Conference in honor of Abderrahmane Bendali

Université de Pau et des Pays de l'Adour

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Local Stabilization of a 2D Fluid - Structure - Interaction system

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Thematic network 'Aeronautic and space' and ANR IFSMACS

Outline of the talk

- Motivations and goals
- The PDE model, issues and method
- Stability and stabilization issues - Eigenvalue analysis
- Control strategy adapted to numerical approximation
- Numerical simulations (2D).

Motivations

• Stabilization of aerodynamic flows around an unstable stationary solution.

Given a fluid flow around an obstacle. Can we determine a control located at the boundary of the obstacle able to stabilize it in presence of disturbances.

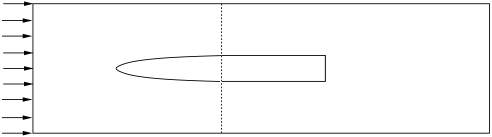
Other project

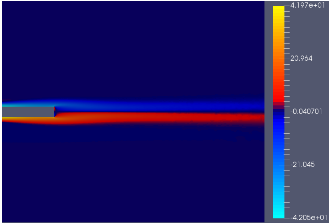
• Autoregulation of either cerebrospinal flows or blood flows in the brain.

Can we model the autoregulation phenomenon by a feedback control corresponding to deformation of vessels ?

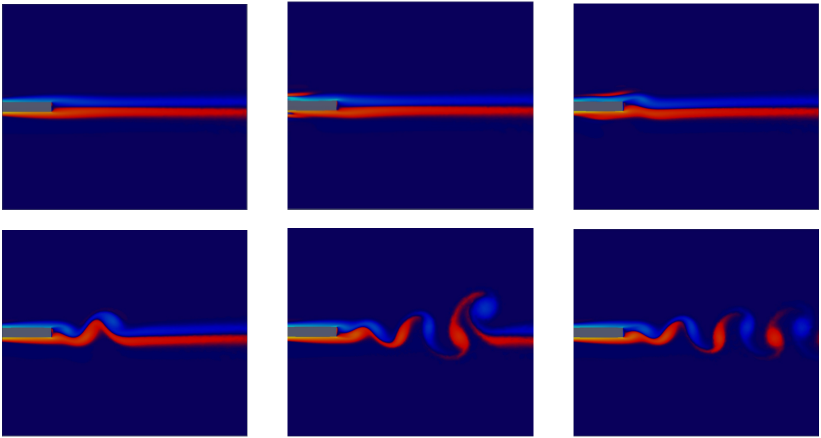
• In both cases we have to deal with mixed boundary conditions.

Wind tunnel - Stabilization of a flow over a thick plate

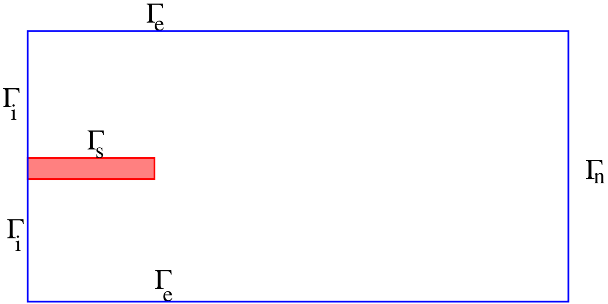




Inflow Perturbation of a stationary flow



The geometrical configuration



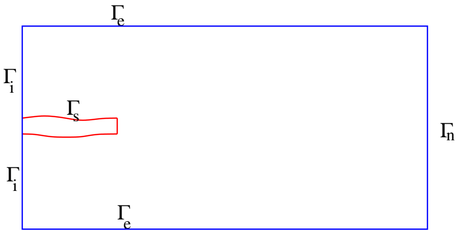
with Dirichlet-Neumann and Navier boundary conditions.

$$(\nu(\nabla u + \nabla u^T) - p\, I)n = \sigma(u,p)n = 0 \quad \text{on} \quad \Gamma_n \times (0,\infty),$$

on $\Gamma_e \times (0,\infty)$ either Navier or Neumann B.C.

For simplicity in writing we shall impose Neumann B.C. on Γ_e.
$\varepsilon(u) = \frac{\nu}{2}(\nabla u + \nabla u^T)$.

Due to the displacement $\eta(t)$ of the structure, the fluid equation is written in a time dependent geometrical domain



We introduce

$$\widetilde{Q} = \cup_{t\geq 0}\,\Omega_{\eta(t)} \times \{t\},$$

$$\widetilde{\Sigma}_s = \cup_{t\geq 0}\,\Gamma_{\eta(t)} \times \{t\}.$$

The model

The fluid equation

$$u_t + (u\cdot\nabla)u - \mathrm{div}\,\sigma(u,p) = 0, \quad \mathrm{div}\,u = 0 \quad \text{in } \widetilde{Q},$$

$$u = \eta_2\vec{e}_2 \ \text{ on } \widetilde{\Sigma}_s, \quad u = g_s + g_d \ \text{ on } \Sigma_i, \quad \sigma(u,p)n = 0 \quad \text{on } \Sigma_n$$

$$u(0) = u_s + v_0 \text{ in } \Omega_0,$$

$$\sigma(u,p) = \nu(\nabla u + \nabla u^T) - p\,I.$$

The beam equation $\eta_{tt} + \alpha\Delta_s^2\eta - \gamma\Delta_s\eta_t = \cdots$

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s,$$

$$\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \gamma\Delta_s\eta_2$$

$$= -\sigma(u,p)(-\eta_{1,x}\vec{e}_1 + \vec{e}_2)\cdot\vec{e}_2 - f_s + f\,\chi_{\Gamma_c} \quad \text{on } \Sigma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s\times(0,\infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.$$

