H-matrices for the BEM in electromagnetism

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LAboratoire PLasmas et Conversion d'Energie (LAPLACE)
INP-ENSEEIHT

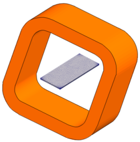
December 13, 2017

join work with R. Perrussel

Context

Boundary Element Method

- Consider the mesh only on the boundary
- drawback : full matrices (N^2 coeff.).
- Compression techniques applied to this full matrix



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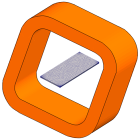

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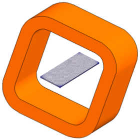
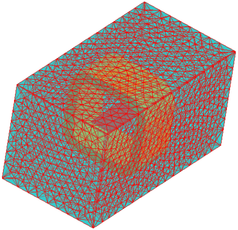


Finite Element Method

- usual method, sparse matrices ($\mathcal{O}(N)$ coefficients for N unknowns).
- disadvantages : volumic mesh, infinite boundary conditions



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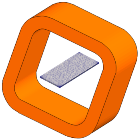

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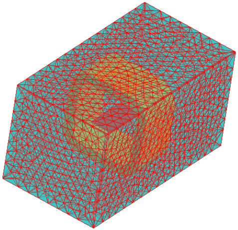

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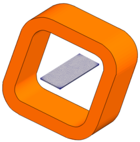

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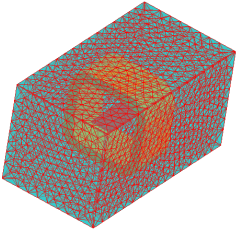

Works in the context of collaborations with

- G2ELab and LAPLACE/GREM3 for low frequency problem
- LAAS-CNRS and IMT for PCF problem

Low rank approximation

Ex. Electrostatic. Unknown $\approx$ surface charges on conductors. Matrix coefficient for 2 packs of N remote charges :

$$A_{i,j} = \frac{1}{\|\mathbf{r}_i - \mathbf{r}_j\|}.$$

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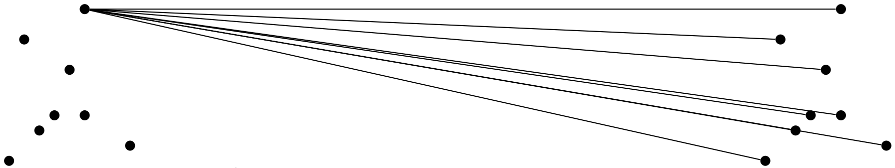
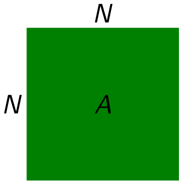


Bloc interactions : N^2 coefficients.



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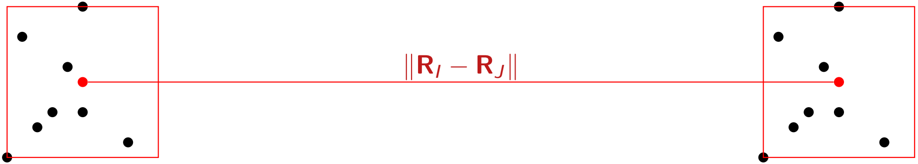
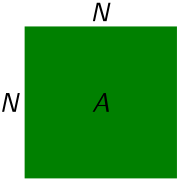
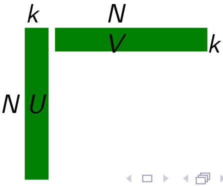


Bloc interactions : N^2 coefficients.

$$A_{i,j} \approx (\|\mathbf{R}_I - \mathbf{R}_J\|)^{-1}.$$





low rank Approx.

