Waves in viscoelastic media

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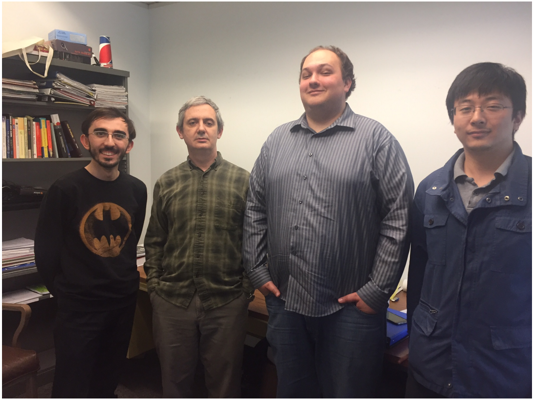
The authors



Left to right, Hasan, me, Tom, Shukai

Team Pancho Summer 2017

Allan, me, Connor, Tom
Hugo, Shukai, Hasan

Contents, disclaimers, warnings

- Work in permanent progress
- Laplace domain analysis
- General Galerkin (FEM) discretization

In a nutshell...

Analysis of FE semidiscretization in space of a class of elastic wave models where the strain-to-stress relation includes memory terms (even using fractional derivatives in time, because fractional derivatives are all the rage)

The mandatory Wiki Photo to claim applications

The model equations and the Laplace domain framework

Model equations

Upper dots are time derivatives
$\mathbf{u}$ is displacement
$\boldsymbol{\sigma}$ is stress

$$
\begin{aligned}
\rho\, \ddot{\mathbf{u}} &= \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} && \text{(Conservation of momentum)} \\
\boldsymbol{\sigma} &= \mathcal{D} * \dot{\varepsilon} && \text{(convolutional material law)} \\
\varepsilon &= \tfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) && \text{(linear strain)}
\end{aligned}
$$

Homogeneous mixed boundary conditions (for simplicity) and homogeneous initial conditions

A simple example

Zener's classical viscoelastic model

$$\boldsymbol{\sigma} + a\dot{\boldsymbol{\sigma}} = \mathrm{C}_0\boldsymbol{\varepsilon} + \mathrm{C}_1\dot{\boldsymbol{\varepsilon}}$$

- C_0, C_1 are 4-order Hookean tensors
- $a \in L^\infty(\Omega)$ is strictly positive
- $\mathrm{C}_{\mathrm{diff}} := \mathrm{C}_1 - a\,\mathrm{C}_0 \geq 0$

In plain symbols

$$\mathrm{C}^{0,1}_{ijkl} = \mathrm{C}^{0,1}_{klij} = \mathrm{C}^{0,1}_{jikl} \in L^\infty(\Omega)$$

$$\mathrm{C}^{0,1}_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq c^{0,1}\epsilon_{ij}\epsilon_{ij} \qquad \forall \epsilon_{ij} = \epsilon_{ji}$$

$$\mathrm{C}^{1}_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq a\,\mathrm{C}^{0}_{ijkl}\epsilon_{ij}\epsilon_{kl} \qquad \forall \epsilon_{ij}$$

Coupled elastic and viscoelastic materials are covered with this

Same thing, bigger words

The tensors $\mathrm{C}^0, \mathrm{C}^1$ define bounded linear operators

$$L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \longrightarrow L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}),$$

which:

- are selfadjoint
- and strictly positive definite
- and commute with multiplication by scalar functions
- and satisfy $\mathrm{C}_{\mathrm{diff}} := \mathrm{C}_1 - a\mathrm{C}_0 \geq 0$

If $\Omega_{\mathrm{el}} \subset \Omega$ is such that

$$\mathrm{C}_{\mathrm{diff}}\Big|_{L^2(\Omega_{\mathrm{el}};\mathbb{R}^{d\times d}_{\mathrm{sym}})} = 0$$

the subdomain Ω_{el} is purely elastic.