Fluid equation

$$(u_s \cdot \nabla)u_s - \operatorname{div}\sigma(u_s, p_s) = 0, \quad \operatorname{div} u_s = 0 \quad \text{in } \Omega,$$

$$u_s = \eta_2 \vec{e}_2 \ \text{ on } \Gamma_s, \quad u_s = g_s \ \text{ on } \Gamma_i, \quad \sigma(u_s, p_s)n = 0 \quad \text{on } \Gamma_n,$$

Beam equation

$$0 = \eta_2 \quad \text{on } \Gamma_s,$$

$$\alpha \Delta_s^2 \eta_1 = -\sigma(u_s, p_s)(-\eta_{1,x}\vec{e}_1 + \vec{e}_2) \cdot \vec{e}_2 - f_s \quad \text{on } \Gamma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial \eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0, \infty).$$

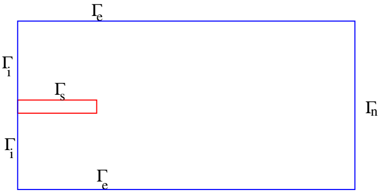
If $f_s = p_s$, then $\eta_1 = \eta_2 = 0$ and

$$(u_s \cdot \nabla)u_s - \operatorname{div}\sigma(u_s, p_s) = 0, \quad \operatorname{div} u_s = 0 \quad \text{in } \Omega,$$

$$u_s = 0 \ \text{ on } \Gamma_s, \quad u_s = g_s \ \text{ on } \Gamma_i, \quad \sigma(u_s, p_s)n = 0 \quad \text{on } \Gamma_n.$$

The change of variable is associated with the *structure displacement:*

$$\text{Reference configuration} \quad \xleftarrow{\mathcal{T}_{\eta(t)}} \quad \text{Deformed configuration}$$



$$\widehat{u}(x,z,t) = u(\mathcal{T}_{\eta(t)}^{-1}(x,z),t), \quad \widehat{p}(x,z,t) = p(\mathcal{T}_{\eta(t)}^{-1}(x,z),t).$$

The system in the reference configuration

Fluid equation

$$\hat{u}_t - \operatorname{div}\sigma(\hat{u},\hat{p}) + (\hat{u}\cdot\nabla)\hat{u} = \mathcal{F}[\hat{u},\hat{p},\eta_1,\eta_2],$$

$$\operatorname{div}\hat{u} = \mathcal{G}[\hat{u},\eta_1] \text{ in } Q = \Omega\times(0,\infty),$$

$$\hat{u} = \eta_2\,\vec{e}_2 \text{ on } \Sigma_s = \Gamma_s\times(0,\infty), \qquad \hat{u} = g_s + g_d \text{ on } \Sigma_i = \Gamma_i\times(0,\infty),$$

$$\sigma(\hat{u},\hat{p})n = 0 \text{ on } \Sigma_n = \Gamma_n\times(0,\infty), \qquad \hat{u}(0) = \hat{u}^0 \text{ in } \Omega,$$

Beam equation

$$\eta_{1,t} = \eta_2 \quad \text{ on } \Sigma_s,$$

$$\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \gamma\Delta_s\eta_2$$

$$= -\sigma(u,p)(-\eta_{1,x}\vec{e}_1 + \vec{e}_2)\cdot\vec{e}_2 - p_s + f\,\chi_{\Gamma_c} \quad \text{ on } \Sigma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{ on } \partial\Gamma_s\times(0,\infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{ on } \Gamma_s.$$

Control strategy

- Linearize the system around the unstable stationary solution.
- Study the stabilizability of the control system.

For numerical guarantee, we look for a feedback control of finite dimension

- Project the control system onto a finite dimensional subspace containing the unstable subspace of the linearized model.
- Determine (numerically) a feedback for which we can guarantee that, when it is applied to the nonlinear PDE system, we still have a local stabilization result.

The linearized system statisfied by $(v, q, \eta_1, \eta_2) = (\widehat{u} - u_s, \widehat{p} - p_s, \eta_1, \eta_2)$ is

Fluid equation

$$v_t - \operatorname{div} \sigma(v,q) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0,$$

$$\operatorname{div} v = A_3\eta_1 \quad \text{in } Q,$$

$$v = \eta_2\, e_2 \quad \text{on } \Sigma_s, \quad v = g_d \quad \text{on } \Sigma_i,$$

$$\sigma(v,q)n = 0 \quad \text{on } \Sigma_n, \quad v(0) = v^0 \text{ in } \Omega,$$

Beam equation

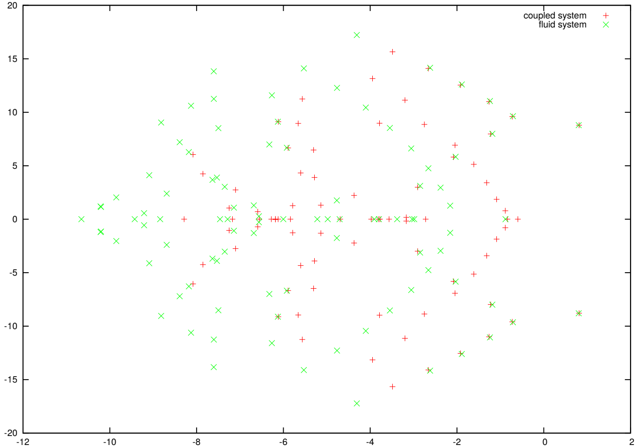
$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s,$$

$$\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \gamma\Delta_s\eta_2 - A_4\eta_1 = q + f \quad \text{on } \Sigma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0, \infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.$$

Spectrum of the coupled system – $\mathrm{Re} = \frac{U_m e}{\nu} = 200$



Are we sure that what we compute is correct ?

