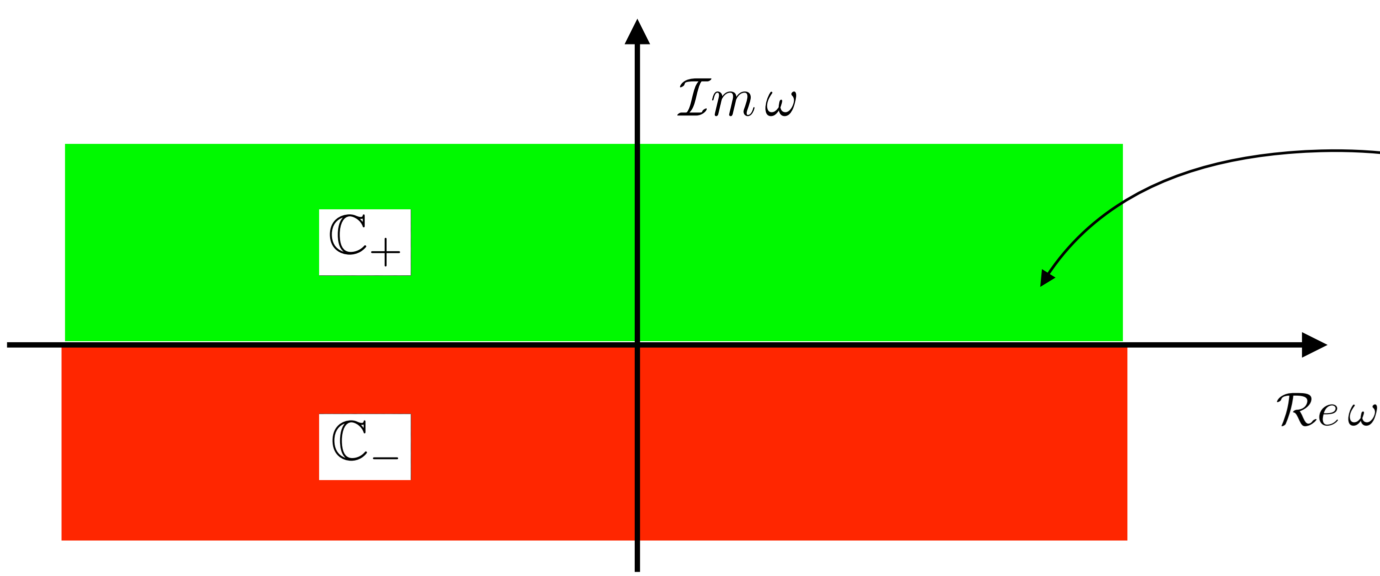
It is a convolution operator characterized by its Fourier Laplace symbol

$$\mathcal{F}: \ \varphi(t) \longrightarrow \boldsymbol{\varphi}(\omega), \quad \omega \in \mathbb{C}^+ \qquad \mathcal{F}\big(\Lambda\varphi\big)(\omega) = \boldsymbol{\Lambda}(\omega)\,\boldsymbol{\varphi}(\omega)$$



$$\boldsymbol{\varphi}(\omega) = \int_0^{+\infty} \varphi(t)\; e^{i\omega t}\; dt$$

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$$\partial_s(\,\mu\,\partial_s \mathbf{u}) + \mu\,\omega^2\,\mathbf{u} = 0 \quad \text{along } \mathcal{T}, \qquad \mathbf{u}(\mathbf{0},\omega) = 1$$

$+$ some condition (C_∞) at infinity (to be made precise)

$$\boldsymbol{\Lambda}(\omega) = \partial_s \mathbf{u}(\mathbf{0},\omega)$$

symbolic notation : $\Lambda \equiv \Lambda(\partial_t)$

Scaling of the DtN operator

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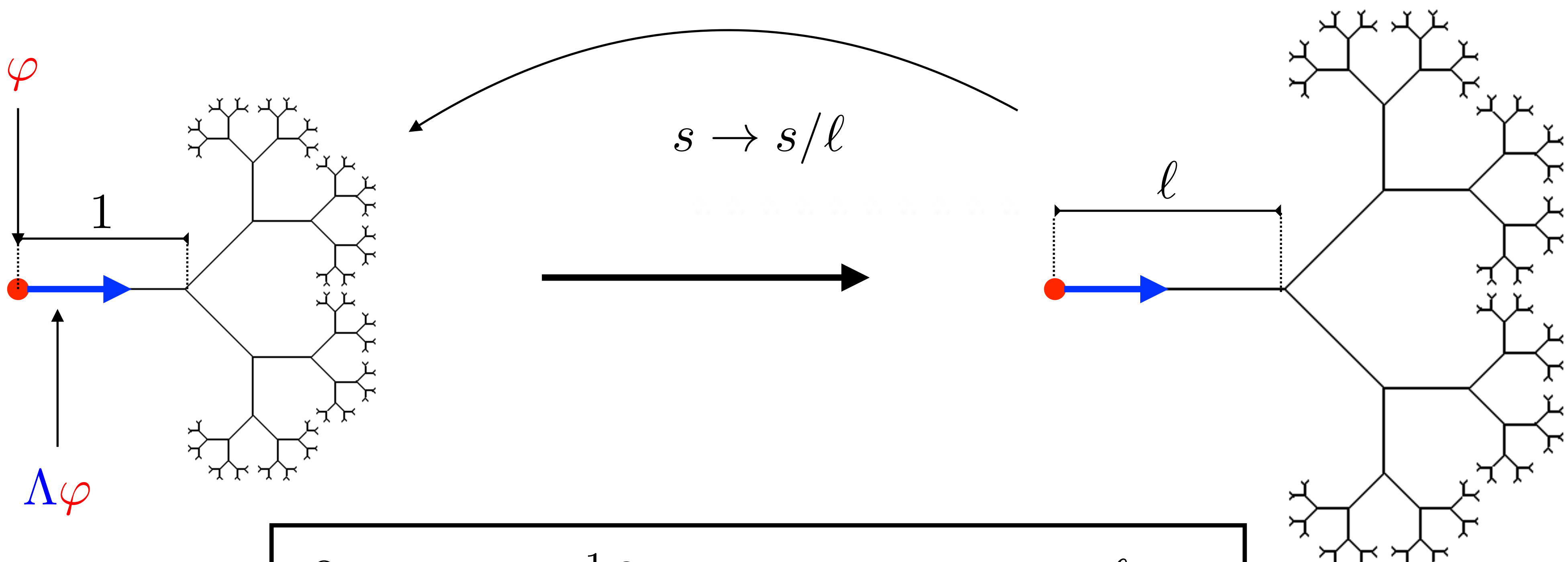

$$\partial_s \longrightarrow \ell^{-1}\partial_s \qquad \omega \longrightarrow \ell\,\omega$$

$$\boldsymbol{\Lambda}(\omega) \longrightarrow \ell^{-1}\boldsymbol{\Lambda}(\omega\ell)$$

The transparent boundary condition

$$\mu\,\partial_t^2 u_\varphi - \partial_s(\,\mu\,\partial_s u_\varphi) = 0 \quad \text{along } \mathcal{T}, \qquad u_\varphi(\mathbf{0}, t) = \varphi(t)$$

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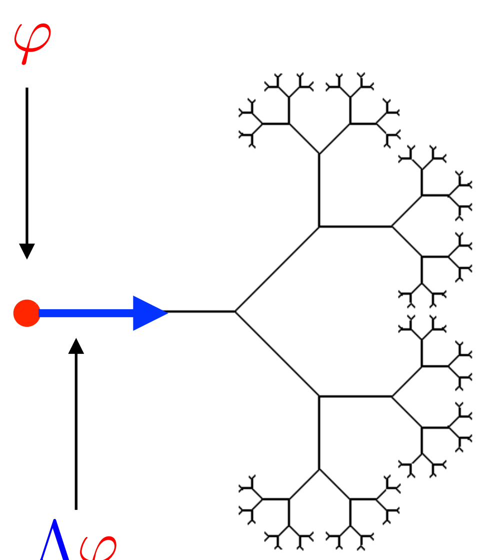
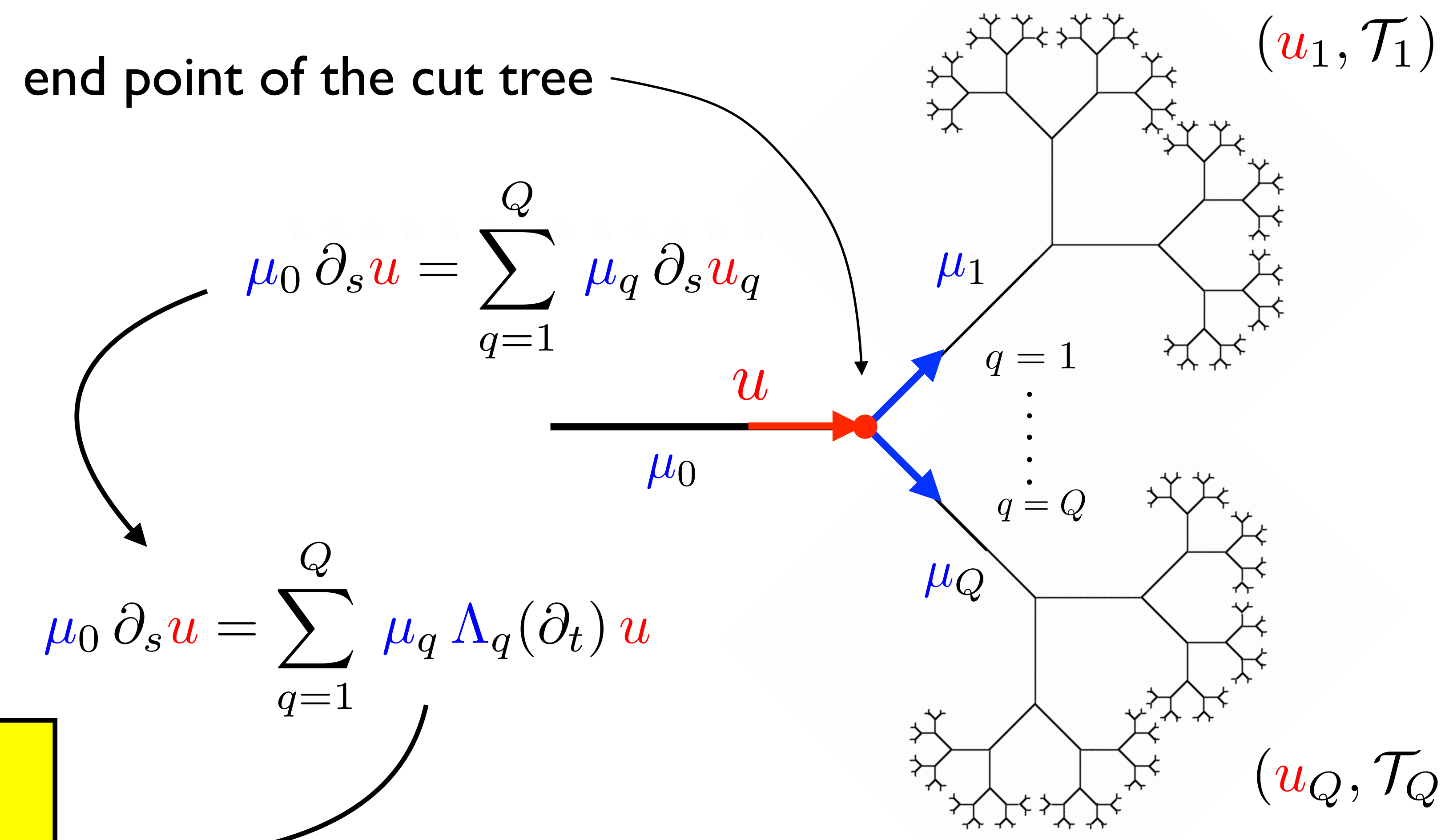