$A \approx UV^T$

$k \ll N$

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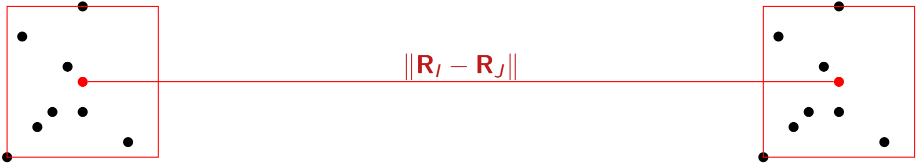

	full matrix	low rank approx.
storage	N^2	$2kN$
matrix-vector product	$\mathcal{O}(N^2)$	$\mathcal{O}(kN)$

Very interesting if $k \ll N$.

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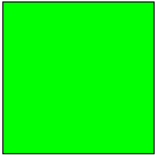

A

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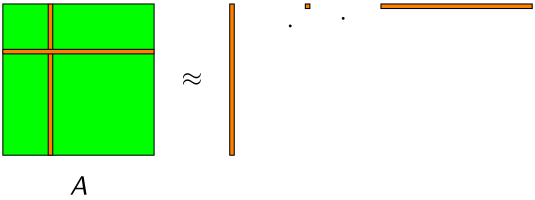

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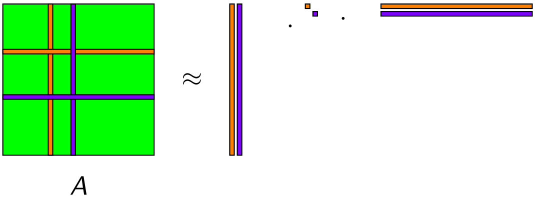

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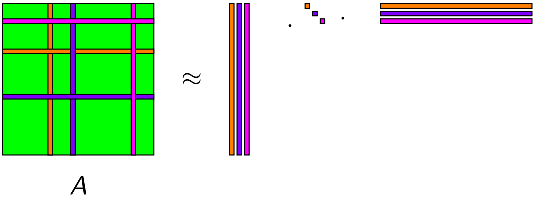

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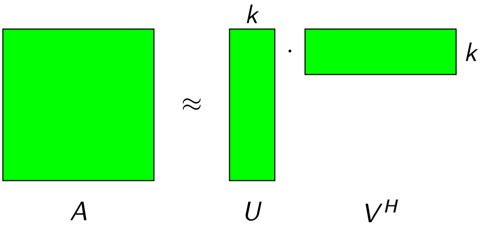

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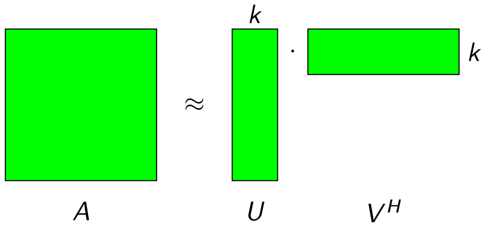

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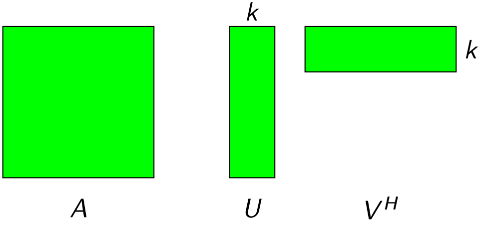

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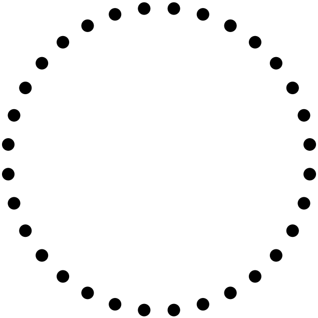
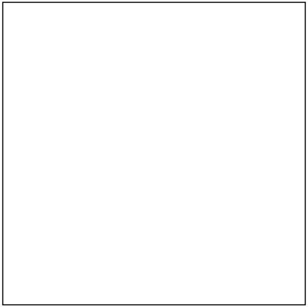
- *Fast Multipole Method* (FMM)
- *Hybrid Cross-Approximation* (HCA)

Clustering

Example : clustering of a metallic cylinder (2D) as

- *Number of DOF :* $N = 32$ *;*
- *Max number of DOF by cluster :* $n_{max} = 5$.