The direct eigenvalue problem

Fluid equation

$$\lambda v - \operatorname{div}\sigma(v,q) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0,$$

$$\operatorname{div} v = A_3\eta_1 \quad \text{in } \Omega,$$

$$v = \eta_2\, e_2 \quad \text{on } \Gamma_s, \quad v = 0 \quad \text{on } \Gamma_i,$$

$$\sigma(v,q)n = 0 \quad \text{on } \Gamma_n,$$

Beam equation

$$\lambda\eta_1 = \eta_2 \quad \text{on } \Gamma_s,$$

$$\lambda\eta_2 + \alpha\Delta_s^2\eta_1 - \gamma\Delta_s\eta_2 - A_4\eta_1 = q \quad \text{on } \Gamma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

Algebraic constraints $\operatorname{div} v = A_3\eta_1$ and $v = \eta_2\, e_2$.

The adjoint eigenvalue problem

Fluid equation

$$\lambda\phi - \mathrm{div}\,\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0,$$

$$\mathrm{div}\,\phi = 0 \quad \text{in } \Omega,$$

$$\phi = \zeta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \qquad \phi = 0 \ \text{ on } \ \Gamma_0,$$

$$\sigma(\phi,\psi)n + u_s\cdot n\,\phi = 0 \ \text{ on } \ \Gamma_n,$$

Beam equation

$$\lambda\zeta_1 + \alpha(\Delta_s^2)^{-1}(\zeta_2 + A_3^*\psi - A_1^*\phi - A_4^*\zeta_2) = 0 \quad \text{in } \Gamma_s,$$

$$\lambda\zeta_2 - \delta\Delta_s\zeta_2 - \alpha\Delta_s^2\zeta_1 - A_2^*\phi = \psi \ \text{ in } \Gamma_s,$$

$$\zeta_1 = 0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

Algebraic constraints $\mathrm{div}\,\phi = 0$ and $\phi = \zeta_2\, e_2$. The algebraic constraints for the direct and adjoint systems are different.

The discrete direct eigenvalue problem

$$\lambda\, M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} = A \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix}, \quad M = \begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{zz} & \widetilde{A}_{z\theta} \\ \widehat{A}_{z\theta}^T & 0 \end{bmatrix},$$

where $\boldsymbol{\theta}$ stands for the pressure $\boldsymbol{p}$ and the multiplier $\boldsymbol{\lambda}$

$$M_{zz} = \begin{bmatrix} M_{vv} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & 0 & M_{\eta\eta} \end{bmatrix}, \quad A_{zz} = \begin{bmatrix} A_{vv} & A_{v\eta_1} & A_{v\eta_2} \\ 0 & 0 & A_{\eta_1\eta_2} \\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2} \end{bmatrix},$$

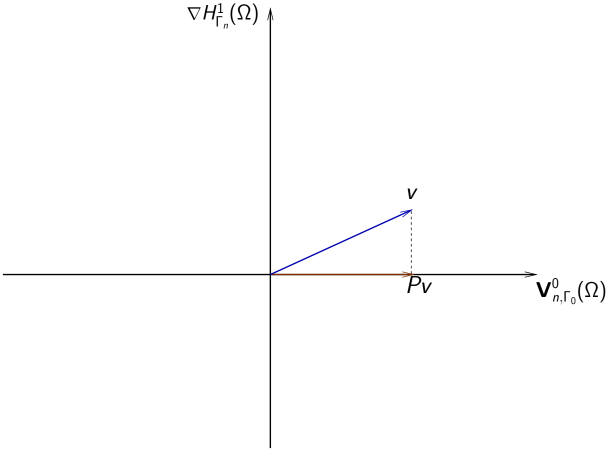
and

$$\widetilde{A}_{z\theta} = \begin{bmatrix} A_{v\theta} \\ 0 \\ A_{\eta_2\theta} \end{bmatrix}, \quad \widehat{A}_{z\theta} = \begin{bmatrix} A_{v\theta} \\ A_{\eta_1\theta} \\ A_{\eta_2\theta} \end{bmatrix}.$$

$$\lambda \begin{bmatrix} M_{vv} & 0 & 0 & 0 \\ 0 & M_{\eta\eta} & 0 & 0 \\ 0 & 0 & M_{\eta\eta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \zeta_1 \\ \zeta_2 \\ \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} A_{zz} & \widehat{A}_{z\theta} \\ \widetilde{A}_{z\theta}^T & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \zeta_1 \\ \zeta_2 \\ \boldsymbol{\rho} \end{bmatrix}.$$

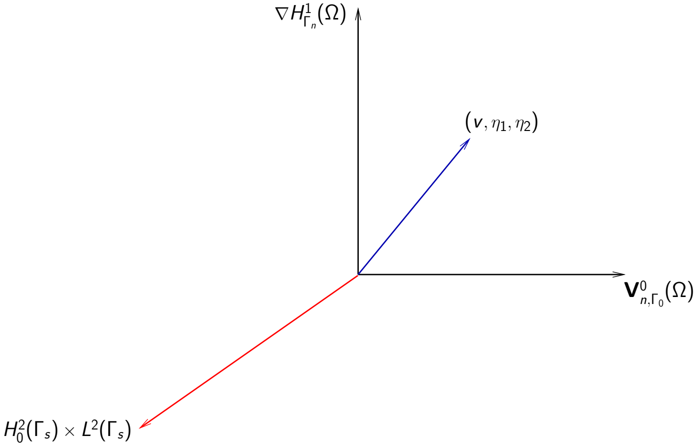
We need to rewrite these problems in the form of equivalent problems with the same algebraic constraints.

