Trefftz-DG solution to the Helmholtz equation involving integral equations

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Conference in honor of Abderrahmane BENDALI

The time harmonic scalar wave equation

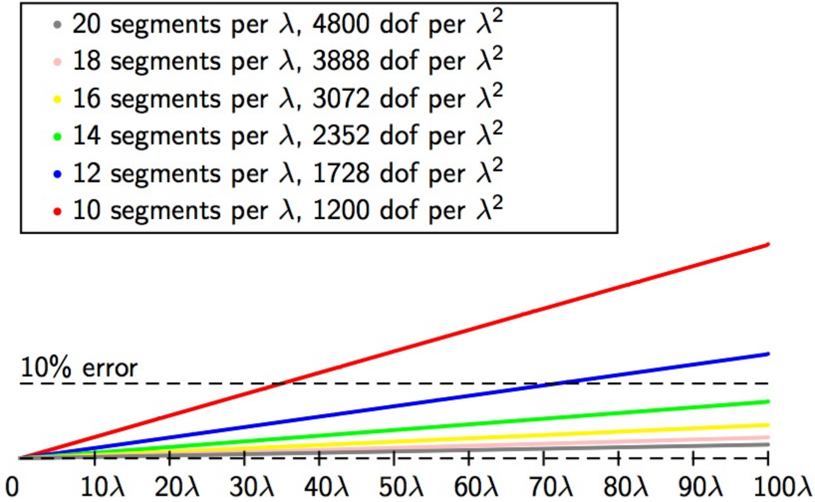
$$
\begin{cases}
\operatorname{div}\Big(\lambda(\mathbf{x})\nabla u(\mathbf{x})\Big) \ + \ \mu(\mathbf{x})u(\mathbf{x}) \ = \ 0 & \text{in } \Omega \\
u(\mathbf{x}) \ = \ g_D(\mathbf{x}) & \text{on } \partial\Omega_D \\
\lambda(\mathbf{x})\dfrac{\partial u}{\partial n}(\mathbf{x}) \ = \ g_N(\mathbf{x}) & \text{on } \partial\Omega_N \\
\lambda(\mathbf{x})\dfrac{\partial u}{\partial n}(\mathbf{x}) + Z(\mathbf{x})u(\mathbf{x}) \ = \ g_F(\mathbf{x}) & \text{on } \partial\Omega_F.
\end{cases}
$$

- Heterogeneities are handled inside the divergence operator
- The physical parameter functions λ and μ are piecewise constant.
- The domain Ω is 2D or 3D with boundary $\partial\Omega$.

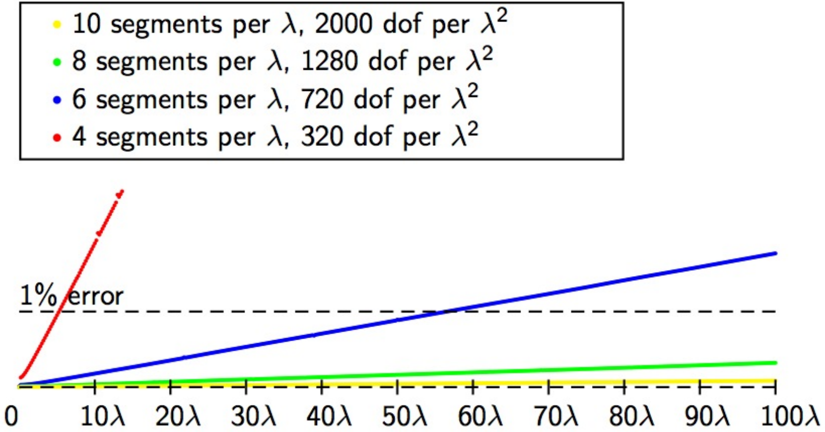
Motivation: Computations on very large domains

- Ω is very large vs the wavelength
- Need to augment the density of nodes to maintain a given level of accuracy

Error at every wavelength in $\mathbb{P}_2$ for 10, 12, 14, 16, 18 segments per λ