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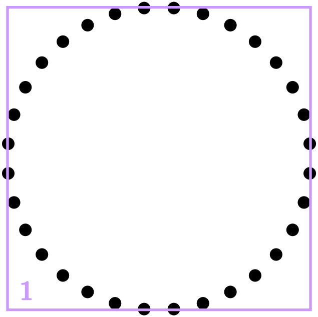

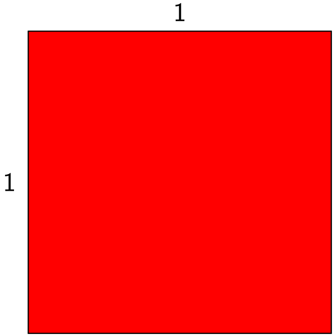

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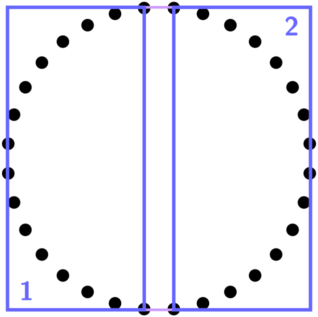

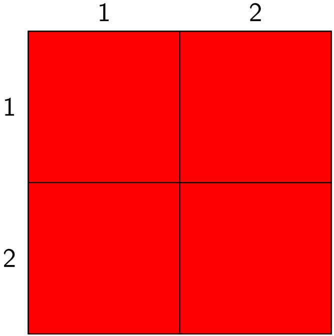

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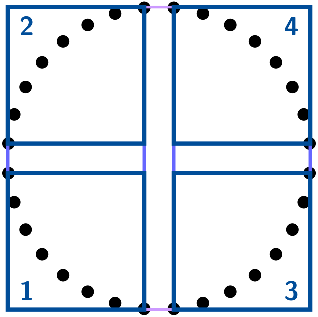

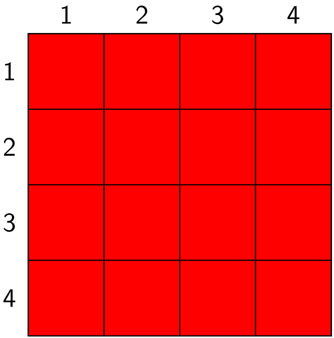

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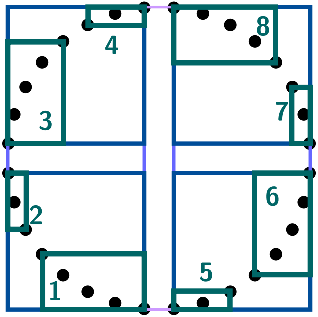
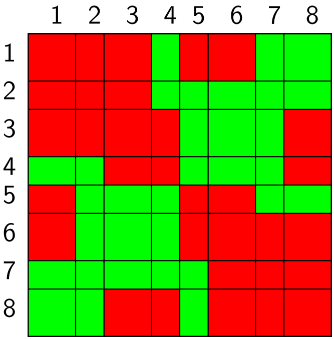
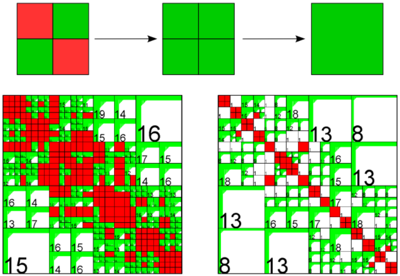




Coarsening



Matrix before and after Coarsening

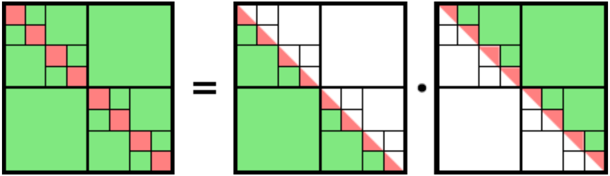
Advantages

- Total storage lowered
- Structure simplified
- Computation times enhanced

Hierarchical preconditionner or direct solver

LU decomposition with a hierarchical matrix

- Redefine each algebraic operation (+,*,inv) with accuracy ε_{LU} for the low rank blocs
- $\mathcal{H}$-LU decomposition is the LU algorithm for this arithmetics