We use the Leray projector for the infinite dimensional problems and its discrete counterpart for the discrete problems.



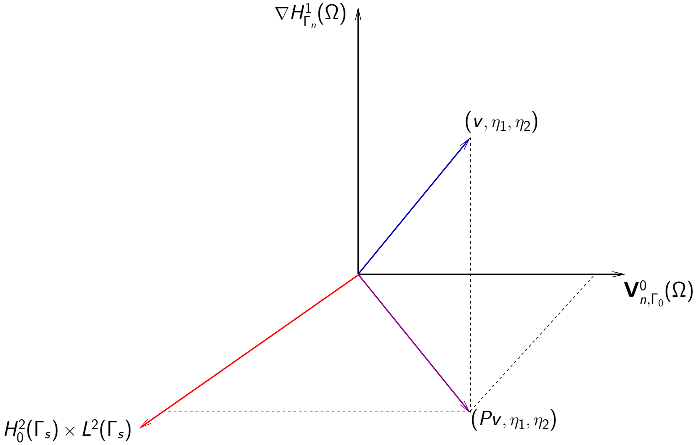
$$V^0_{n,\Gamma_d}(\Omega) = \Big\{ v \in L^2(\Omega;\mathbb{R}^2) \mid \text{ div } v = 0,\ \ v \cdot n = 0 \text{ on } \Gamma_d \Big\},$$

$$L^2(\Omega;\mathbb{R}^2) = V^0_{n,\Gamma_d}(\Omega) \oplus \nabla H^1_{\Gamma_n}(\Omega).$$



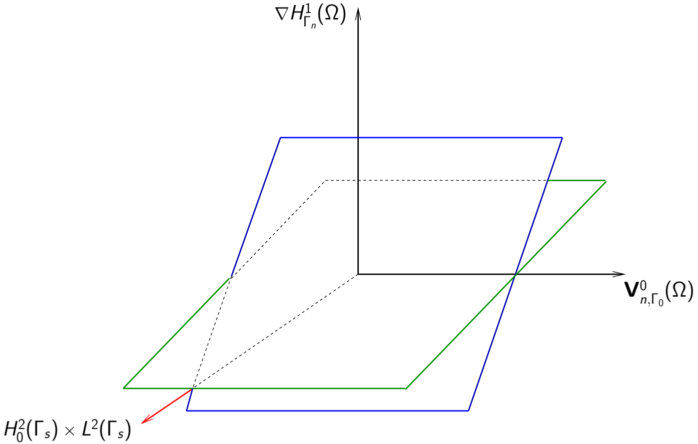
$$V^0_{n,\Gamma_d}(\Omega) = \left\{ v \in L^2(\Omega;\mathbb{R}^2) \mid \ \text{div } v = 0, \ v \cdot n = 0 \text{ on } \Gamma_d \right\},$$

$$L^2(\Omega;\mathbb{R}^2) = V^0_{n,\Gamma_d}(\Omega) \oplus \nabla H^1_{\Gamma_n}(\Omega).$$



$$V^0_{n,\Gamma_d}(\Omega) = \left\{ v \in L^2(\Omega; \mathbb{R}^2) \mid \text{ div } v = 0, \ v \cdot n = 0 \text{ on } \Gamma_d \right\},$$

$$L^2(\Omega; \mathbb{R}^2) = V^0_{n,\Gamma_d}(\Omega) \oplus \nabla H^1_{\Gamma_n}(\Omega).$$



$$V^0_{n,\Gamma_d}(\Omega) = \left\{ v \in L^2(\Omega; \mathbb{R}^2) \mid \text{ div } v = 0, \ v \cdot n = 0 \text{ on } \Gamma_d \right\},$$

$$L^2(\Omega; \mathbb{R}^2) = V^0_{n,\Gamma_d}(\Omega) \oplus \nabla H^1_{\Gamma_n}(\Omega).$$

The fluid equations

$$v_t - \operatorname{div}\sigma(v,q) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0,$$

$$\operatorname{div} v = A_3\eta_1 \quad \text{in } Q,$$

$$v = \eta_2\, e_2 \quad \text{on } \Sigma_s, \quad v = g_d \quad \text{on } \Sigma_i,$$

$$\sigma(v,q)n = 0 \quad \text{on } \Sigma_n, \quad v(0) = v^0 \text{ in } \Omega.$$

A the Oseen operator, P the Leray projector,

The algebraic differential system

$$Pv' = APv + CouplingTerms(\eta_1,\eta_2) + NonHomTerm(g_d),$$

$$Pv(0) = Pv^0, \quad (I-P)v = (I-P)L(\eta_2, A_3\eta_1).$$

L is a lifting operator.

The Stokes operator

$$D(A_0) = \Big\{ v \in V^1_{\Gamma_d}(\Omega) \mid$$

$$\exists p \in L^2(\Omega) \text{ s. t. } \mathrm{div}\sigma(v,p) \in L^2(\Omega;\mathbb{R}^2)$$

$$\text{and } \mathrm{div}\sigma(v,p)\, n = 0 \quad \text{on} \quad \Gamma_n \Big\},$$

$$A_0 v = P(\mathrm{div}\sigma(v,p)) \quad (\text{does not depend on } p).$$

The Oseen operator $(A, D(A))$ is defined by

$$D(A) = D(A_0) \quad \text{and} \quad Av = A_0 v + P\left((u_s \cdot \nabla)v + (v \cdot \nabla)u_s\right).$$