$$\mu_0\,\partial_s u = \sum_{q=1}^{Q} \mu_q\,\partial_s u_q$$

$$\mu_0\,\partial_s u = \sum_{q=1}^{Q} \mu_q\,\Lambda_q(\partial_t)\,u$$

$$\boxed{T = \mu_0^{-1}\,\sum_{q=1}^{Q} \mu_q\,\Lambda_q(\partial_t)}$$



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$+$ some condition (C_∞) at infinity (to be made precise)

$$\boldsymbol{\Lambda}(\omega) = \;\partial_s \mathbf{u}(\mathbf{0},\omega)$$

symbolic notation : $\Lambda \equiv \Lambda(\partial_t)$

Dirichlet or Neumann condition at ∞ are defined in a variational way

Weighted Sobolev spaces in p-adyc s.s. trees

Notation : $$\int_{\mathcal{T}} \mu\, f\, ds \;:=\; \sum_{n\geq 0}\sum_{j=1}^{p^n} \mu_{n,j} \int_{e_{n,j}} f\, ds$$

Weighted broken H^1 - norm : $$\|u\|^2_{\mathbf{H}^1_\mu} \;=\; \int_{\mathcal{T}} \mu\, |\partial_s u|^2\, ds + \int_{\mathcal{T}} \mu\, |u|^2\, ds$$

Associated Sobolev spaces

$$\mathbf{H}^1_\mu(\mathcal{T}) = \{ v \in C^0(\mathcal{T}) \;/\; \|v\|^2_{\mathbf{H}^1_\mu} < \infty \}$$ (for Neumann)

$$\mathbf{H}^1_{\mu,0}(\mathcal{T}) = \overline{\mathbf{H}^1_{\mu,c}(\mathcal{T})}^{\;\mathbf{H}^1_\mu(\mathcal{T})}$$ (for Dirichlet)

where $\mathbf{H}^1_{\mu,c}(\mathcal{T}) = \{ \mathbf{v} \in \mathbf{H}^1_\mu(\mathcal{T})$ such that $\exists\, N \;/\; \mathbf{v} = 0$ in $\mathcal{T} \setminus \mathcal{T}_N \}$

Dirichlet and Neumann Helhmoltz problems

Given $\omega \notin \mathbb{R}$, we define the Dirichlet and Neumann (at ∞) problems

$(\mathcal{P}_{\mathfrak{d}})$ Find $\mathbf{u} \in \mathbf{H}^1_{\mu,0}(\mathcal{T})\ /\ \mathbf{u}(\mathbf{0}) = 1$, such that

$$\int_{\mathcal{T}} \mu\, \mathbf{u}'\mathbf{v}' - \omega^2 \int_{\mathcal{T}} \mu\, \mathbf{u}\, \mathbf{v} = 0, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\mu,0}(\mathcal{T}) \text{ such that } \mathbf{v}(\mathbf{0}) = 0$$

$(\mathcal{P}_{\mathfrak{n}})$ Find $\mathbf{u} \in \mathbf{H}^1_{\mu}(\mathcal{T})\ /\ \mathbf{u}(\mathbf{0}) = 1$, such that

$$\int_{\mathcal{T}} \mu\, \mathbf{u}'\mathbf{v}' - \omega^2 \int_{\mathcal{T}} \mu\, \mathbf{u}\, \mathbf{v} = 0, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\mu}(\mathcal{T}) \text{ such that } \mathbf{v}(\mathbf{0}) = 0$$

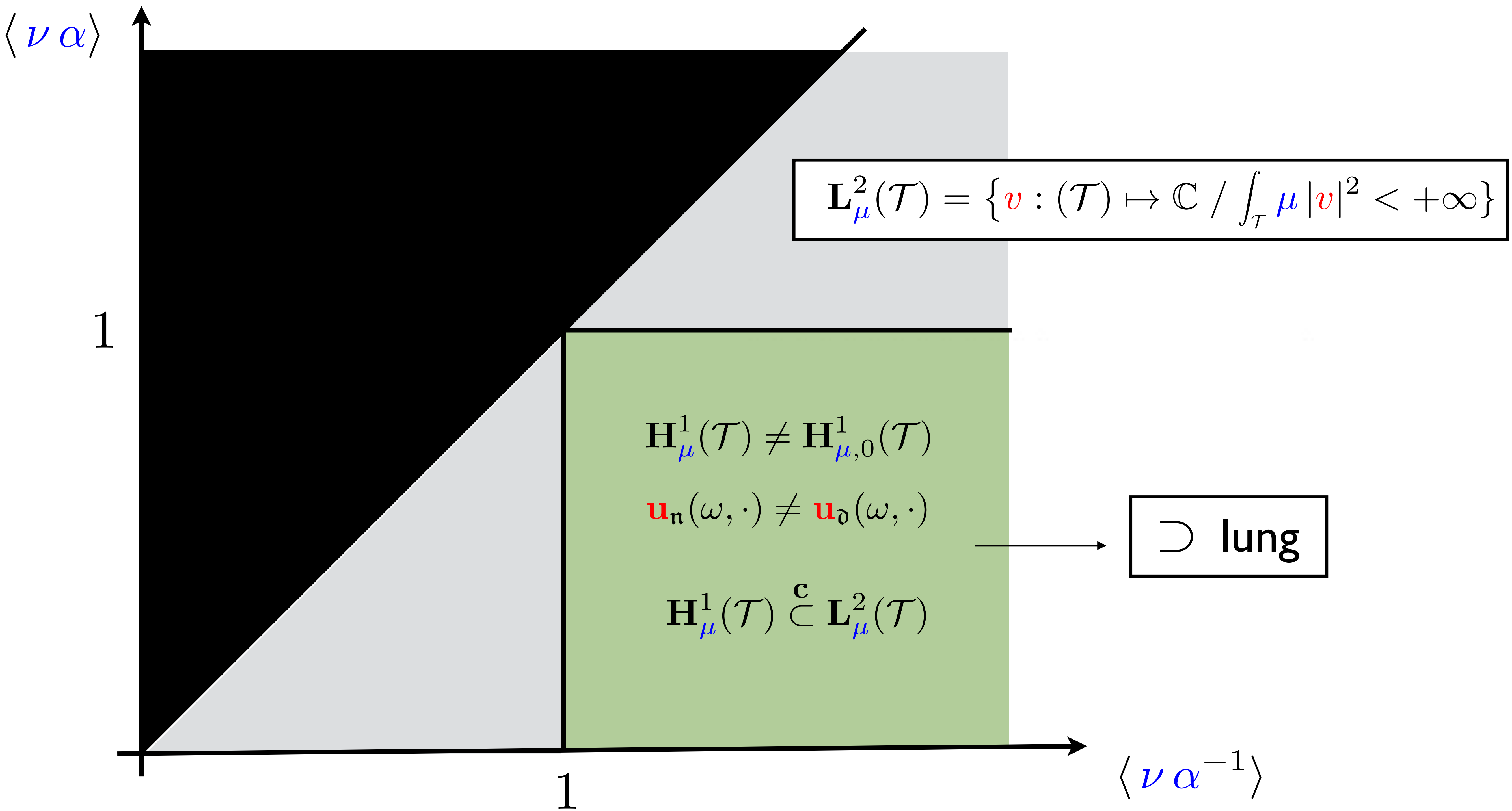
For each $\omega \notin \mathbb{R}$, $(\mathcal{P}_{\mathfrak{d}})$ (resp. $(\mathcal{P}_{\mathfrak{n}})$) admits a unique solution denoted

$\mathbf{u}_{\mathfrak{d}}(\omega, \cdot)$ (resp. $\mathbf{u}_{\mathfrak{n}}(\omega, \cdot)$)

Definition : $\boldsymbol{\Lambda}_{\mathfrak{d}}(\omega) := \partial_s u_{\mathfrak{d}}(\omega, 0)$ $\boldsymbol{\Lambda}_{\mathfrak{n}}(\omega) := \partial_s u_{\mathfrak{n}}(\omega, 0)$