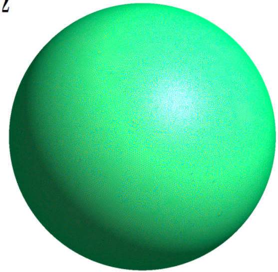
$\mathcal{H}$-LU decomposition: $\approx LU$ with accuracy ε_{LU}

Electrostatic and magnetostatic problems (with O. Chadebec and J. Siau)

$$G\,q = V$$

$$G_{i,j} = \frac{1}{4\pi}\int_{\Omega}\int_{\Omega}\varphi_i(\mathbf{r})\frac{1}{\|\mathbf{r}-\mathbf{r}'\|}\varphi_j(\mathbf{r}')\mathrm{d}\Omega'\mathrm{d}\Omega$$

$$V_i = \int_{\Omega}\varphi_i(\mathbf{r})V_0(\mathbf{r})\mathrm{d}\Omega$$



ACA and FMM

Electrostatic without preconditionner

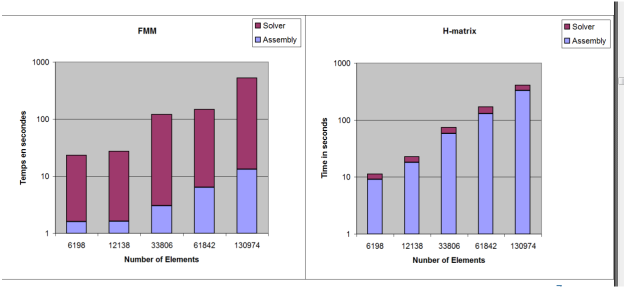


Figure : Comparison HCA/FMM

Comments

- ACA Assembly is slower than FMM.
- but solution is faster

Coarsening result on a magnetostatic example

Table : Computing times, Storage and error without coarsening

NbDof	Assemb.	Storage (kb/ddl)	Erreur
3846	12 mn	15.4	1.53%
12938	2 h	24.3	0.85%
48154	15 h	37.6	0.37%

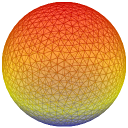
Table : Computing times, Storage and error with coarsening

NbDof	Assemb.	Storage (kb/ddl)	Erreur
3846	16 mn	7	1.66%
12938	2 h 45	10.3	0.27%
48154	20 h	14.9	0.45%

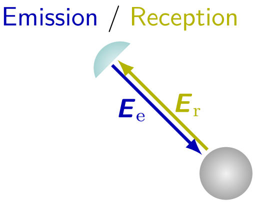
comments

- Compression higher
- Slow increase of Assembly

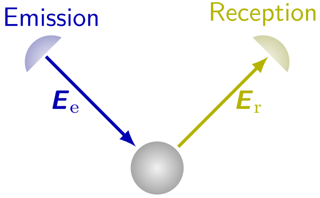
Scattering problem and RCS



- Frequency $f = 300\,\text{MHz}$
- discretisation $h < \frac{\lambda}{10}$



monostatic Radar



Bistatic Radar

- bistatique RCS : 1 RHS
- monistatic RHS : multiple RHS

Scattering problem and RCS

The scattering problem

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0 \text{ dans } \Omega \\ \nabla \times \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0 \text{ dans } \Omega \\ \mathbf{E} \times \mathbf{n} = 0 \quad \textit{sur} \quad \Gamma \\ \nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{H} = 0 \quad \text{dans } \Omega \\ \lim\limits_{|x| \to \infty} |x| \left\{ \sqrt{\mu}\mathbf{H}(x) \times \frac{x}{|x|} - \sqrt{\varepsilon}\mathbf{E}(x) \right\} = 0 \end{cases}$$