Due to the corners, we have

$$D(A_0) \subset H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2), \qquad \varepsilon_0 > 0.$$

$$
\begin{aligned}
&\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s,\\
&\eta_{2,t} - \beta\Delta_s\eta_1 + \alpha\Delta_s^2\eta_1 - \gamma\Delta_s\eta_2 - A_4\eta_1 = q + f \quad \text{on } \Sigma_s,\\
&\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0,\infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.
\end{aligned}
$$

The evolution equation for the beam

$$
\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix} 0 & I \\ A_4 - \alpha\Delta_s^2 & \gamma\Delta_s\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}0\\ q+f\end{pmatrix},
$$

$$
\begin{pmatrix}\eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix}\eta_1^0\\ \eta_2^0\end{pmatrix}.
$$

Expression of the pressure - 1

$$-\Delta q = A_3 \eta_{1,t} + \mathrm{div}((u_s \cdot \nabla) v + (v \cdot \nabla) u_s) - \mathrm{div}(A_1 \eta_1) - \mathrm{div}(A_2 \eta_2)$$

$$q = 2\nu\, \varepsilon(v) n \cdot n \qquad \text{on } \Gamma_n,$$

$$\frac{\partial q}{\partial n} = 2\nu\, \mathrm{div}\varepsilon(v) \cdot n - v_t \cdot n = 2\nu\, \mathrm{div}\varepsilon(v) \cdot n - \eta_{2,t} \quad \text{on } \Gamma \setminus \Gamma_n.$$

Thus

$$q = -N_s(\eta_{2,t}) + N_d(A_3\, \eta_{1,t}) + N_v(v) + N(A_1 \eta_1) + N(A_2 \eta_2).$$

$$-\Delta q = \mathrm{div}((u_s \cdot \nabla)v + (v \cdot \nabla)u_s),$$

$$q = 2\nu\, \boldsymbol{\varepsilon}(v)n \cdot n \quad \text{on } \Gamma_n, \quad \frac{\partial q}{\partial n} = 2\nu\, \mathrm{div}\boldsymbol{\varepsilon}(v) \cdot n \quad \text{on } \Gamma \setminus \Gamma_n.$$

Find $q \in L^2(\Omega)$ such that

$$\int_\Omega q\zeta = 2\nu \langle \varepsilon(v), \nabla^2\chi \rangle_{\mathbf{L}^2_{-\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} - 2\nu \int_{\Gamma_0} \varepsilon(v)n \cdot \nabla\chi$$
$$-2\nu \int_{\Gamma_0} \varepsilon(v)n \cdot \nabla\chi + \int_\Omega \Big[(u_s \cdot \nabla)v + (v \cdot \nabla)u_s\Big] \cdot \nabla\chi,$$

for all $\zeta \in L^2(\Omega)$,

where $\Delta\chi = \zeta$ in Ω, $\quad \frac{\partial \chi}{\partial n} = 0$ on $\Gamma \setminus \Gamma_n$, $\quad \chi = 0$ on Γ_n.

A posteriori $q \in H^{1/2+\varepsilon_0}(\Omega_0)$ for some $\varepsilon_0 > 0$.

Coming back to the beam equation

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{pmatrix}0 & I\\ A_4-\alpha\Delta_s^2 & \gamma\Delta_s\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

$$+\begin{pmatrix}0\\ f - N_s(\eta_{2,t}) + N_d(A_3\,\eta_{1,t}) + N_v(v) + N(A_1\eta_1) + N(A_2\eta_2)\end{pmatrix},$$

The added mass operator

$$\frac{d}{dt}\begin{pmatrix}\eta_1\\ N_s(\eta_2) - N_d(A_3\,\eta_1) + \eta_2\end{pmatrix}$$

$$= \begin{pmatrix}0 & I\\ A_4 + NA_1 - \alpha\Delta_s^2 & NA_2\gamma\Delta_s\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} + \begin{pmatrix}0\\ f + N_v(v)\end{pmatrix}.$$

Differential algebraic formulation versus PDE formulation

$$M_a z' = \widehat{\mathcal{A}} z + \mathcal{B} f, \quad z(0) = z_0,$$

$$(I - P)v(t) = (I - P)L(\eta_2(t)\, e_2, A_3 \eta_1(t)),$$

$$z = (Pv, \eta_1, \eta_2)^T, \quad \mathcal{B} = (0\ 0\ \chi_{\Gamma_c})^T,$$

L is the lifting operator of the divergence and Dirichlet boundary condition.

For numerical purpose –> PDE system

For the control strategy and stability analysis –> Differential algebraic evolution equation

Equivalent formulation of the fluid-structure system

$$M_a z' = \widehat{\mathcal{A}} z + \mathcal{B} h, \quad z(0) = z_0,$$

$$(I - P)v(t) = (I - P)L(\eta_2(t)\, e_2, A_3 \eta_1(t)),$$

$$z = (Pv, \eta_1, \eta_2)^T, \quad \mathcal{B} = (0\ 0\ \chi_{\Gamma_c})^T,$$

L is the lifting operator of the divergence and Dirichlet boundary condition, and

$$\widehat{\mathcal{A}} = \begin{pmatrix} A & (PA_1 - APL(0, A_3)) & (PA_2 - APL(\cdot, 0)) \\ 0 & 0 & I \\ \gamma_s N_s & -\alpha \Delta_s^2 + \cdots & \gamma \Delta + \cdots \end{pmatrix},$$

and

$$M_a = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma_s N_d A_3 & I + \gamma_s N_s \end{pmatrix}.$$

The added mass operator, M_a, is no longer symmetric.