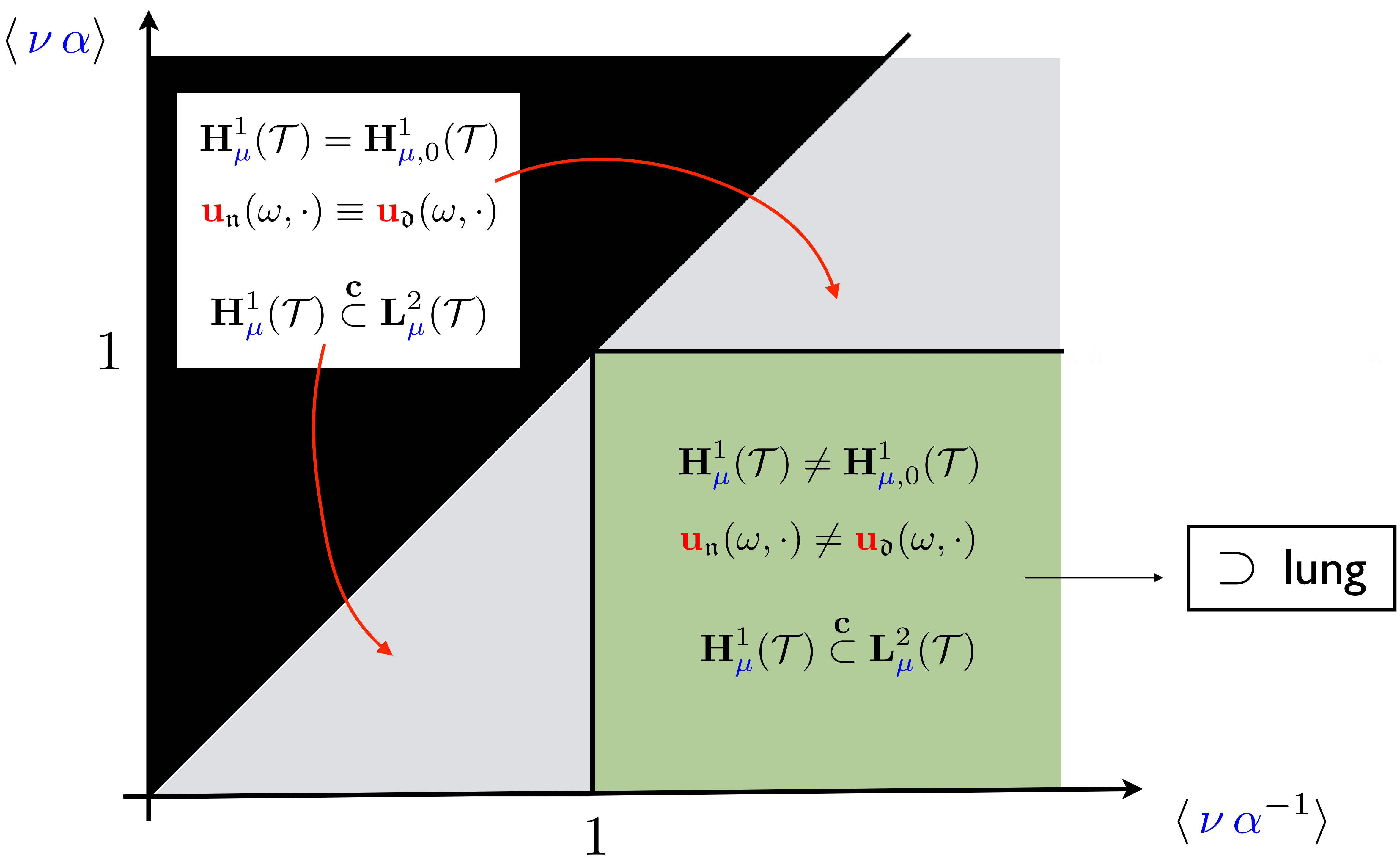
Some mathematical analysis

$$\langle \nu\, \alpha^{-1} \rangle := \sum \nu_i\, {\alpha_i}^{-1} \quad > \quad \langle \nu\, \alpha \rangle := \sum \nu_i\, \alpha_i \qquad \left(\max_{1 \le i \le p} \alpha_i < 1 \right)$$



Some mathematical analysis

$$\langle \nu\,\alpha^{-1}\rangle := \sum \nu_i\,{\alpha_i}^{-1} \;>\; \langle \nu\,\alpha\rangle := \sum \nu_i\,\alpha_i \qquad \left(\max_{1\le i\le p} \alpha_i < 1\right)$$

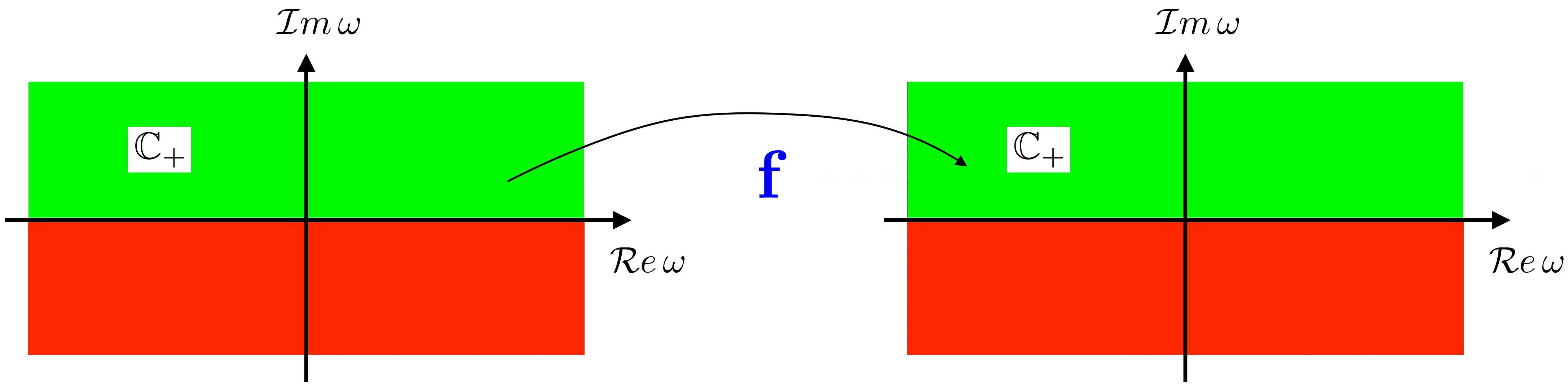


Some mathematical analysis

Definition : $\mathbf{\Lambda}_{\mathfrak{d}}(\omega) := \partial_s u_{\mathfrak{d}}(\omega, 0) \qquad \mathbf{\Lambda}_{\mathfrak{n}}(\omega) := \partial_s u_{\mathfrak{n}}(\omega, 0)$

Theorem : $\mathbf{f}_{\mathfrak{a}}(\omega) := \omega^{-1}\mathbf{\Lambda}_{\mathfrak{a}}(\omega), \mathfrak{a} = \mathfrak{d}, \mathfrak{n}$ are Herglotz functions

$\mathbf{f}$ is a Herglotz function $\Longleftrightarrow$ $\mathbf{f}$ is analytic from $\mathbb{C}_+$ into $\mathbb{C}_+$



Time domain version : $\displaystyle\int_0^T \Lambda_{\mathfrak{a}}\varphi(t)\;\partial_t\varphi(t)\;dt \geq 0 \qquad \forall\, T > 0,\ \forall\, \varphi(t)$

Some mathematical analysis

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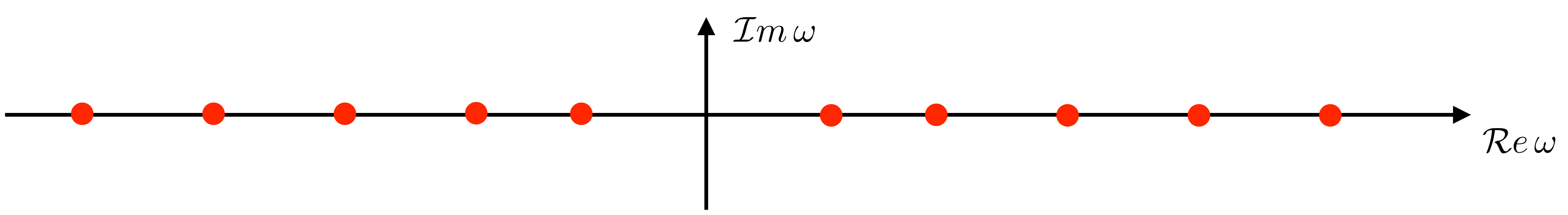
Theorem : $(\mathcal{P}_{\mathfrak{d}})$ and $(\mathcal{P}_{\mathfrak{n}})$ are well-posed except for a sequence of real frequencies $\pm\,\omega_{\mathfrak{d}}^n \in \mathbb{R}_*^+,\ \omega_{\mathfrak{d}}^n \to +\infty$ and $\pm\,\omega_{\mathfrak{n}}^n \in \mathbb{R}_*^+,\ \omega_{\mathfrak{n}}^n \to +\infty$ and $\omega \to u_{\mathfrak{d}}(\omega, \cdot)$ and $u_{\mathfrak{n}}(\omega, \cdot)$ are meromorphic, poles $\{\pm\,\omega_{\mathfrak{d}}^n\}$ and $\{\pm\,\omega_{\mathfrak{n}}^n\}$

Proof : spectral theory of self-adjoint operators with compact resolvent



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Corollary : $\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega) = \boldsymbol{\Lambda}_{\mathfrak{a}} - \omega^2 \sum_{n=0}^{+\infty} \frac{\Omega_{\mathfrak{a},n}^2}{\omega_{\mathfrak{a},n}^2 - \omega^2}$

How to compute $\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega)$ in practice ?