Error at every wavelength in $\mathbb{P}_3$ for 4, 6, 8, 10 segments per λ

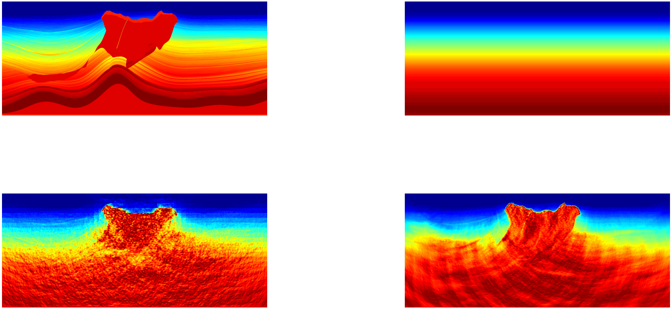


Motivation: Computations on very large domains

- Ω is very large vs the wavelength
- Need to augment the density of nodes to maintain a given level of accuracy
- Exceed the storage capacities.
- Discontinuous Galerkin methods do resist better to pollution effect

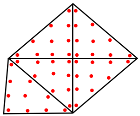
FWI using DG or FD

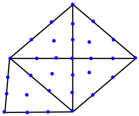
Seam acoustic model of size 35×15 km. Exact same FWI algorithm (n iterations, frequencies, ...), no initial information.



General setting

- Studies show that DG weak inter-element continuity contributes to fight the pollution effect
- But DG approximations imply to increase the number of nodes significantly





- Lead to Trefftz methods
- In particular, Ultra-Weak-Variational-Formulations proposed by B. Desprès.
- Trefftz method: shape functions are solutions to the problem
- Set on a single element K: Trefftz formulation reduces to the boundary of the element

General setting

- Boundary Integral Equations (BIE) lead to less pollution effect than FEMs
- Recently, Hofreither et al. (2015) have proposed a FEM in which local shape functions are obtained on the basis of a BIE.
- In the same spirit, we propose a DG method using local shape solutions to the Helmholtz problem that are matched at the interface of the mesh thanks to the Dirichlet-to-Neumann (DtN) operator which is computed with a BIE.

Outline

- The IPDG method set in a Trefftz space
- The DtN approximation
- Numerical experiments
- Conclusions and perspectives

The Classical IPDG method

The Interior Penalty Discontinuous Galerkin method

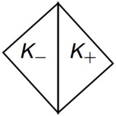
- Arnold in 1982
- It has been intensively studied during the last decade
- The main Advantages in frequency domain
  - High oscillations of the coefficients can be considered
  - Every element is connected only to its neighbors (important for direct methods like LU)
  - Less dispersive than Continuous Finite Element

The IPDG method

The classical variational formulation involves the bilinear form a:

$$
\left\{
\begin{array}{rcl}
\mathrm{a}(u,v) & = & \displaystyle\sum_K \int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) - \mu_K u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} \\
& - & \displaystyle\sum_T \int_T \{p\}[v] ds_{\mathbf{x}} \\
& - & \displaystyle\sum_T \int_T \{q\}[u] ds_{\mathbf{x}} \quad \text{with } q = \lambda \nabla v \cdot \mathbf{n} \\
& + & \displaystyle\sum_T \alpha_T \int_T [u][v] ds_{\mathbf{x}}
\end{array}
\right. \tag{1}
$$

$$[v] = v_+ - v_- \text{ and } v = \frac{v_+ + v_-}{2}$$



Rewrite the IPDG formulation in a different context

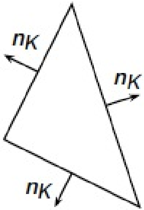
- The shape functions are quasi solutions to the Helmholtz equation, constructed thanks to a Boundary element method
- This is a Trefftz-like method
- Classically, plane wave bases or Bessel functions inside each element.
- Here use of BEM to compute local solutions.

We call it **BEM-STDG method**

Trial and test functions

Trial and test functions are solutions to the Helmholtz equation in each element

$$\mathrm{div}\Big(\lambda_K \nabla u_K(\mathbf{x})\Big) \;+\; \mu_K u_K(\mathbf{x}) \;=\; 0 \text{ in } K$$



u_K is uniquely defined by its Dirichlet trace if K is small enough (geometrical criterion)

$$u_K \in H^{1/2}(\partial K)$$

The discrete variational space is then obtained by considering a discrete trace space