We set

$$\mathcal{A} = M_a^{-1}\,\widehat{\mathcal{A}},$$

and, due to the corners

$$D(\mathcal{A}) = \Big\{(Pv,\eta_1,\eta_2) \in H^{1/2+\varepsilon_0}(\Omega;\mathbb{R}^2) \times (H^4 \cap H_0^2)(\Gamma_s) \times H_0^2(\Gamma_s) \;\Big|\; Pv - PL(\eta_2\,\vec{e}_2, A_3\eta_1) \in D(A_0)\Big\}.$$

Theorem. The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $Z = V^0_{n,\Gamma_d}(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s)$.

Analyticity of the Oseen operator + Analyticity for the structure (Chen-Triggiani) + perturbation arguments

Theorem. The resolvent of $(\mathcal{A}, D(\mathcal{A}))$ is compact in Z.

Consequence. To stabilize the linearized model - with a prescribed exponential decay rate - it is necessary and sufficient to control a finite dimension space.

Project the Differential algebraic evolution equation onto a finite dimensional subspace to determine a feedback control law: Jordan decomposition of an algebraic differential system – Link between PDE and DAE.

Come back to the control strategy

- $Z = V^0_{n,\Gamma_d}(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s)$
- Choose

$$Z_u = \oplus_{j \in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{with} \quad Z = Z_u \oplus Z_s$$

and

$$Z_u^* = \oplus_{j \in J_u} G^*_{\mathbb{R}}(\lambda_j) \quad \text{with} \quad Z^* = Z = Z_u^* \oplus Z_s^*.$$

- Determine a basis $\{e_1, \cdots, e_{d_u}\}$ of Z_u and a basis $\{\Phi_1, \cdots, \Phi_{d_u}\}$ of Z_u^* satisfying

$$(e_i, \Phi_j)_Z = \delta_{i,j}.$$

- These bases are used to determine the projected linearized system and therefore the feedback operator.

(Pv, η_1, η_2) is an eigenvector (or a generalized eigenvector) for $\mathcal{A}$ associated with λ and $(I - P)v = (I - P)L(\eta_2, A_3\eta_1)$ iff

(v, η_1, η_2) is an eigenvector (or a generalized eigenvector) for the direct PDE system, associated with λ.

$M_a^*(P\phi, \zeta_1, \zeta_2)$ is an eigenvector (or a generalized eigenvector) for $\mathcal{A}^*$ associated with λ and $(I - P)\phi = (I - P)L(\zeta_2, 0)$ iff

(ϕ, ζ_1, ζ_2) is an eigenvector (or a generalized eigenvector) for the adjoint PDE system, associated with λ.

The bi-orthogonality condition for eigenfunctions of the PDE systems is equivalent to the bi-orthogonality condition for eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$.

$$((v_i, \eta_{1,i}, \eta_{2,i}), (\phi_j, \zeta_{1,j}, \zeta_{2,j}))_{L^2} = \delta_{i,j}$$

is equivalent to

$$((Pv_i, \eta_{1,i}, \eta_{2,i}), M_a^*(P\phi_j, \zeta_{1,j}, \zeta_{2,j}))_Z = \delta_{i,j}$$

Stabilizability of the linearized F–S system

- The boundary control is chosen of the form

$$f(x,t) = \sum_{i=1}^{N_c} f_i(t)\,\xi_i(x).$$

- The functions $(\xi_i)_{1\le i\le N_c}$ can be chosen to prove that the linearized system around an unstable stationary solution is stabilizable (under investigation – verified by numerical calculations).

We have to check the following unique continuation property.

If $(\lambda, \phi, \psi, \zeta_1, \zeta_2)$ is solution to the following eigenvalue problem

$$\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \quad \text{and} \quad \operatorname{div}\phi = 0 \quad \text{in } \Omega,$$
$$\phi = \zeta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \phi = 0 \ \text{ on } \ \Gamma_i, \quad \sigma(\phi,\psi)n + u_s\cdot n\,\phi = 0 \ \text{ on } \ \Gamma_n,$$
$$\lambda\zeta_1 + \zeta_2 + A_1^*\phi + A_3^*\psi + B_1^*\zeta_2 = 0 \quad \text{in } \Gamma_s,$$
$$\lambda\zeta_2 + \beta\Delta_s\zeta_1 - \delta\Delta_s\zeta_2 - \alpha\Delta_s^2\zeta_1 - A_2^*\phi = \psi \ \text{ in } \Gamma_s,$$
$$\zeta_1 = 0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

with $\operatorname{Re}\lambda \geq -\omega$ and

$$B^*(P\phi, \zeta_1, \zeta_2) = \zeta_2\,\chi_{\Gamma_c} = 0,$$

then

$$\phi = 0, \quad \psi = 0, \quad \zeta_1 = \zeta_2 = 0.$$

Regularity results in the presence of mixed B.C.

If $v_0 \in H^1_{\Gamma_i}(\Omega;\mathbb{R}^2)$, $\mathrm{div} v_0 = 0$, $\eta_1^0 \in H^3(\Gamma_s) \cap H^2_0(\Gamma_s)$, $\eta_2^0 \in H^1_0(\Gamma_s)$, $v_0 = \eta_2^0 \vec{e}_2$ on Γ_s, the solution to the linearized system belongs to

$$
\begin{aligned}
& v \in L^2(0,T;H^2_\delta(\Omega;\mathbb{R}^2)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^2)) = H^{2,1}_\delta(Q_T;\mathbb{R}^2), \\
& p \in L^2(0,T;H^1_\delta(\Omega;\mathbb{R}^2)), \\
& \eta_1 \in L^2(0,T;H^4(\Gamma_s)) \cap H^2(0,T;L^2(\Gamma_s)) = H^{4,2}(\Sigma^T_s), \\
& \eta_2 \in L^2(0,T;H^2(\Gamma_s)) \cap H^1(0,T;L^2(\Gamma_s)) = H^{2,1}(\Sigma^T_s).
\end{aligned}
$$

$$
\begin{aligned}
\|v\|^2_{H^2_\delta(\Omega;\mathbb{R}^2)} &= \textstyle\sum_{|k|=0}^2 \sum_{i=1}^2 \int_\Omega r^{2\delta} |\partial_k v|^2 dx, \\
\|p\|^2_{H^1_\delta(\Omega)} &= \textstyle\sum_{|k|=0}^1 \sum_{i=1}^2 \int_\Omega r^{2\delta} |\partial_k p|^2 dx,
\end{aligned}
$$

where r stands for the distance to the corners.