Laplace domain hypotheses

$$\boldsymbol{\sigma} = \mathcal{D} * \dot{\varepsilon} \qquad \mathrm{C}(s) := s\,\mathcal{L}\{\mathcal{D}\}(s)$$

- Well defined in $\mathbb{C}_+ := \{s \,:\, \mathrm{Re}\, s > 0\}$
- $\mathrm{C}_{ijkl}(s) = \mathrm{C}_{jikl}(s) = \mathrm{C}_{klij}(s)$
- $\|\mathrm{C}_{ijkl}(s)\|_{L^\infty(\Omega)} \lesssim 1/\min\{1, \mathrm{Re}\, s\}^2$
- Positivity

$$\mathrm{Re}\Big(\overline{s}\,\mathrm{C}_{ijkl}(s)\xi_{ij}\overline{\xi_{kl}}\Big) \geq c_0\, \mathrm{Re}\, s\, \xi_{ij}\overline{\xi_{ij}} \qquad \forall \xi_{ij} = \xi_{ji}$$

Zener's model: $\mathrm{C}(s) = (1 + a\,s)^{-1}(\mathrm{C}_0 + s\,\mathrm{C}_1)$

And I didn't forget a conjugation...

A generalization

Fractional Zener's model

In the Laplace domain

$$\mathrm{C}(s) = \mathrm{C}_{\mathrm{Zener}}(s^\nu) = (1 + a s^\nu)^{-1}(\mathrm{C}_0 + s^\nu \mathrm{C}_1) \qquad \nu \in (0,1)$$

and in the time domain

$$\boldsymbol{\sigma} + a\,\partial^\nu \boldsymbol{\sigma} = \mathrm{C}_0 \varepsilon + \mathrm{C}_1 \partial^\nu \varepsilon$$

Simple observations... (1) The Laplace domain bounds for the Fractional Zener Model are exactly those for the classical model. (2) The analysis allows for combinations of different fractional models on different subdomains.

Which fractional derivative?

The distributional kind

FEM semi-discretization and transfer function analysis

Finite element discretization

$\mathbf{V}_h$ — Finite element space satisfying the homogeneous Dirichlet boundary conditions

For all $t \geq 0$,

$$\begin{aligned}
&\mathbf{u}_h(t) \in \mathbf{V}_h \\
&(\rho\, \ddot{\mathbf{u}}_h(t), \mathbf{v})_\Omega + (\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_\Omega = (\mathbf{f}(t), \mathbf{v})_\Omega \qquad \forall \mathbf{v} \in \mathbf{V}_h \\
&\boldsymbol{\sigma}(t) = \mathcal{D} * \varepsilon(\dot{\mathbf{u}}_h(t)) \\
&\mathbf{u}_h(0) = \dot{\mathbf{u}}_h(0) = 0
\end{aligned}$$

FAQ

How do you handle the convolution term? CQ. Plenty of advantages

Laplace transform of the above

For all $s \in \mathbb{C}_+$

$$\mathbf{U}_h(s) \in \mathbf{V}_h$$

$$(\rho\, s^2 \mathbf{U}_h(s), \mathbf{v})_\Omega + (\mathrm{C}(s)\varepsilon(\mathbf{U}_h(s)), \varepsilon(\mathbf{v}))_\Omega = (\mathbf{F}(s), \mathbf{v})_\Omega \qquad \forall \mathbf{v} \in \mathbf{V}_h$$

Sesquilinear form

$$a(\mathbf{u}, \mathbf{v}; s) := (\rho\, s^2 \mathbf{u}, \mathbf{v})_\Omega + (\mathrm{C}(s)\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_\Omega$$

If there are non-vanishing IC, they appear on the right hand side, but again...