One can make explicit as an integral time convolution operator

$$\Lambda_{\mathfrak{a}} u = \boldsymbol{\Lambda}_{\mathfrak{a}} u + \boldsymbol{K}(0)\,\partial_t^2 u + \int_0^t \mathbf{K}(t-\tau)\,\partial_t^3 u(\tau)\,d\tau \qquad \mathbf{K}(t) := \sum_{n=0}^{+\infty} \frac{\Omega_{\mathfrak{a},n}^2}{\omega_{\mathfrak{a},n}^2} \cos\omega_{\mathfrak{a},n}(t)$$

Some mathematical analysis

Definition : $\boldsymbol{\Lambda}_{\mathfrak{d}}(\omega) := \partial_s u_{\mathfrak{d}}(\omega, 0) \qquad \boldsymbol{\Lambda}_{\mathfrak{n}}(\omega) := \partial_s u_{\mathfrak{n}}(\omega, 0)$

Theorem : $\mathbf{f}_{\mathfrak{a}}(\omega) := \omega^{-1}\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega), \mathfrak{a} = \mathfrak{d}, \mathfrak{n}$ are Herglotz functions

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Corollary : $$\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega) = \boldsymbol{\Lambda}_{\mathfrak{a}} - \omega^2 \sum_{n=0}^{+\infty} \frac{\Omega_{\mathfrak{a},n}^2}{\omega_{\mathfrak{a},n}^2 - \omega^2}$$

How to compute $\boldsymbol{\Lambda}_{\mathfrak{a}}(\omega)$ in practice ?

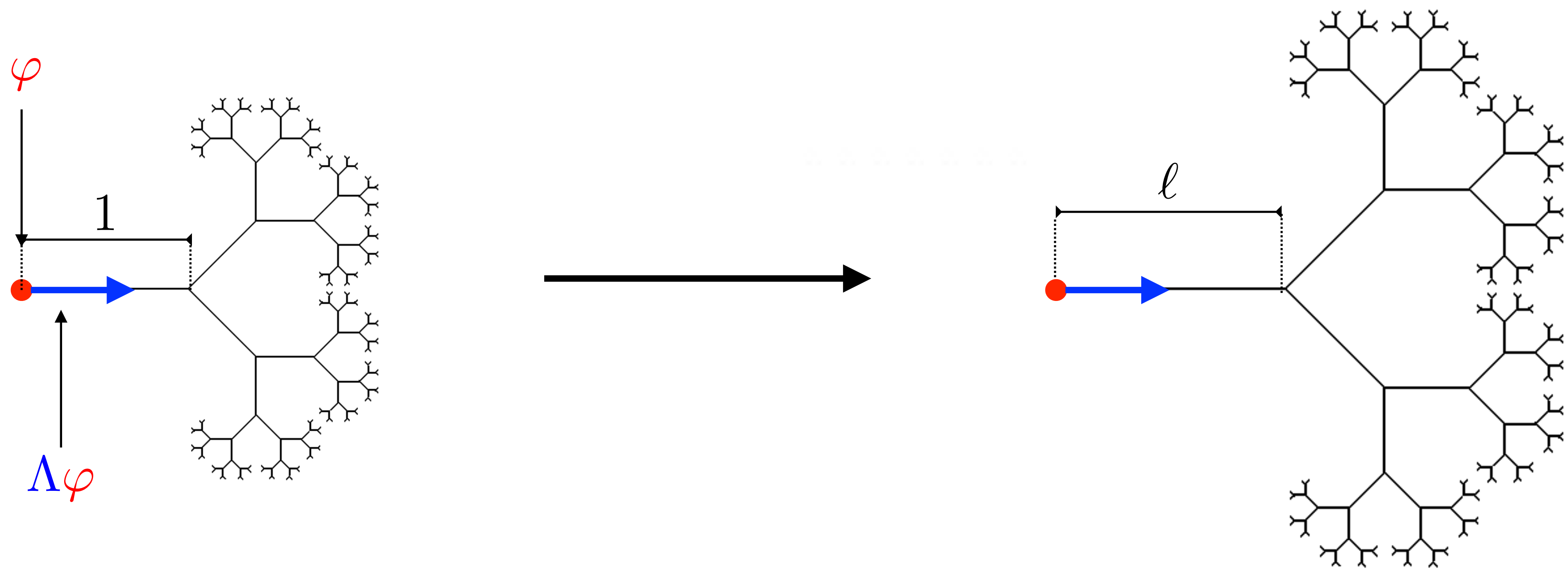
This is where we shall really exploit the self-similar nature of the tree

Scaling of the DtN operator

$$\partial_t^2 u_\varphi - \partial_s(\,\mu\,\partial_s u_\varphi) = 0 \quad \text{along } \mathcal{T}, \qquad u_\varphi(\mathbf{0},t) = \varphi(t)$$

$+$ some condition at infinity (to be made precise)

The Dirichlet-to-Neumann operator : $\Lambda\varphi(t) := \partial_s u_\varphi(\mathbf{0},t)$

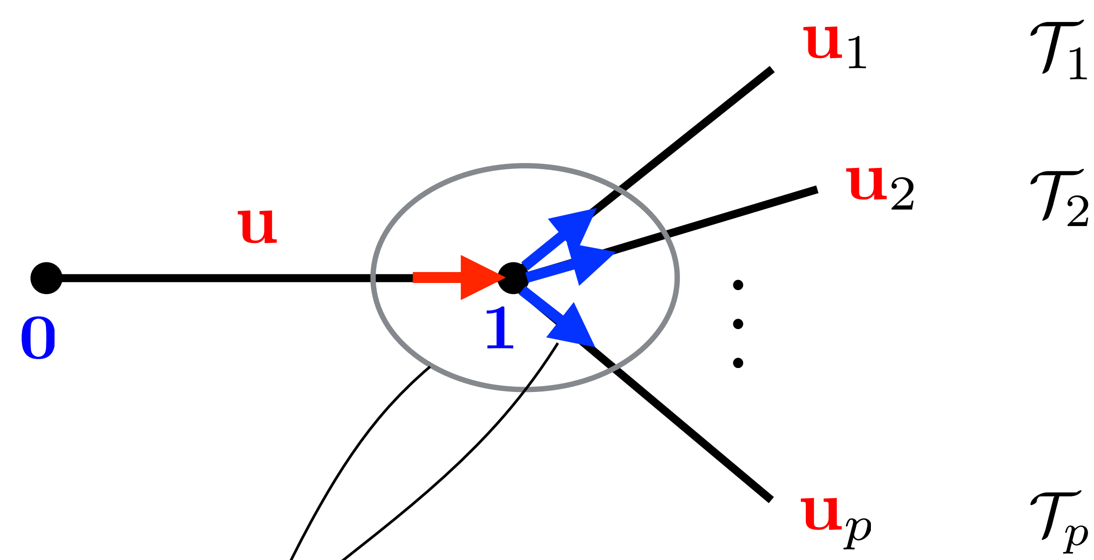


$$\mathbf{\Lambda}(\omega) \longrightarrow \ell^{-1}\mathbf{\Lambda}(\omega\ell)$$

A characterization of the function $\boldsymbol{\Lambda}(\omega)$

$$\partial_s(\,\mu\,\partial_s\mathbf{u}) + \mu\,\omega^2\ \mathbf{u} = 0 \qquad \mathcal{T}$$

$$\begin{cases} \mathbf{u}(\mathbf{0},\omega) = 1 \\ \mathbf{u}'(\mathbf{0},\omega) = \boldsymbol{\Lambda}(\omega) \end{cases}$$



Along the first branch : $\quad \partial_s^2\mathbf{u} + \omega^2\ \mathbf{u} = 0$

$$\Longrightarrow \quad \mathbf{u}(s,\omega) = \cos(\omega\, s) + \frac{\boldsymbol{\Lambda}(\omega)}{\omega}\sin(\omega\, s)$$

$$\Longrightarrow \begin{cases} \mathbf{u}(\mathbf{1},\omega) = \cos(\omega) + \dfrac{\boldsymbol{\Lambda}(\omega)}{\omega}\sin(\omega) \\ \mathbf{u}'(\mathbf{1},\omega) = \boldsymbol{\Lambda}(\omega)\cos(\omega) - \omega\sin(\omega) \end{cases}$$

Kirchhoff condition at point $\mathbf{1}$: $\quad \mathbf{u}'(\mathbf{1},\omega) = \sum\ \nu_i\,\mathbf{u}_i'(\mathbf{1},\omega)$

Scaling argument (previous slide) : $\quad$ DtN for $\mathcal{T}_j$ $\quad \mathbf{u}_j'(\mathbf{1},\omega) = \alpha_j^{-1}\boldsymbol{\Lambda}(\alpha_j\omega)\,\mathbf{u}(\mathbf{1},\omega)$

$$\boldsymbol{\Lambda}(\omega)\cos(\omega) - \omega\sin(\omega) = \sum\ \frac{\nu_i}{\alpha_i}\Big(\cos(\omega) + \frac{\boldsymbol{\Lambda}(\omega)}{\omega}\sin(\omega)\Big)\ \boldsymbol{\Lambda}(\alpha_i\,\omega)$$

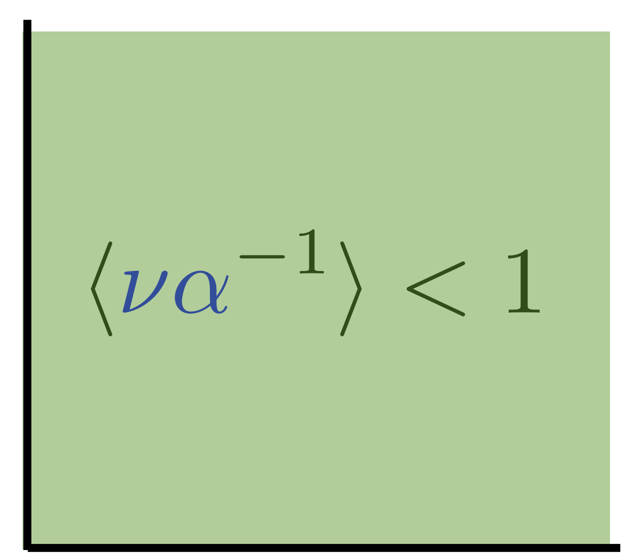
A characterization of the function $\boldsymbol{\Lambda}(\omega)$