Due to the right angles at the Neumann-Dirichlet junctions and Dirichlet-Dirichlet junctions, we have

$$
\begin{aligned}
&H^2_\delta(\Omega;\mathbb{R}^2) \subset H^{3/2+\varepsilon_0}(\Omega;\mathbb{R}^2),\\
&H^1_\delta(\Omega) \subset H^{1/2+\varepsilon_0}(\Omega;\mathbb{R}^2), \quad \text{for some } \varepsilon_0 > 0.
\end{aligned}
$$

If $u_0 \in H^1_{\Gamma_i}(\Omega;\mathbb{R}^2)$, $\eta^0_1 = 0$, $\eta^0_2 \in H^1_0(\Gamma_s)$, $u_0 = \eta^0_2\,\vec{e}_2$ on Γ_s,

and if $(u_0 - u_s, \eta^0_2)$ is small enough in $H^1_{\Gamma_i}(\Omega;\mathbb{R}^2) \times H^1_0(\Gamma_s)$,

if g_d is small enough in an appropriate space,

then the closed loop nonlinear system admits a solution decaying

exponentially to the stationary solution in

$H^{2,1}_\delta(Q;\mathbb{R}^2) \times H^{4,2}(\Sigma_s) \times H^{2,1}(\Sigma_s)$.

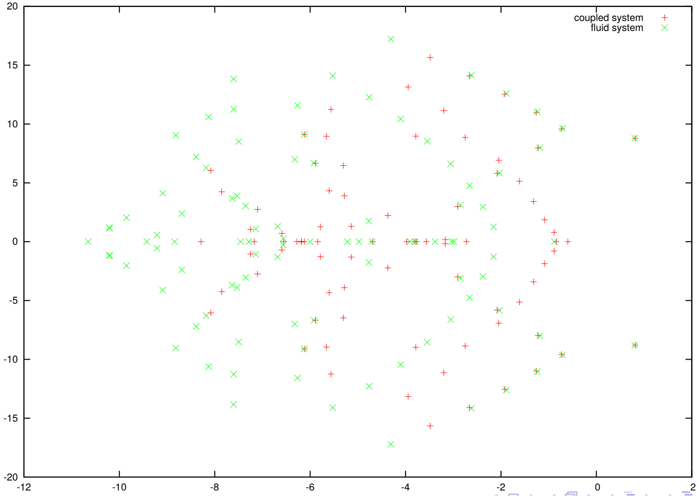
$P_2 \times P_1 \times P_1$ or $P_2 \times P_1 \times P_2$ for the velocity, the pressure, the Lagrange multipliers, and H_3 for the beam displacement.

The Inf-Sup condition for this system reduces to an Inf-Sup condition for the fluid, that is satisfied.

We have performed numerical tests with triangular meshes of 89418 cells, 75846 cells and 283956 cells to study the convergence of the numerical spectrum of the linearized model.

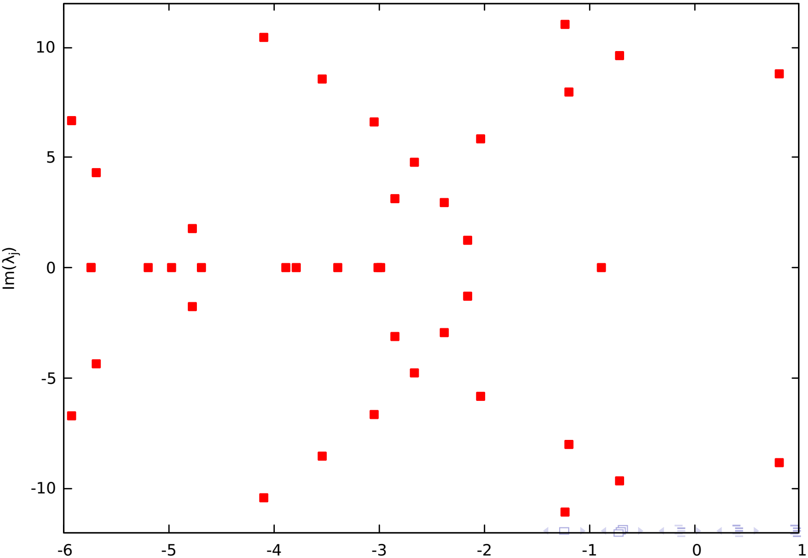
89418 cells corresponds to 406339 dof.