Stability

Energy norm (kinetic + elastic)

$$\|\!|\mathbf{u}|\!\|^2_{|s|} := \|s\,\rho^{1/2}\mathbf{u}\|^2_\Omega + \|\varepsilon(\mathbf{u})\|^2_\Omega$$

Laplace Domain Estimate

$$\|\!|\mathbf{U}_h(s)|\!\|_{|s|} \lesssim \frac{1}{\mathrm{Re}\, s}\|\mathbf{F}(s)\|_\Omega$$

The proof

$$
\begin{aligned}
(\mathrm{Res})|||\mathbf{U}_h(s)|||^2_{|s|} \lesssim \mathrm{Re}\Big(\overline{s}\,(\mathrm{C}(s)\varepsilon(\mathbf{U}_h(s)), \varepsilon(\mathbf{U}_h(s)))_\Omega \\
+ s|s|^2(\rho\,\mathbf{U}_h(s), \mathbf{U}_h(s))_\Omega\Big) \\
= \mathrm{Re}\Big((\mathrm{C}(s)\varepsilon(\mathbf{U}_h(s)), \varepsilon(s\mathbf{U}_h(s)))_\Omega \\
+ (s^2\rho\mathbf{U}_h(s), s\mathbf{U}_h(s))_\Omega\Big) \\
= \mathrm{Re}(\mathbf{F}(s), s\mathbf{U}_h(s))_\Omega \\
\le \|\mathbf{F}(s)\|_\Omega |||\mathbf{U}_h(s)|||_{|s|}
\end{aligned}
$$

$\square$

Solvability from Lax-Milgram

Semidiscretization 'error'

Laplace Domain Estimate

$$\|\!|\mathbf{U}(s) - \mathbf{U}_h(s)\|\!|_{|s|} \lesssim \frac{1}{\mathrm{Re}\, s \min\{1, \mathrm{Re}\, s\}^2} \|\!| s\mathbf{U}(s) \|\!|_{|s|}$$

Proof:

$$\begin{aligned}
(\mathrm{Re} s)\|\!|\mathbf{U}(s) - \mathbf{U}_h(s)\|\!|^2_{|s|} \\
&\lesssim |a(\mathbf{U}(s) - \mathbf{U}_h(s), s(\mathbf{U}(s) - \mathbf{U}_h(s)); s)| \\
&= |a(\mathbf{U}(s) - \mathbf{U}_h(s), s\mathbf{U}(s)); s)| \\
&\lesssim 1/\min\{1, \mathrm{Re}\, s\}^2 \|\!|\mathbf{U}(s) - \mathbf{U}_h(s)\|\!|_{|s|} \|\!| s\mathbf{U}(s) \|\!|_{|s|}
\end{aligned}$$

□

You can easily get bounds for

$$\|\mathrm{C}(s)\mathbf{U}_h(s)\|_\Omega \qquad \text{and} \qquad \|\mathrm{C}(s)\mathbf{U}(s) - \mathrm{C}(s)\mathbf{U}_h(s)\|_\Omega$$

Time domain estimates

Laplace 2 Time (an inversion theorem)

Abstract linear system:

$$\text{X-valued data } g \qquad \longmapsto \qquad \text{Y-valued solution } v$$

Transfer function (preprocess data, solve):

$$\mathrm{A}(s)\mathrm{V}(s) = \mathrm{B}(s)\mathrm{G}(s)$$

Hypothesis:

$$\|\mathrm{A}^{-1}(s)\mathrm{B}(s)\|_{X\to Y} \le m(\mathrm{Re}\, s)|s|^{\mu}, \qquad \forall s \in \mathbb{C}_+$$

with μ non-negative integer and $m : (0,\infty) \to (0,\infty)$ non-increasing and rationally unbounded at zero.
Thesis:

$$\|v(t)\|_Y \le c_{\mathrm{univ}} m(t^{-1}) \frac{t}{1+t} \sum_{\ell=\mu}^{\mu+2} \int_0^t \|g^{(k)}(\tau)\|_X \mathrm{d}\tau.$$

Stability

Recall the energy norm

$$\|\!|\mathbf{u}|\!\|_{|s|}^2 := \|s\,\rho^{1/2}\mathbf{u}\|_\Omega^2 + \|\varepsilon(\mathbf{u})\|_\Omega^2$$