EFIE

$$\mathbf{E}(\mathbf{r}') = \nabla_{r'} \int_\Gamma \frac{-1}{j\omega} \nabla_\Gamma \cdot \mathbf{J}(r) \frac{e^{-jkR}}{4\pi R} d\gamma(r) + j\omega \int_\Gamma \mu \frac{e^{-jkR}}{4\pi R} \mathbf{J}(\mathbf{r}) d\gamma(r)$$

Hierarchical preconditionner and direct solver

Metallic sphere monostatic RCS

multiple RHS -$p = 360$

- Frequency: $f = 300\,\text{MHz}$;
- Compression :($\varepsilon_{\text{ACA}} = 10^{-4}$)
- $\varepsilon_{\text{rés.}} = 10^{-4}$

	GMRES	GMRES précond.		$\mathcal{H}$-LU
		10^{-1}	10^{-3}	
coarsening ($\varepsilon_{\text{préc.}}$ (s)	$\emptyset$	36,152	71,884	$\emptyset$
$\mathcal{H}$-LU decomposition (s)	$\emptyset$	124,800	852,980	1622,080
GMRES (s)	109 164,62	16 370,384	1394,260	$\emptyset$
solving $\mathcal{H}$-LU (s)	$\emptyset$	$\emptyset$	$\emptyset$	226,808
Temps total (s)	**109 164,62**	**16 532,192**	**2320,428**	**2048,888**
total iterations	241 630	17 993	720	$\emptyset$
memory cost (Mo)	854,912	1036,7	1422,5	879,25

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GMRES (s)	109 164,62	16 370,384	1394,260	$\emptyset$
solving $\mathcal{H}$-LU (s)	$\emptyset$	$\emptyset$	$\emptyset$	226,808
Temps total (s)	**109 164,62**	**16 532,192**	**2320,428**	**2048,888**
total iterations	241 630	17 993	720	$\emptyset$
memory cost (Mo)	854,912	1036,7	1422,5	879,25

TTIL RSMOSG : LAPLACE/LAAS/IMT

- Optical fiber consisting of a cladding and C cores
- Study of the propagation of light in the fiber

Purpose: To find the refractive index n_{eff} characterizing the fiber

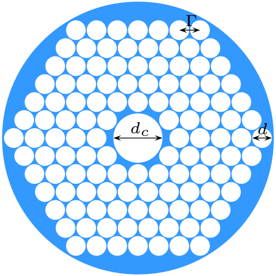


Figure : Example of a fiber with 121 cores

The transmission problem

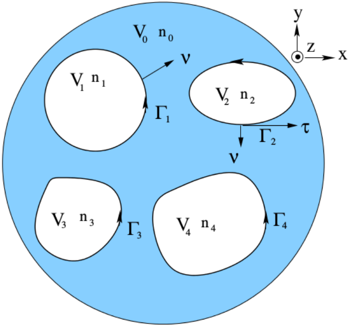
$E = \boldsymbol{E}_z$ and $H = \sqrt{\frac{\varepsilon_0}{\mu_0}}\boldsymbol{H}_z$, $k_0 = \frac{2\pi}{\lambda_0}$, $k = k_0 n$ and $\beta = k_0 n_{\text{eff.}}$.