The spectrum of the controlled system – $\mathrm{Re} = \frac{U_m\, e}{\nu} = 200.$



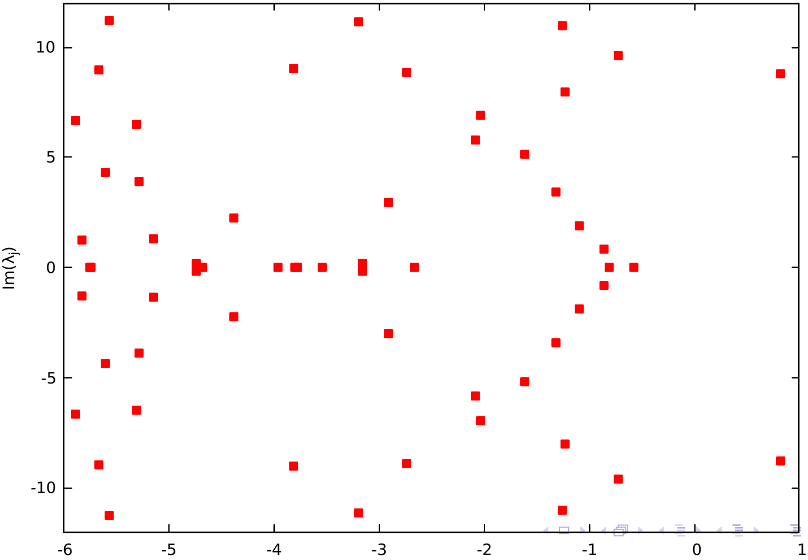
Numerical results

The spectrum of the controlled system – $\mathrm{Re} = \frac{U_m e}{\nu} = 200.$



Numerical results

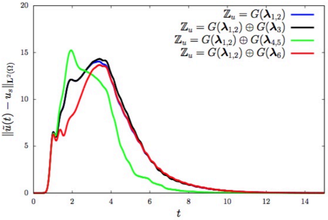
The spectrum of the controlled system – $\mathrm{Re} = \frac{U_m\, e}{\nu} = 200.$



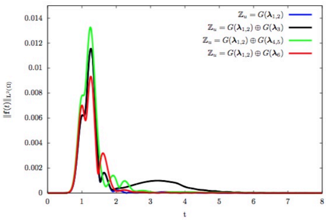
Method.

- We project the linearized system onto the sum of the generalized eigenspace generated by the 2 unstable eigenvalues (4 unstable eigenvalues). It is of dimension 2 and the control is of dimension 4 (2 controls on each beam). (When the projected system is of dimension 4, the control space is of dimension 2.)
- We determine a feedback stabilizing the projected system.
- The feedback law is next applied to the full nonlinear system.

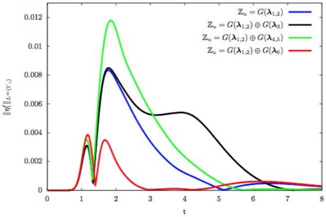
Different choices of Z_u – Amplitude 0.1



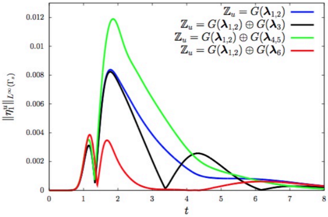
(a) Evolutions of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolutions of the L^2-norm of the controls.

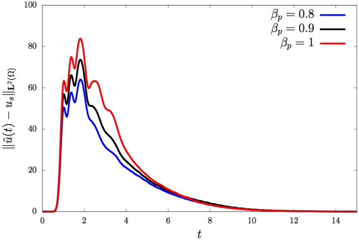


(c) Evolution of the L^∞-norm of η_1 (lower part).

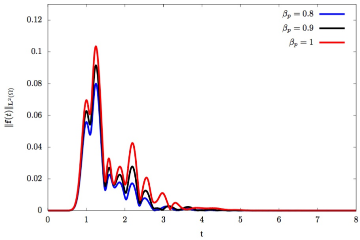


(d) Evolution of the L^∞-norm of η_1 (upper part).

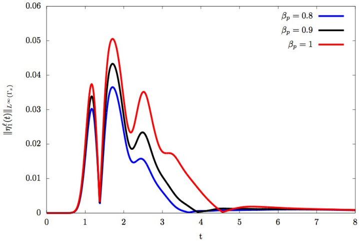
$Z_u = G(\lambda_{1,2}) \oplus G(\lambda_6)$ – Amplitude up to 1.



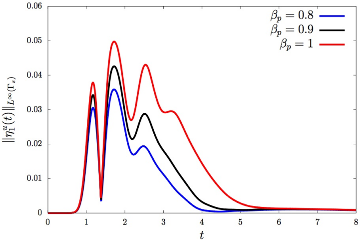
(a) Evolutions of the L^2-norm of $\tilde{u} - u_s$.



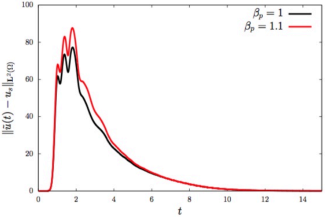
(b) Evolutions of the L^2-norm of the controls.



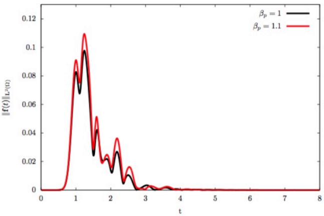
(c) Evolution of the L^∞-norm of η_1 (lower part).



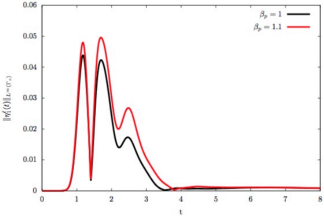
(d) Evolution of the L^∞-norm of η_1 (upper part).



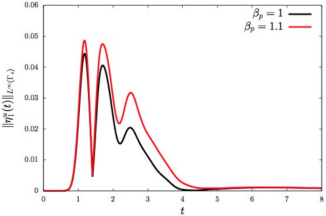
(a) Evolutions of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolutions of the L^2-norm of the controls.



(c) Evolution of the L^∞-norm of η_1 (lower part).



(d) Evolution of the L^∞-norm of η_1 (upper part).

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Thank you for your attention