Laplace Domain Estimate

$$\|\!|\mathbf{U}_h(s)|\!\|_{|s|} \lesssim \frac{1}{\mathrm{Re}\, s}\|\mathbf{F}(s)\|_\Omega$$

Time Domain Estimate

$$\|\rho^{1/2}\dot{\mathbf{u}}_h(t)\|_\Omega + \|\varepsilon(\mathbf{u}_h)(t)\|_\Omega \lesssim \frac{t^2}{1+t}\sum_{\ell=0}^{2}\int_0^t \|\mathbf{f}^{(\ell)}(\tau)\|_\Omega \mathrm{d}\tau$$

Semidiscrete error estimate

Laplace Domain Estimate

$$\|\mathbf{U}(s)-\mathbf{U}_h(s)\|_{|s|} \lesssim \frac{1}{\operatorname{Re} s \min\{1, \operatorname{Re} s\}^2} \|s\mathbf{U}(s)\|_{|s|}$$

Time Domain Estimate

$$\|\rho^{1/2}(\dot{\mathbf{u}}(t)-\dot{\mathbf{u}}_h(t))\|_\Omega + \|\varepsilon(\mathbf{u})(t)-\varepsilon(\mathbf{u}_h)(t)\|_\Omega$$

$$\lesssim \frac{t^2}{1+t}\max\{1,t^2\}\sum_{\ell=1}^{3}\int_0^t \|\mathbf{u}^{(\ell+1)}(\tau)\|_\Omega \mathrm{d}\tau + \|\varepsilon(\mathbf{u}^{(\ell)})(\tau)\|_\Omega \mathrm{d}\tau$$

That doesn't look like an error estimate...

Well, it is!

By 'Galerkin orthogonality,' it doesn't really matter to study the error when $\mathbf{u}(t)$ is the exact solution, or when $\mathbf{u}(t) - \pi_h \mathbf{u}(t)$ is the exact solution...

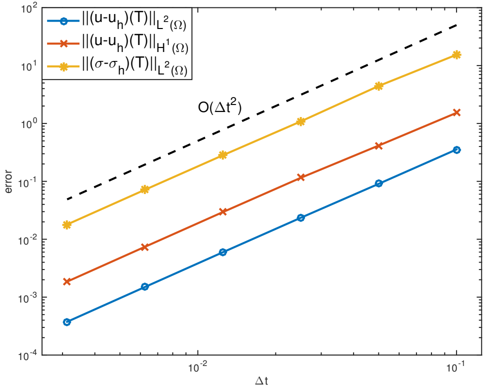
$$\|\rho^{1/2}(\dot{\mathbf{u}}(t) - \dot{\mathbf{u}}_h(t))\|_\Omega + \|\varepsilon(\mathbf{u})(t) - \varepsilon(\mathbf{u}_h)(t)\|_\Omega$$

$$\lesssim \frac{t^2}{1+t} \max\{1, t^2\} \sum_{\ell=1}^{3} \int_0^t \|\mathbf{u}^{(\ell+1)}(\tau) - \pi_h \mathbf{u}^{(\ell+1)}(\tau)\|_\Omega \mathrm{d}\tau + \|\varepsilon(\mathbf{u}^{(\ell)})(\tau) - \varepsilon(\pi_h \mathbf{u}^{(\ell)})(\tau)\|_\Omega \mathrm{d}\tau$$

What else?