$$u_K \text{is } \mathbb{P}_r\text{-continuous on } \partial K.$$

The symmetric variational formulation

Having test functions solutions to the Helmholtz equation, we get

$$\left\{\begin{array}{rcl} a(u,p;v,q) & = & \displaystyle\int_\Gamma \{u\}\{q\} + \{p\}\{v\} - \{p\}[v] - [u]\{p\} ds_{\mathbf{x}} \\ & - & \displaystyle\int_{\partial\Omega_D} pv + uq ds_{\mathbf{x}} \\ & + & \displaystyle\int_{\partial\Omega_N} pv + uq ds_{\mathbf{x}} \\ & + & \displaystyle\int_{\partial\Omega_F} pv + uq + 2Zuv ds_{\mathbf{x}} \end{array}\right.$$

$$\ell_1(v,q) = -2\int_{\partial\Omega_D} g_D q ds_{\mathbf{x}} + 2\int_{\partial\Omega_N} g_N v ds_{\mathbf{x}} + 2\int_{\partial\Omega_F} g_F v ds_{\mathbf{x}}$$

The symmetric variational formulation

Adding the penalization terms ($[u] = 0$ on Γ and $u = g$ on $\partial\Omega$):

$$\underbrace{\int_\Gamma \alpha[u][v] + \int_{\partial\Omega_D} \alpha u v}_{b(u,p;v,q)} = \underbrace{\int_{\partial\Omega_D} \alpha g_D v}_{\ell_2(v)}$$

This leads to the Trefftz-DG formulation

$$a(u, p; v, q) + b(u, p; v, q) = \ell_1(q) + \ell_2(v).$$

Why the symmetry is important ?

- for the linear algebra solver: it needs less memory
- it has been observed that BIE methods are more stable.

Now, the unknowns are u and $p = \partial_\nu u$ on each face of the mesh. One may be removed.

The DtN operator

Let

$$u_K(\mathbf{x}) \text{ be given on } \partial K.$$

The Neumann trace

$$p_K \;=\; \lambda_K \, \nabla u_K \cdot \mathbf{n}_K \text{ on } \partial K$$

may then be deduced thanks to the Dirichlet-to-Neumann operator

$$DtN : \begin{cases} H^{\frac{1}{2}}(\partial K) \longrightarrow H^{-\frac{1}{2}}(\partial K) \\ u_K \longmapsto p_K \end{cases}$$

and we end up with a system involved unknowns defined on the boundary of each element.

The problem to be addressed: compute the DtN operator

The DtN operator approximation: include an auxiliary numerical method

We can think about different methods like:

- finite element/finite difference method based on the velocity/pressure formulation
- Boundary element method

Why BEM? They do resist very well to pollution effect

The secondary numerical method: BEM

$$\frac{V_K p_K}{\lambda_K} = \frac{M_K u_K}{2} - N_K u_K$$

- u_K is approximated by a $\mathbb{P}_r$-continuous function
- p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function
- V_K and N_K are the single layer and double layer operators.

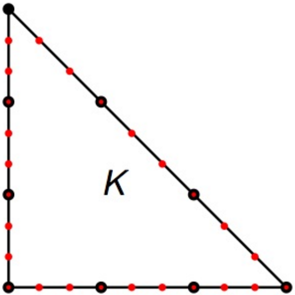
$$\left\{ \begin{array}{lcl} (M_K u_K, q_k)_{\partial K} & = & \displaystyle\int_{\partial K} u_K(\mathbf{x})\; q_K(\mathbf{x}) ds_{\mathbf{x}}, \\ (V_K p_K, q_k)_{\partial K} & = & \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\; G_K(\mathbf{x}-\mathbf{y})\; q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}}, \\ (N_K u_K, q_k)_{\partial K} & = & \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\; \frac{\partial G_K}{\partial \mathbf{n_y}}(\mathbf{x}-\mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}} \end{array} \right.$$

with

$$G(\mathbf{x}) = \frac{\exp(ik_K\|\mathbf{x}\|)}{4\pi\|\mathbf{x}\|} \text{ with } k_K = \sqrt{\frac{\mu_K}{\lambda_K}}$$

The boundary element method

- u_K is approximated by a $\mathbb{P}_r$-continuous function
- p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function



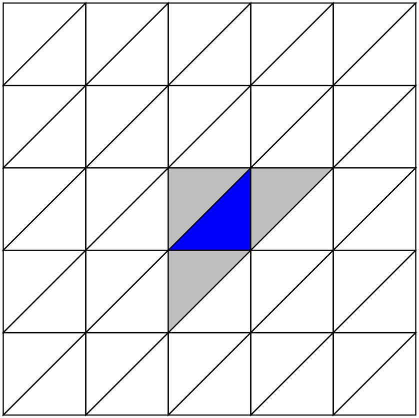
- geometric nodes for p_K
- geometric nodes for u_K

We can use different meshes for u_K and p_K.