$$\left.\begin{aligned} \left(\Delta + (k^2 - \beta^2)\right)E(x) &= 0 \\ \left(\Delta + (k^2 - \beta^2)\right)H(x) &= 0 \end{aligned}\right\} \forall x \in \bigcup_{i=0}^{M} \Omega_i$$

$$\left.\begin{aligned} [E] &= 0 \\ [H] &= 0 \\ \left[\frac{n_{\text{eff.}}}{n^2 - n_{\text{eff.}}^2}\frac{\partial E}{\partial \tau}\right] - \left[\frac{1}{n^2 - n_{\text{eff.}}^2}\frac{\partial H}{\partial \nu}\right] &= 0 \\ \left[\frac{n_{\text{eff.}}}{n^2 - n_{\text{eff.}}^2}\frac{\partial H}{\partial \tau}\right] + \left[\frac{n^2}{n^2 - n_{\text{eff.}}^2}\frac{\partial E}{\partial \nu}\right] &= 0 \end{aligned}\right\} \text{in } \Gamma = \bigcup_{i=1}^{M} \Gamma_i$$

- Helmholtz equation on E and H;
- Transmission condition on E, H and derivative.

BEM for PCF (with P. Daquin, J. Vincent)



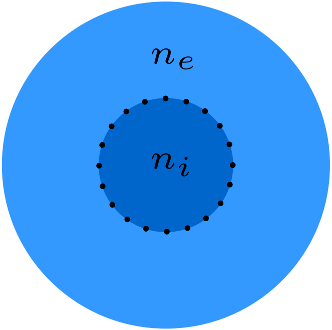
- Boundary conditions : $[X]_\Gamma = X_e - X_i$

$$\left[\frac{n_{eff}}{n^2 - n_{eff}^2}\partial_\tau E\right]_\Gamma = \left[\frac{1}{n^2 - n_{eff}^2}\partial_\nu H\right]_\Gamma$$

$$\left[\frac{n_{eff}}{n^2 - n_{eff}^2}\partial_\tau H\right]_\Gamma = -\left[\frac{n^2}{n^2 - n_{eff}^2}\partial_\nu E\right]_\Gamma$$

$\mathbf{E}$, $\partial_{\nu_\mathbf{i}}\mathbf{E}$, $\partial_{\nu_\mathbf{e}}\mathbf{E}$, $\mathbf{H}$, $\partial_{\nu_\mathbf{i}}\mathbf{H}$, $\partial_{\nu_\mathbf{e}}\mathbf{H}$, $\partial_\tau\mathbf{E}$ and $\partial_\tau\mathbf{H}$ are the initial unknowns

The operators



- Inside

$$(Id - J_i)\,\partial_{\nu i} u = -W_i u \tag{1}$$

- Outside

$$(Id + J_e)\,\partial_{\nu e} u = W_e u \tag{2}$$

$u = E$ or H

T is the tangential derivative operator

$$\left[\frac{1}{n_i^2 - n_{eff}^2} - \frac{1}{n_e^2 - n_{eff}^2}\right] n_{eff}\, TE = \frac{1}{n_i^2 - n_{eff}^2}\partial_{\nu i} H - \frac{1}{n_e^2 - n_{eff}^2}\partial_{\nu e} H \tag{3}$$

$$\left[\frac{1}{n_i^2 - n_{eff}^2} - \frac{1}{n_e^2 - n_{eff}^2}\right] n_{eff}\, TH = \frac{1}{n_e^2 - n_{eff}^2}\partial_{\nu e} E - \frac{1}{n_i^2 - n_{eff}^2}\partial_{\nu i} E \tag{4}$$

The involved problem

After discretization we can build the following 6 unknowns system

$$\begin{bmatrix} Wi & Id - Ji & 0 & 0 & 0 & 0 \\ We & 0 & -(Id + Je) & 0 & 0 & 0 \\ 0 & -\frac{n_c^2}{n_c^2 - n_{eff}^2} Id & \frac{n_g^2}{n_g^2 - n_{eff}^2} Id & \left(\frac{n_{eff}}{n_g^2 - n_{eff}^2} - \frac{n_{eff}}{n_c^2 - n_{eff}^2}\right) T & 0 & 0 \\ 0 & 0 & 0 & Wi & Id - Ji & 0 \\ 0 & 0 & 0 & We & 0 & -(Id + Je) \\ \left(\frac{n_{eff}}{n_g^2 - n_{eff}^2} - \frac{n_{eff}}{n_g^2 - n_{eff}^2}\right) T & 0 & 0 & 0 & \frac{1}{n_c^2 - n_{eff}^2} Id & -\frac{1}{n_g^2 - n_{eff}^2} Id \end{bmatrix} \cdot \begin{bmatrix} E \\ \partial_{\nu i} E \\ \partial_{\nu e} E \\ H \\ \partial_{\nu i} H \\ \partial_{\nu e} H \end{bmatrix} = 0$$