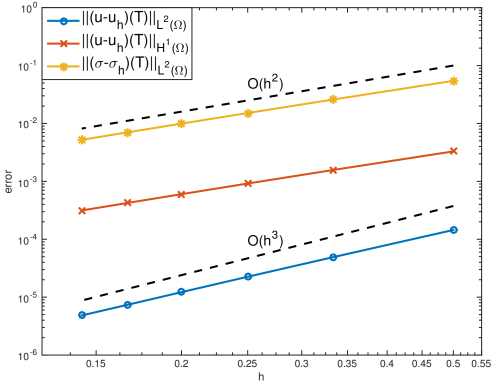
- Non-homogeneous Dirichlet and Neumann boundary conditions (Neumann simpler than Dirichlet)
- Full discretization using MS-CQ (use results from Lubich 1994) or RK-CQ (Banjai, Lubich, Melenk 2010)
- Nice features:
  - extremely easy to change the viscoelastic model
  - joint treatment of classical viscoelastic models and their fractional counterparts
  - coupled problems handled naturally
- Time domain analysis, only available at the semidiscrete stage so far for the classical viscoelastic model (with possible purely elastic zones)
- Pending: visco-poroelastic model

Experiment # 1



Uniform mesh of a cube with $\mathcal{P}_2$ elements. Trapezoidal rule. Refinement in time and not in space

Experiment # 2

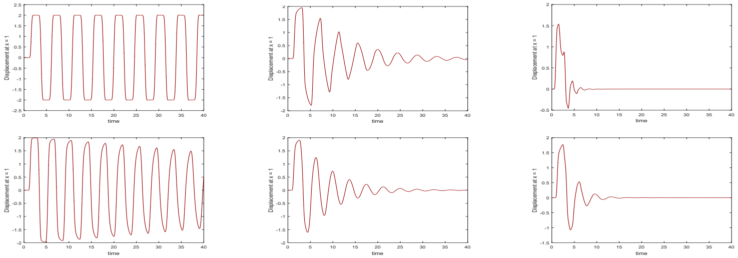


Sequence of uniform meshes for a cube. $\mathcal{P}_2$ elements. Trapezoidal rule with 500 time steps.

One-dimensional examples

The setting

One dimensional viscoelastic ‘beam,’ no loads, shaken on one side, end left free on the other. We observe the displacement on the free tip. High order in space FEM with TR-CQ in time.

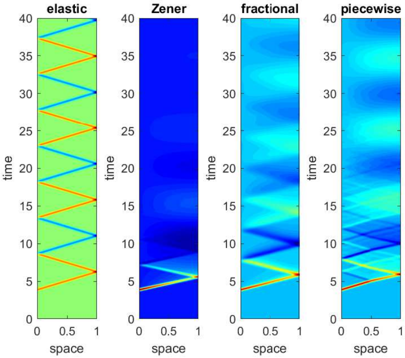


1D code by Connor Swalm and Hugo Diaz

One-dimensional examples

The setting

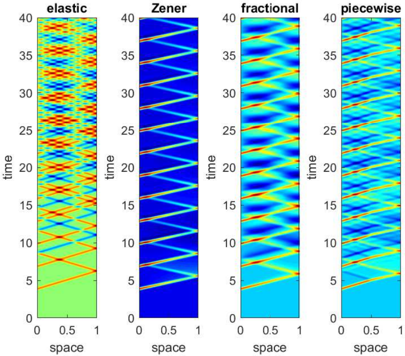
One dimensional viscoelastic 'beam,' no loads, single pulse on one side, end left free on the other. We observe the displacement in space-time for four different material properties. High order in space FEM with TR-CQ in time.



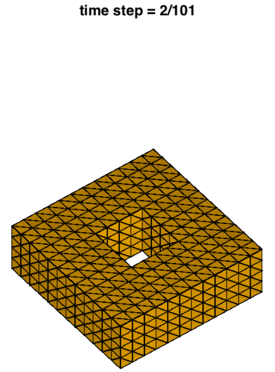
One-dimensional examples

The setting

One dimensional viscoelastic 'beam,' no loads, train of pulses on one side, end left free on the other. We observe the change from transient to time-harmonic regime. High order in space FEM with TR-CQ in time.