Both functions $\boldsymbol{\Lambda}_{\mathfrak{d}}(\omega)$ and $\boldsymbol{\Lambda}_{\mathfrak{n}}(\omega)$ solve the quadratic functional equation

$$\text{(E)} \qquad \boldsymbol{\Lambda}(\omega)\cos(\omega) - \omega\sin(\omega) = \Big(\sum \frac{\nu_i}{\alpha_i}\boldsymbol{\Lambda}(\alpha_i\omega)\Big)\Big(\cos(\omega) + \frac{\boldsymbol{\Lambda}(\omega)}{\omega}\sin(\omega)\Big)$$

The frequency $\omega = 0$ plays a particular role : $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(0)$ satisfies

$$\boldsymbol{\Lambda} = \langle \nu\alpha^{-1}\rangle(1+\boldsymbol{\Lambda})\,\boldsymbol{\Lambda} \implies \begin{cases} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathfrak{n}} := 0 \quad (\iff u_{\mathfrak{n}}(0) = 1) \\ \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathfrak{d}} := \langle \nu\alpha^{-1}\rangle^{-1}\big(1 - \langle \nu\alpha^{-1}\rangle\big) < 0 \end{cases}$$



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$$\text{(E)} \qquad \boldsymbol{\Lambda}(\omega)\cos(\omega) - \omega\sin(\omega) = \Big(\sum \frac{\nu_i}{\alpha_i}\boldsymbol{\Lambda}(\alpha_i\omega)\Big)\Big(\cos(\omega) + \frac{\boldsymbol{\Lambda}(\omega)}{\omega}\sin(\omega)\Big)$$

The frequency $\omega = 0$ plays a particular role : $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(0)$ satisfies

$$\boldsymbol{\Lambda} = \langle \nu\alpha^{-1}\rangle(1+\boldsymbol{\Lambda})\,\boldsymbol{\Lambda} \implies \begin{cases} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathfrak{n}} := 0 \quad (\iff u_{\mathfrak{n}}(0) = 1) \\ \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\mathfrak{d}} := \langle \nu\alpha^{-1}\rangle^{-1}\big(1 - \langle \nu\alpha^{-1}\rangle\big) < 0 \end{cases}$$

Theorem: $\boldsymbol{\Lambda}_{\mathfrak{d}}(\omega)$ is the unique meromorphic function solution of (E) s. t.

$$\boldsymbol{\Lambda}_{\mathfrak{d}}(0) = \boldsymbol{\Lambda}_{\mathfrak{d}}$$

$\boldsymbol{\Lambda}_{\mathfrak{n}}(\omega)$ is the unique meromorphic function solution of (E) s. t.

$$\boldsymbol{\Lambda}_{\mathfrak{n}}(0) = \boldsymbol{\Lambda}_{\mathfrak{n}}$$

An algorithm for the computation of $\mathbf{\Lambda}(\omega)$

(E) $$\mathbf{\Lambda}(\omega)\cos(\omega) - \omega\sin(\omega) = \Big(\sum \frac{\nu_i}{\alpha_i}\mathbf{\Lambda}(\alpha_i\omega)\Big)\Big(\cos(\omega) + \frac{\mathbf{\Lambda}(\omega)}{\omega}\sin(\omega)\Big)$$

Setting $\mathbf{f}(\omega) := \omega^{-1}\mathbf{\Lambda}(\omega)$ and $\mathbf{f}_N(\omega) = \tan(\omega)$, (E) can be rewritten as :

$$\mathbf{f}(\omega) = \frac{\mathbf{f}_b(\omega) + \mathbf{f}_N(\omega)}{1 - \mathbf{f}_b(\omega)\,\mathbf{f}_N(\omega)} \qquad \mathbf{f}_b(\omega) := \sum \nu_i\,\mathbf{f}(\alpha_i\omega)$$

(*) considering as an equation for $\mathbf{\Lambda}(\omega)$, assuming that the $\mathbf{\Lambda}(\alpha_i\omega)$'s are known

An algorithm for the computation of $\mathbf{\Lambda}(\omega)$

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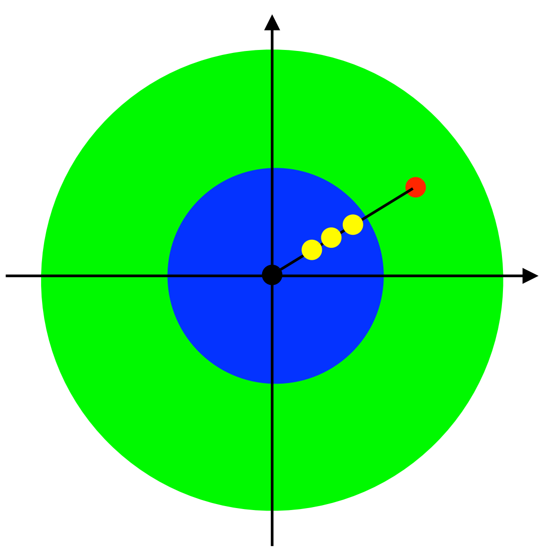
$$\mathbf{f}(\omega) = \frac{\mathbf{f}_b(\omega) + \mathbf{f}_N(\omega)}{1 - \mathbf{f}_b(\omega)\,\mathbf{f}_N(\omega)} \qquad \mathbf{f}_b(\omega) := \sum \nu_i\, \mathbf{f}(\alpha_i \omega)$$

Setting $\alpha^+ = \max\ \alpha_i < 1$ and knowing $\mathbf{f}(\omega)$ in $B_\delta^+ = \{|z| < \delta\} \cap \mathbb{C}^+$, this allows to compute $\mathbf{f}(\omega)$ in the larger domain $(\alpha^+)^{-1}\ B_\delta^+$:



By induction, one can then determinate $\mathbf{f}(\omega)$ in all $\mathbb{C}^+$

The algorithm transmits (locally) the Herglotz property

One initiates by a Taylor expansion in B_δ^+ for δ small enough

Taylor expansion : Substituting $\mathbf{\Lambda}(\omega) = \sum \Lambda_{2n}\ \omega^{2n}$ in (E) allows to compute Λ_{2n} by induction:

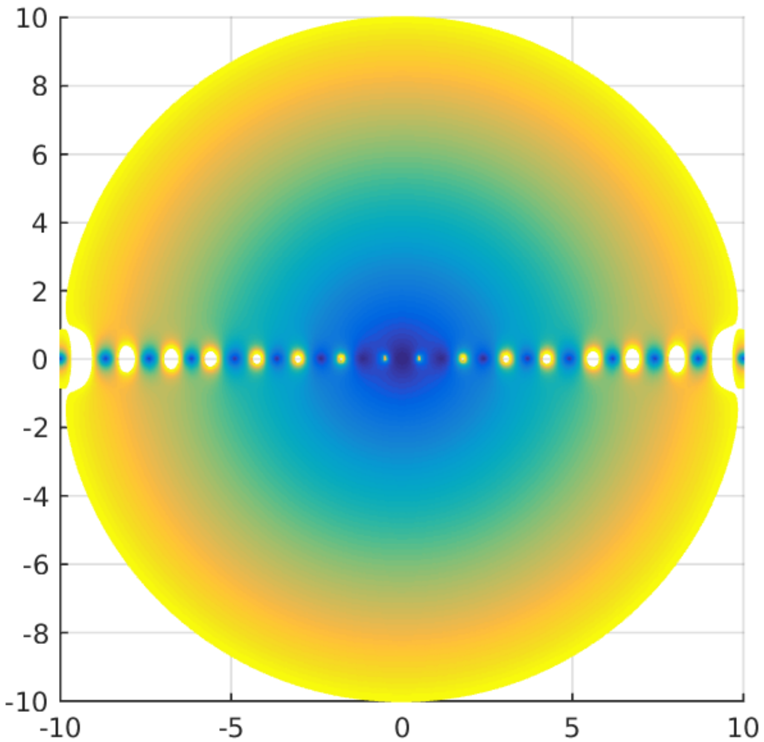
$\Lambda_{2n} = g_n(\Lambda_0, \cdots, \Lambda_{2n-2})$ with g_n known explicitly

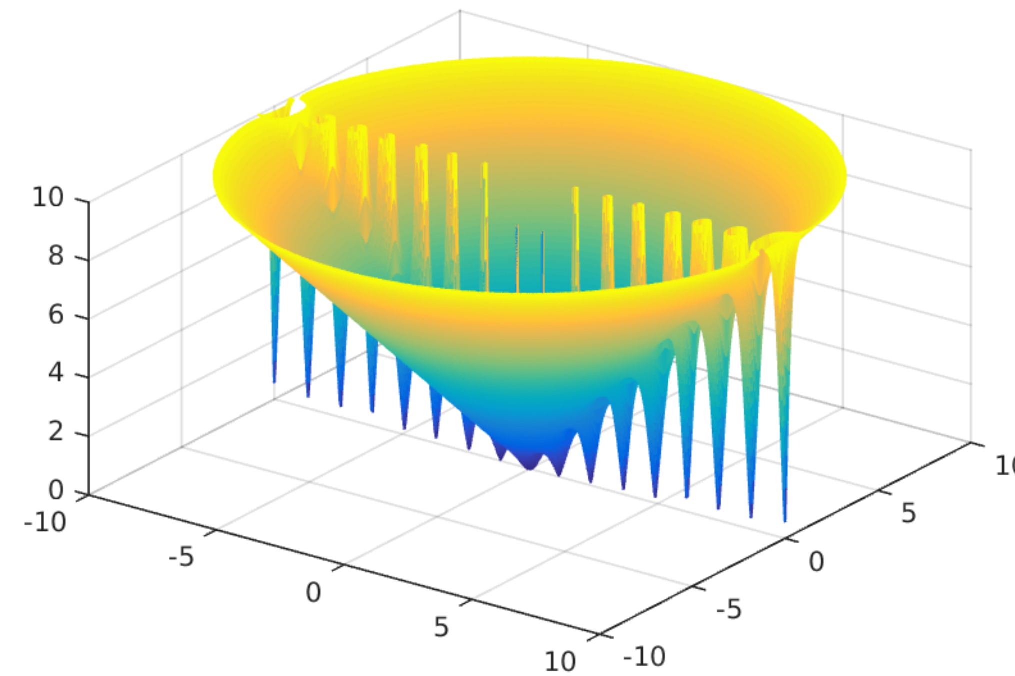
This is initiated with $\Lambda_0 = \Lambda_{\mathfrak{d}}$ or $\Lambda_0 = \Lambda_{\mathfrak{n}}$ which allows to distinct between Dirichlet and Neumann