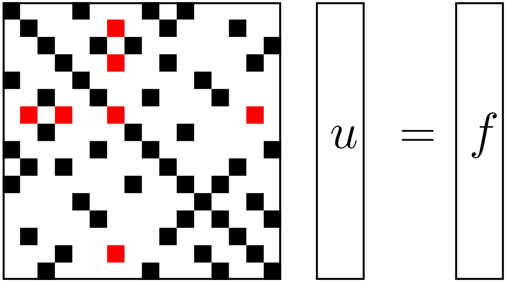
Idea: the Neumann trace must be computed accurately

Remark: p_K is discontinuous, only at the geometric singularities.

Connection of the elements

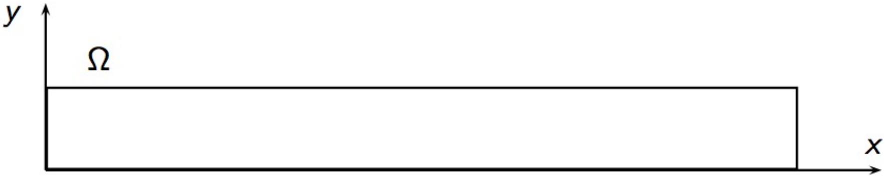


The final formulation



- Symmetric block sparse matrix
- full small blocks

A numerical simulation



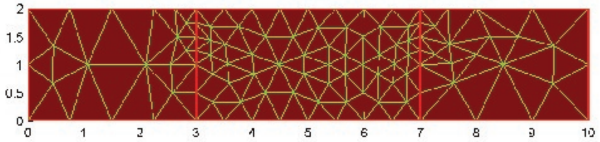
$$\begin{cases} \Delta u(\mathbf{x}) \; + \; k^2 \, u(\mathbf{x}) \; = \; 0 & \text{in } \Omega \\ u(\mathbf{x}) = 1 & \text{at } x = 0, \\ \dfrac{\partial u}{\partial n}(\mathbf{x}) = 0 & \text{at } \mathbf{x} = N\lambda \\ \frac{\partial u}{\partial n}(\mathbf{x}) + iku(\mathbf{x}) = 0 & \text{at } y = 0 \text{ and } N\lambda \end{cases} \qquad (2)$$

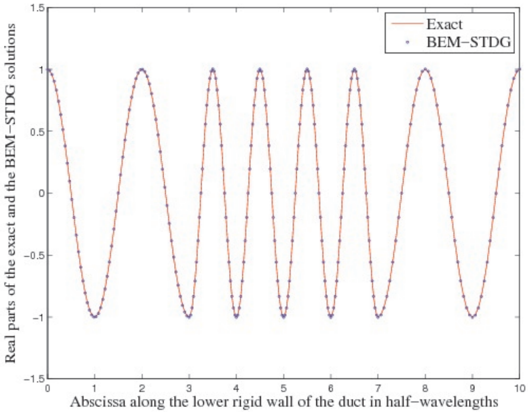
Poly degree	nodes per λ	Method	Error at 175λ for $\mathbb{P}_2$ Error at 500λ for $\mathbb{P}_3$
$m = 2$	12	IPDG	72 %
		BEM-STDG	22 %
	16	IPDG	67 %
		BEM-STDG	5.6 %
	24	IPDG	13 %
		BEM-STDG	0.8 %
$m = 3$	12	IPDG	19 %
		BEM-STDG	1.6 %
	18	IPDG	1.7 %
		BEM-STDG	0.1 %
	24	IPDG	0.3 %
		BEM-STDG	0.02 %