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Reduction to a 4 unknowns problem with

$$\left(\frac{n_g^2}{n_g^2-n_{eff}^2}Id\right)\partial_{\nu e}E = \left(\frac{n_c^2}{n_c^2-n_{eff}^2}Id\right)\partial_{\nu i}E - \left(\left(\frac{n_{eff}}{n_g^2-n_{eff}^2}-\frac{n_{eff}}{n_c^2-n_{eff}^2}\right)T\right)H$$

$$\left(\frac{1}{n_g^2-n_{eff}^2}Id\right)\partial_{\nu e}H = \left(\frac{1}{n_c^2-n_{eff}^2}Id\right)\partial_{\nu i}H + \left(\left(\frac{n_{eff}}{n_g^2-n_{eff}^2}-\frac{n_{eff}}{n_g^2-n_{eff}^2}\right)T\right)E$$

The 4 unknowns problem

$$\begin{bmatrix} W_e & -\frac{n_i^2}{n_e^2}\frac{n_e^2-n_{eff}^2}{n_i^2-n_{eff}^2}(Id+J_e) & \frac{n_{eff}}{n_e^2}\left(1+\frac{n_e^2-n_{eff}^2}{n_i^2-n_{eff}^2}\right)(Id+J_e)T & 0 \\ W_i & Id-J_i & 0 & 0 \\ n_{eff}\left(1+\frac{n_e^2-n_{eff}^2}{n_i^2-n_{eff}^2}\right)(Id+J_e)T & 0 & W_e & -\frac{n_e^2-n_{eff}^2}{n_i^2-n_{eff}^2}(Id+J_e) \\ 0 & 0 & W_i & Id-J_i \end{bmatrix} \cdot \begin{bmatrix} E \\ \partial_{\nu_i}E \\ H \\ \partial_{\nu_i}H \end{bmatrix} = 0$$

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Non linear problem

The is solving by seeking the zeros of f define by:

$$f(n_{\text{eff.}}) = \frac{1}{\Psi^T F^{-1}\Phi}$$

where Ψ and Φ are two random vectors.
$n_{\text{eff.}}$ solution of the problem leads to $\|F^{-1}\| \to +\infty$.

The algebraic problem

$$F\boldsymbol{u} = \boldsymbol{\Phi} \tag{5}$$

$$\begin{bmatrix} W_e & K_e & K_e T & \\ W_i & K_i & & \\ \alpha K_e T & & W_e & \beta K_e \\ & & W_i & K_i \end{bmatrix}$$

The algebraic problem

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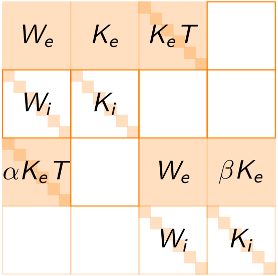

F sparse matrix of size $(4NC)^2$ with

- 6 blocs 0
- 4 sparse blocs : diagonal blocs
- **6 full blocs**

The algebraic problem

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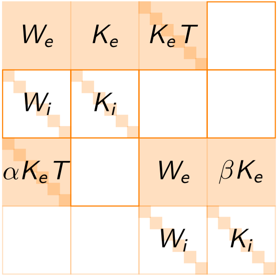

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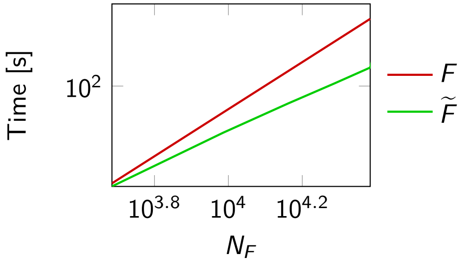
- **6 full blocs**