$p = 2 \quad \alpha_1 = \alpha_2 = 0.4 \quad \mu_1 = \mu_2 = 0.4$

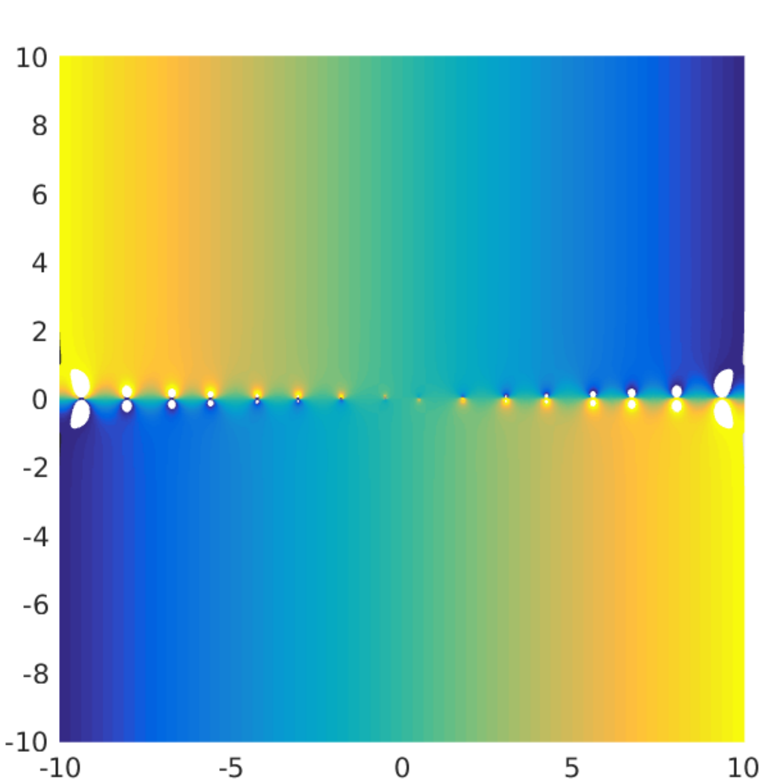
Neumann case

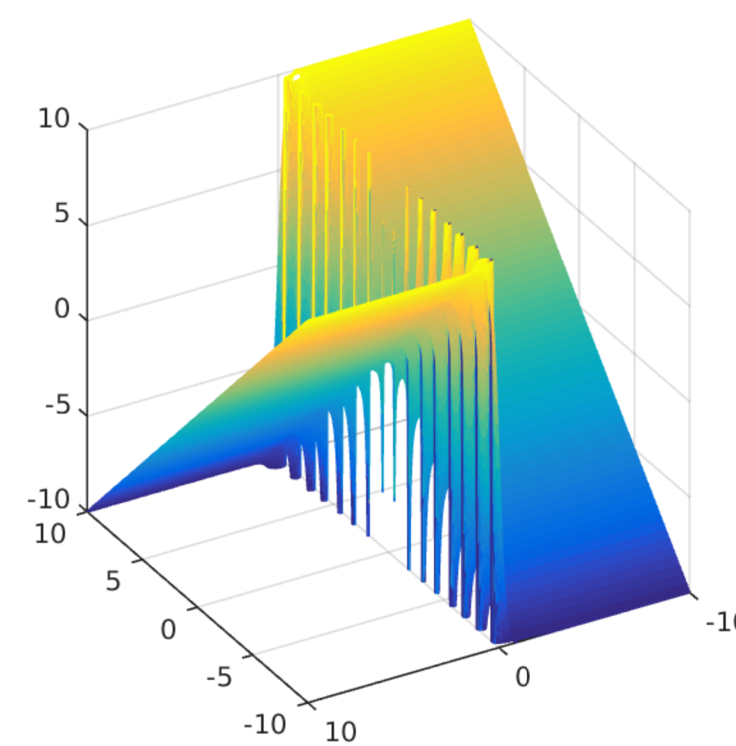
Modulus





Imaginary part

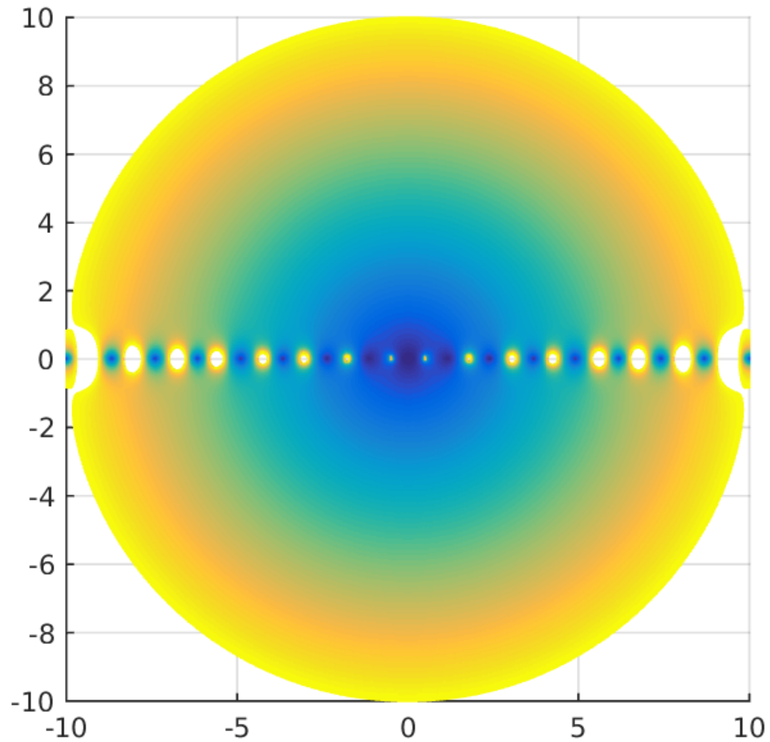


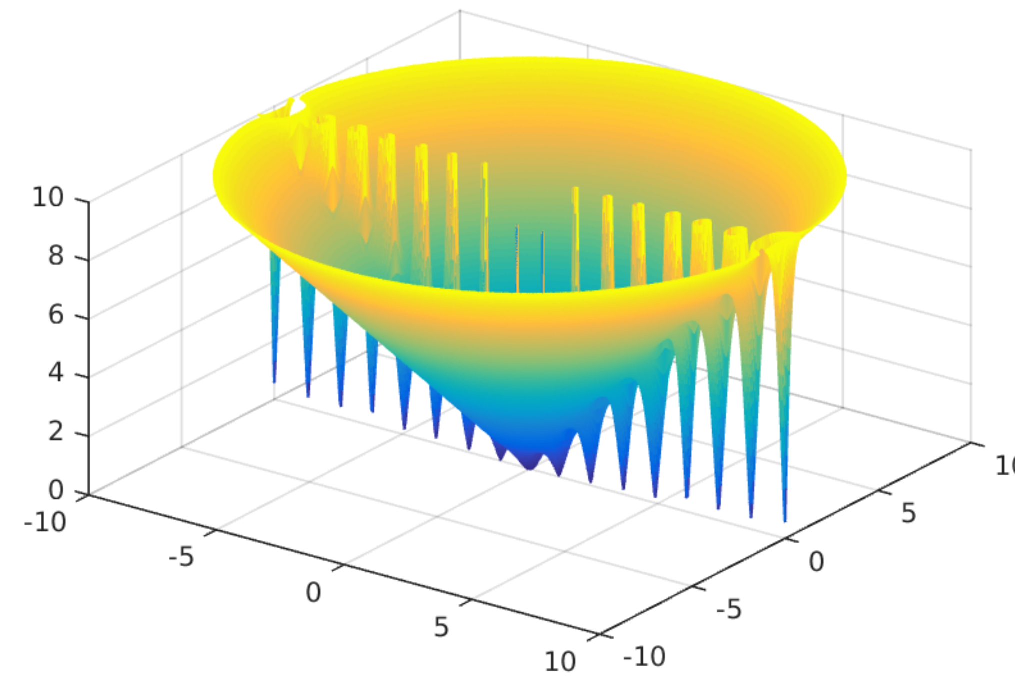


$p = 2 \quad \alpha_1 = \alpha_2 = 0.4 \quad \mu_1 = \mu_2 = 0.4$

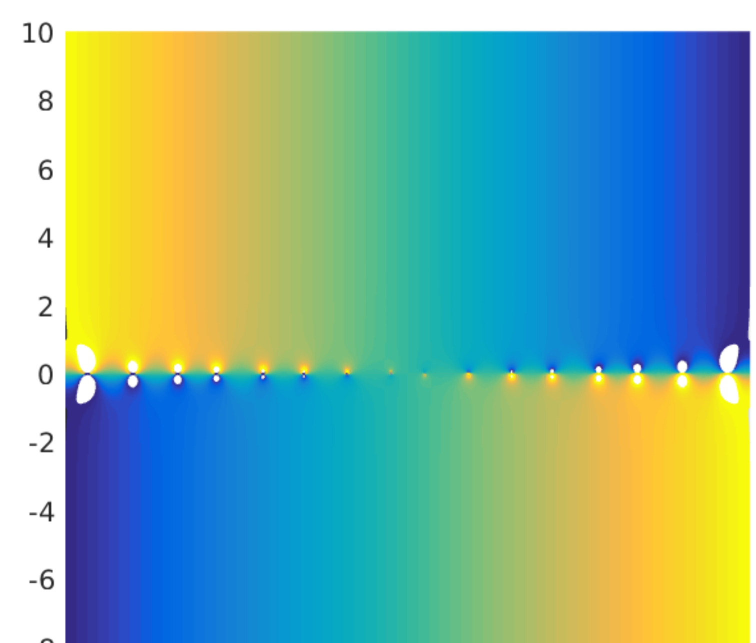
Neumann case

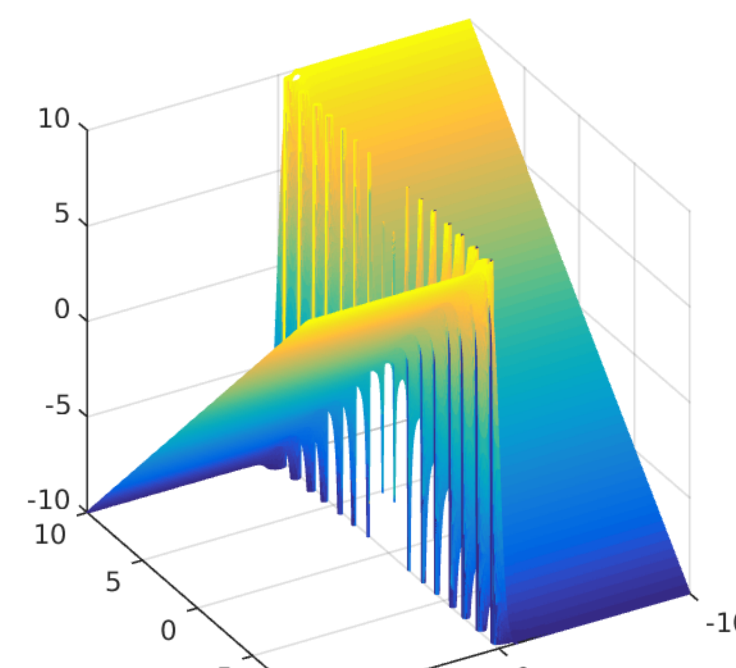
Modulus





Imaginary part





$$\left|\boldsymbol{\Lambda}(\omega) - i\,\omega\,\mathrm{sgn}\big(\mathcal{I}m\,\omega\big)\right| \leq C\,|\omega|^2\ \max\left(1, \frac{1}{|\mathcal{I}m\,\omega|^2}\right)\, e^{-2|\mathcal{I}m\,\omega|}$$

Discretization: convolution quadrature approach

The continuous Dirichlet problem : $\varphi := \varphi(t) \longrightarrow u_\varphi := u_\varphi(s,t)$

$$\boxed{\mu\, \partial_t^2 u_\varphi - \partial_s\big(\mu\, \partial_s u_\varphi\big) = 0 \qquad u_\varphi(\mathbf{0},t) = \varphi(t) \qquad + (C_\infty)}$$

The semi-discrete Dirichlet problem : $\varphi^{\Delta t} := \{\varphi^n\} \longrightarrow u_\varphi^{\Delta t} := \{u_\varphi^n(s)\}$

$$\mu\, \frac{u_\varphi^{n+1} - 2u_\varphi^n + u_\varphi^{n-1}}{\Delta t^2} - \partial_s\big(\mu\, \partial_s u_\varphi^{n,\frac14}\big) = 0 \qquad u_\varphi^n(\mathbf{0}) = \varphi^n \qquad + (C_\infty)$$

with $u_\varphi^{n,\frac14} := \dfrac{u_\varphi^{n+1} + 2u_\varphi^n + u_\varphi^{n-1}}{4}$ (non-dissipative and unconditionally stable scheme)

The discrete DtN operator $\Lambda^{\Delta t}$: $\big(\Lambda^{\Delta t}\varphi\big)^n := \partial_s u_\varphi^n(\mathbf{0})$

Discrete positivity property $\displaystyle\int_0^T \Lambda_{\mathfrak{a}}\varphi(t)\, \partial_t\varphi(t)\, dt \geq 0$ crete DtN condition