Degree	Density (nodes/λ)	Method	Condition number 50 λ	Condition number 500 λ	CPU time 500 λ
$m = 2$	12	IPDG	$8.8\,10^{10}$	$4.7\,10^{11}$	2.54
		BEM-STDG	$6.06\,10^{7}$	$1.00\,10^{8}$	4.76
	24	IPDG	$1.2\,10^{12}$	$2.76\,10^{12}$	19.03
		BEM-STDG	$5.96\,10^{8}$	$9.42\,10^{9}$	8.5
$m = 3$	12	IPDG	$4.2\,10^{11}$	$8.9\,10^{11}$	2.13
		BEM-STDG	$2.1\,10^{8}$	$1.5\,10^{9}$	4.75
	24	IPDG	$2.0\,10^{12}$	$6.2\,10^{13}$	20.81
		BEM-STDG	$9.52\,10^{8}$	$8.07\,10^{10}$	8.4
$m = 4$	8	IPDG	$1.43\,10^{11}$	$3.78\,10^{11}$	0.66
		BEM-STDG	$1.13\,10^{8}$	$1.08\,10^{8}$	3.89
	24	IPDG	$2.38\,10^{12}$	$2.41\,10^{14}$	17.91
		BEM-STDG	$1.7\,10^{9}$	$1.7\,10^{11}$	8.41

Condition number and CPU time for h p refinements

Case of an unstructured mesh





Conclusion and perspectives

- Trefftz IPDG formulation combined with BEM reduces the pollution effect
- Numerical experiments are in progress to test the limit of the method on very long ducts including contrasts that justify unstructured meshes
- The method has also been implemented in 3D, numerical validation is in progress
- On-going works: accurate Neumann (or Dirichlet) traces obtained by using a FEM locally.
- The future: extension to elastic waves. Not that obvious when considering BEM or FEM...

BEM and elasticity

- **Discontinuity of Neumann data to be statisfied**
- For Acoustics, with FEM, we can apply a mixed finite element approach which does not work for elastic waves except with structured meshes. It turns out that spectral methods developed by Trefethen are very efficient but the implementation is tricky.
- For Elastic waves, it is still an open question because of the symmetry of the strain tensor. Mixed finite elements can be used but on structured meshes only.
- BEM are not that easy to implement for higher orders of approximation. Moreover, the singularities of the kernels are difficult to handle.
- That is why we are considering DG approximation to compute the Neumann data.
- In the case of a source, a preprocessing is performed at the level of the element by using DG with homogeneous Dirichlet condition to come back to the case where the source is zero.

The IPDG method

On each element K, we write a variational equation

$$\mathrm{a}_K(u,v) \;=\; 0$$

with the bilinear form

$$\left\{ \begin{array}{rcl} \mathrm{a}_K(u,v) & = & \displaystyle\int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} \;-\; \int_{\partial K} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}}, \\ & - & \displaystyle\int_K \mu_K u(\mathbf{x}) \; v(\mathbf{x}) \; d\mathbf{x} \end{array} \right.$$

and

$$p_K(\mathbf{x}) = \lambda_K \nabla u(\mathbf{x}) \cdot \mathbf{n}_K$$

The IPDG method

Summing over all the elements K of the mesh, we have

$$\mathrm{a}(u,v) \;=\; 0 \tag{3}$$

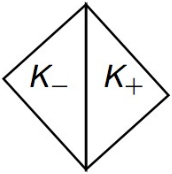
with the bilinear form

$$\left\{\begin{array}{rcl} \mathrm{a}(u,v) & = & \displaystyle\sum_K \int_K \lambda_K \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} \\ & - & \displaystyle\sum_K \int_{\partial K_{\mathrm{int}}} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}}, \\ & - & \displaystyle\sum_K \int_K \mu_K u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}. \end{array}\right. \tag{4}$$

The IPDG (Neumann boundary condition)

Interior elements share a face T which implies

$$\sum_K \int_{\partial K_{\mathrm{int}}} p_K(\mathbf{x}) v(\mathbf{x}) ds_{\mathbf{x}} = \sum_T \int_T [p(\mathbf{x}) v(\mathbf{x})] ds_{\mathbf{x}} \qquad (5)$$



with

$$[p(\mathbf{x}) v(\mathbf{x})] = p_+ v_+(\mathbf{x}) + p_- v_-(\mathbf{x}) \qquad (6)$$

now that the test function v is discontinuous across the interface.

The IPDG method

Introduce now the mean value p. We have that

$$[p(\mathbf{x})v(\mathbf{x})] = \{p\}\,[v] \tag{7}$$

with

$$\left\{\begin{array}{rcl} [v] & = & v_+ + v_-; \\ \{p\} & = & \dfrac{p_+(\mathbf{x}) - p_-(\mathbf{x})}{2} \end{array}\right. \tag{8}$$

since p is continuous across the elements while v is discontinuous.