$\rightarrow$ **Compress the full blocs with ACA**
$\rightarrow$ **Apply $\mathcal{H}$-matrices strategy to the full matrix**

Assembling Time

Assembly time of the matrix F for a fiber with 121 cores with or without compression as a function of the size $N_F = 4NC$ of the problem

- Compression : ACA ($\varepsilon_{\text{ACA}} = 10^{-5}$).



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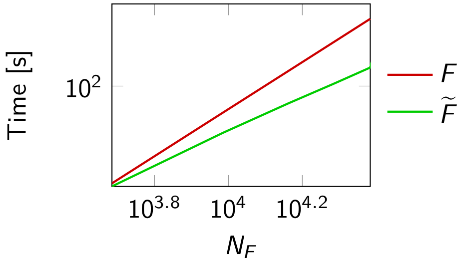

Improved assembly time by applying hierarchical format

Direct Solver $\mathcal{H}-$LU / GMRES

the GMRES solver has very slow convergence

Table : Number of iterations for the GMRES solver

$\varepsilon_{\text{GMRES}}$	1452 ddl	2420 ddl	4840 ddl
10^{-3}	1347	1822	4789
10^{-4}	1354	1838	4798
10^{-5}	1361	1850	4801
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$\mathcal{H}-$LU solver needs more accuracy than in the conventional cases

Table : Relative error $\|x - x_{ref}\|/\|x_{ref}\|$

$\varepsilon_{\text{rés.}}$ or ε_{LU}	$\mathcal{H}$-LU	GMRES
10^{-3}	1.42	$1.81\ 10^{-5}$
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10^{-5}	$6.31\ 10^{-2}$	$2.97\ 10^{-8}$
10^{-6}	$4.25\ 10^{-3}$	$6.82\ 10^{-9}$
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GMRES with $\mathcal{H}$-LU preconditionning

To solve the previous difficulties it is convenient to use an iterative solution with $\mathcal{H}$-LU preconditionning

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10^{-2}	4546	4133
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The value of ε_{LU} has to be lower that the desired accuracy to be efficicient and the coarsening is very efficient for accurate values

Computation Times

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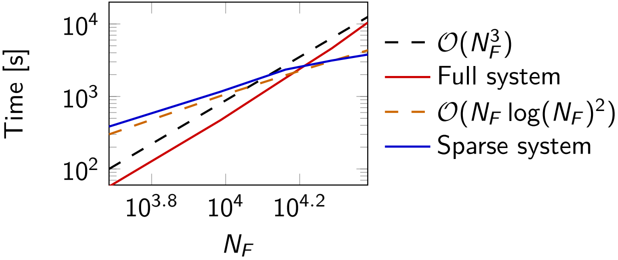

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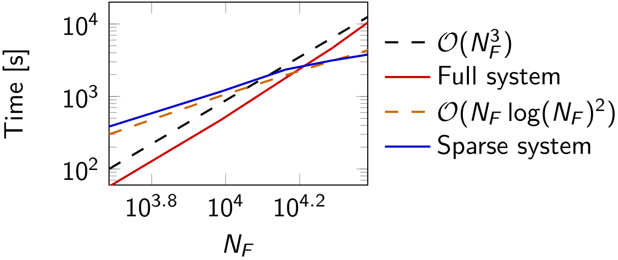

Improved solution time by applying hierarchical format

Conclusion and futur prospects

- Wave problems and low frequency application
  - Very efficient for the low frequency applications
  - Implemented on the improved G2ELab code
  - Good results but that need to be improved for the high frequency case

  - study of the H2-Matrix format
- PCF
  - Code and tools not very optimized
  - Currently in progress

  - scaling of the coefficients
  - smart pivoting of the blocks
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