$$\boxed{\sum_{n=1}^{N-1} \big(\Lambda^{\Delta t}\varphi\big)^{n,\frac14} \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t}\, \Delta t = \frac12 \sum_{n=0}^{N-1} \int_{\mathcal{T}} \mu \left\{ \left|\frac{u_\varphi^{n+1} - u_\varphi^n}{\Delta t}\right|^2 + \left|\partial_s\Big(\frac{u_\varphi^{n+1} + u_\varphi^n}{2}\Big)\right|^2 \right\} \Delta t}$$

The convolution quadrature approach

Discrete symbol : $\mathcal{F}\big(\Lambda^{\Delta t}\varphi\big)(\omega) = \boldsymbol{\Lambda}^{\Delta t}(\omega)\,\boldsymbol{\varphi}(\omega)$ (discrete Fourier)

$$\boldsymbol{\Lambda}^{\Delta t}(\omega) = \boldsymbol{\Lambda}\Big(\frac{i}{\Delta t}\Big(\frac{1-z}{1+z}\Big)\Big) \qquad z = e^{i\omega\Delta t} \qquad \text{shift operator}$$

$$\omega \in \mathbb{C}^+ \iff |z| < 1 \iff \frac{i}{\Delta t}\Big(\frac{1-z}{1+z}\Big) \in \mathbb{C}^+ \qquad \text{(unconditional stablility)}$$

$$\boldsymbol{\Lambda}^{\Delta t}(\omega) = \sum \lambda^m(\Delta t)\; z^m \implies \big(\Lambda^{\Delta t}\varphi\big)^n = \sum_{q=0}^{n} \lambda^q(\Delta t)\;\varphi^{n-q}$$

Convolution weights : $(z = \rho\, e^{i\theta}, \rho < 1 \implies$ Fourier series in θ)

$$\forall\, \rho < 1, \qquad \lambda^m(\Delta t) = \frac{\rho^{-m}}{2\pi}\int_0^{2\pi} \boldsymbol{\Lambda}\Big(\frac{i}{\Delta t}\Big(\frac{1-\rho\, e^{i\theta}}{1+\rho\, e^{i\theta}}\Big)\Big)\; e^{-im\theta}\; d\theta$$

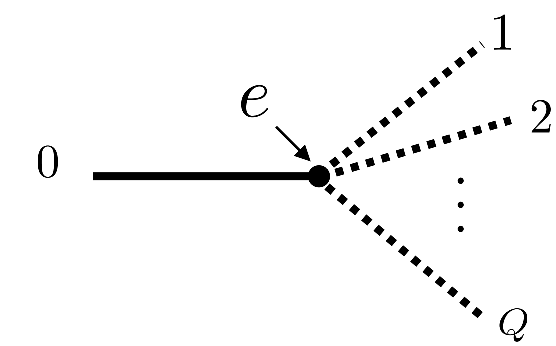
Fourier coefficients

FFT algorithm

The convolution quadrature approach

Recap : transparent DtN operator at each end point e

$$T = \mu_0^{-1} \sum_{q=1}^{Q} \mu_q \, \Lambda_q(\partial_t) \qquad \boldsymbol{\Lambda}_q(\omega) = \ell_q^{-1} \boldsymbol{\Lambda}(\ell_q \, \omega)$$



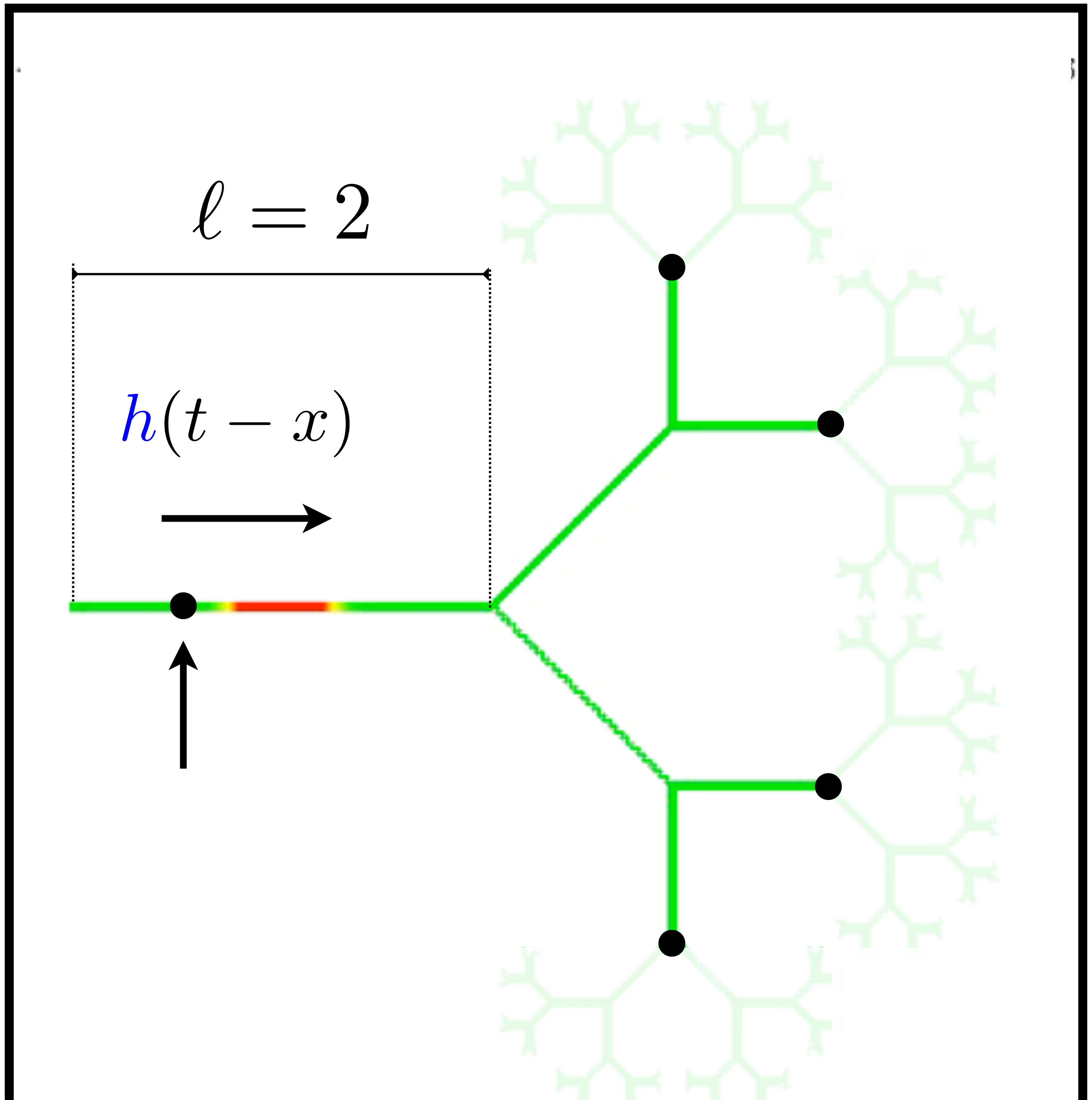
The fully discrete truncated tree problem:

$$\int_{\mathcal{T}_c} \frac{u_h^{n+1} - 2u_n^n + u_h^{n-1}}{\Delta t^2} v_h + \int_{\mathcal{T}_c} \mu \, \partial_s u_h^n \; \partial_s v_h^n \longleftarrow \text{explicit}$$

$$\text{implicit} \longrightarrow \quad + \sum_{e \in \mathcal{E}} \big(T_{\Delta t} \, u_h\big)^{n,\frac{1}{4}} \, v_h = 0,$$

$$\big(T_{\Delta t} \, u\big)^{n,\frac{1}{4}} = \mu_0^{-1} \sum_{q=1}^{Q} \big(\mu_q \, \Lambda_q^{\Delta t} u\big)^{n,\frac{1}{4}} \qquad \Lambda_q^{\Delta t} = \ell_q^{-1} \, \Lambda^{\Delta t_q}, \;\; \Delta t_q = \Delta t / \ell_q$$

Numerical simulations



t=0

Neumann case

$$p = 2$$

$$\alpha_1 = \alpha_2 = 0.4$$

$$\mu_1 = \mu_2 = 0.4$$

$$\mathrm{supp}\ h(\omega) \subset \{|\omega| \subset \Omega_{max}\}$$

$$\Omega_{max} \approx 20\,\pi$$

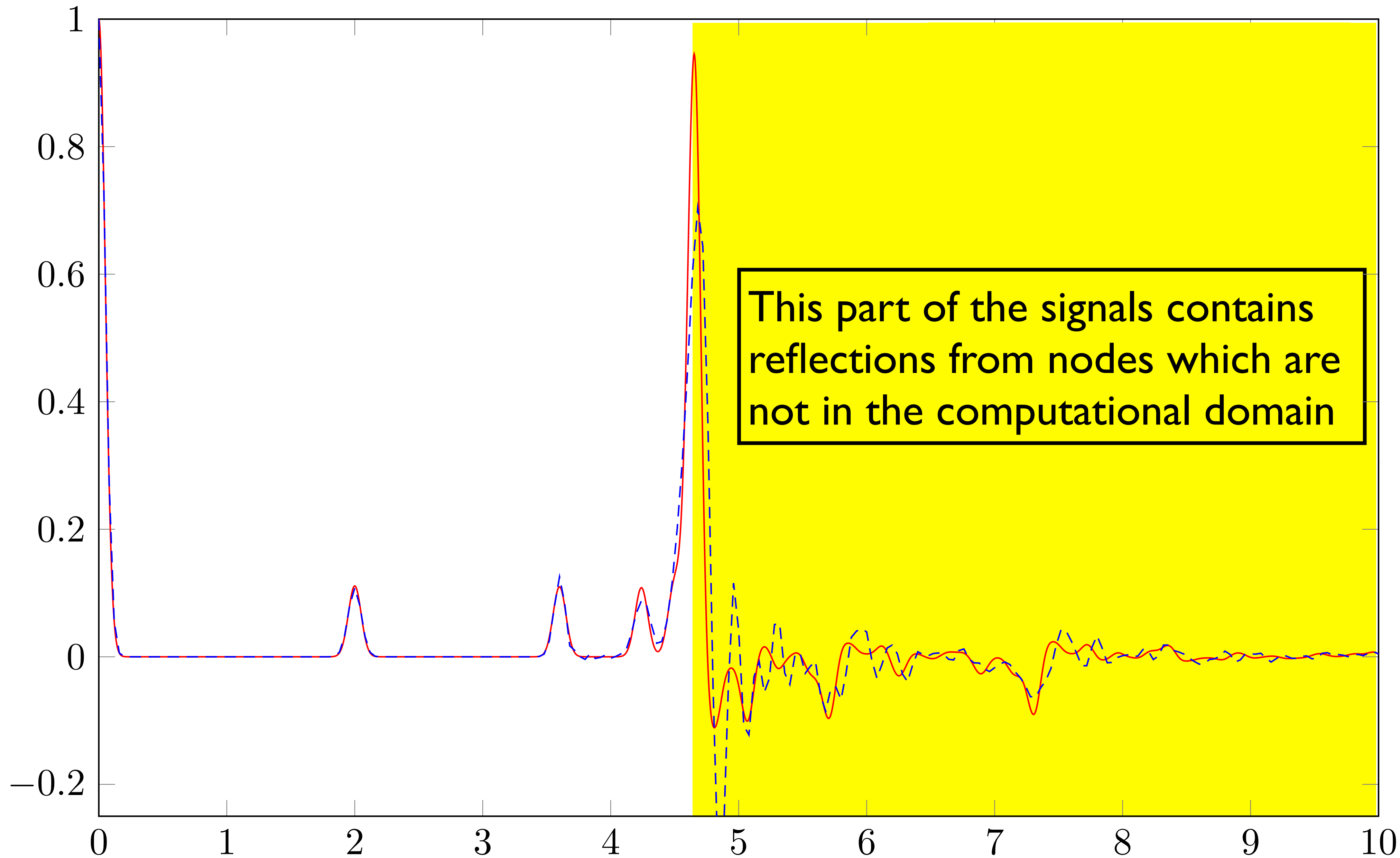
$$T_{min} \approx 0.1$$

$$\lambda_{min} \approx 0.1$$

Numerical simulations

Reference solution : computed with brute force (one week of computation)

Approximate solution : computed with $\Delta t = 0.04 \quad (T_{min} \approx 0.1)$

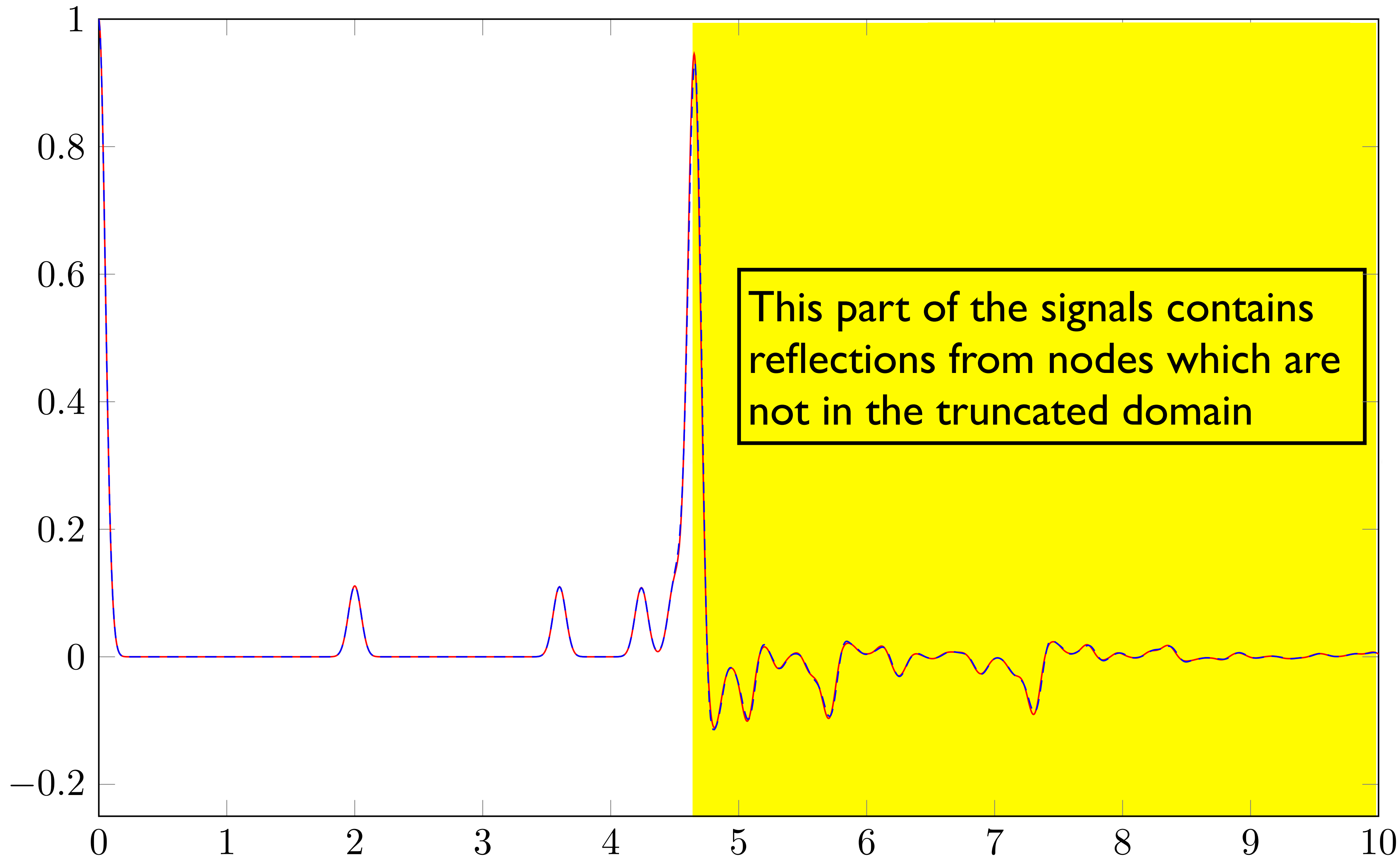


The convolution quadrature approach

Numerical simulations

Reference solution : computed with brute force (one week of computation)

Approximate solution : computed with $\Delta t = 0.01 \;\; (T_{min} \approx 0.1)$



The convolution quadrature approach