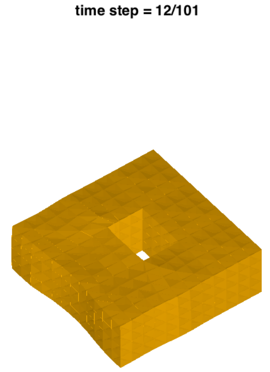


Several snapshots of a simulation



The bottom of the squared donut is gently shaken with a plane wave. No volume forcing. All other boundaries are traction free.

Several snapshots of a simulation



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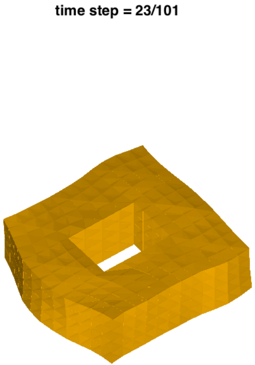
Several snapshots of a simulation



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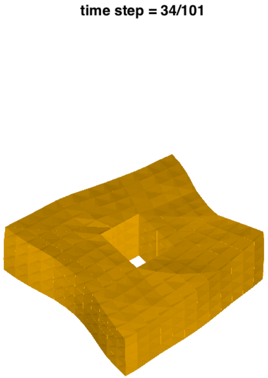

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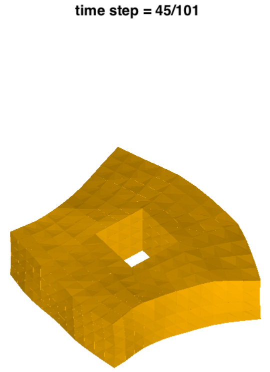

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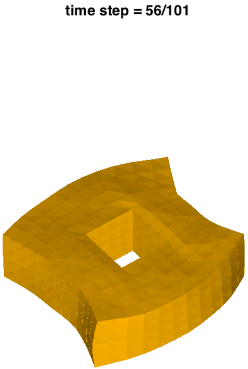

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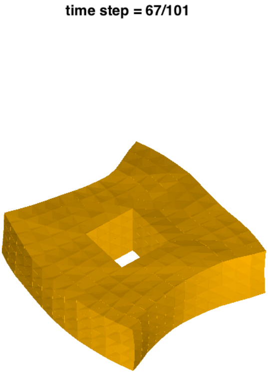

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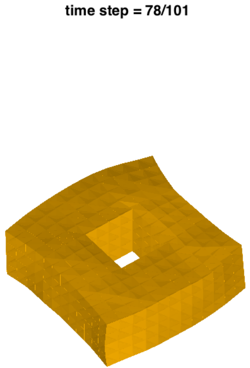
Several snapshots of a simulation



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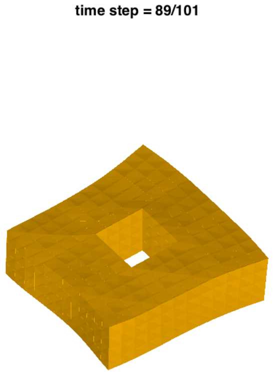

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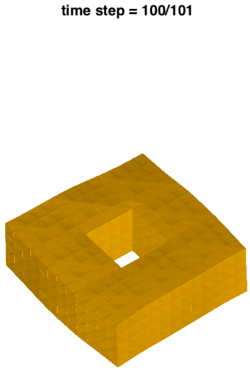

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Conclusions

- Easy treatment of a large class of viscoelastic models (FEM-CQ discretization), including coupled models
- Joint analysis using Laplace transform tools
- Improved estimates in some cases using energy techniques (semigroup theory)
- Tested 3D code and 1D toy-code with parameter derivatives
- To do: integrated fractional model, visco-poroelastic problems (theory and simulation), model adjustment (using optimization tools),...

What do you get with Google-Image?

This photo from a couple of websites

What do you get with Google-Image?

Giving a talk with Jean-Claude Nédélec in the audience

What do you get with Google-Image?

Some matheux who happen to work in Pau...

What do you get with Google-Image?

And then there's 'comedy stage hypnotist' Ben Dali

Bon Anniversaire!

ou

Bonne retraite!

Thanks to Hélène, Julien, and Sébastien