Explicit variational forms for the inverses of integral operators for the Laplace equation in the exterior of a flat disk in $\mathbb{R}^3$

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Abstract

We introduce variational formulations for the weakly- and hyper-singular operators (as well as for their corresponding inverses) associated to the Laplace operator in the domain of R^3 exterior to a flat open disk in $\mathbb{R}^3$. Using adequate basis functions on the disk, we obtain an exact expression for the associated kernels. This work is an extension to R^3 of the article by Jerez-Hanckes and Nédélec (2012, Explicit variational forms for the inverses of integral logarithmic operators over an interval ([3])).

Log-Kernel

Consider first the isotropic space $\mathbb{R}^2$ divided into two half-planes:

$$\pi_{\pm} := \left\{ \mathbf{x} \in \mathbb{R}^2 \;:\; x_2 \lesseqgtr 0 \right\} \tag{1}$$

with interface Γ given by the line $x_2 = 0$. The interface is further divided into the open disjoint segments $\Gamma_m := (-1, 1) \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_m$. Consequently, we have defined the domain $\Omega := \mathbb{R}^2 \setminus \bar{\Gamma}_m$. We seek u such that

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega \\ \quad u = g & \text{for} \quad \mathbf{x} \in \Gamma_m; \quad \text{with} \quad g \in H^{1/2}(\Gamma_m). \end{cases} \tag{2}$$

Then, the potential u can be represented as a single layer potential:

$$u(\mathbf{x}) = L_1\varphi = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} \,, \qquad \text{for} \quad \mathbf{x} \in \Omega \,, \tag{3}$$

Then φ is the solution of the logarithmic integral equation:

$$g(\mathbf{x}) = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} \qquad \text{for} \quad \mathbf{x} \in \Gamma \,. \tag{4}$$

The equation (4) has a variational formulation in the space $\widetilde{H}_0^{-1/2}(\Gamma_m)$ which is:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi^t(\tau)dtd\tau=\int_{\Gamma_m}g(\tau)\varphi^t(\tau)d\tau,\forall\varphi^t\in\widetilde{H}_0^{-1/2}(\Gamma_m) \tag{5}$$

This operator is a bijection between $\widetilde{H}_0^{-1/2}(\Gamma_m)$ and the space $H_*^{1/2}(\Gamma_m)$ of functions in $H^{1/2}(\Gamma_m)$ satisfying $\displaystyle\int_{\Gamma_m}\frac{1}{\sqrt{1-t^2}}\,g(t)\,dt\;=\;0$. and we have

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi(\tau)dtd\tau\geq C\,\|\varphi\|^2_{\widetilde{H}_0^{-1/2}(\Gamma_m)},\forall\varphi\in\widetilde{H}_0^{-1/2}(\Gamma_m). \tag{6}$$

The inverse operator is a bijection of $H_*^{1/2}(\Gamma_m)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_m)$. This operator N_1 is symmetric and coercive in the space $H_*^{1/2}(\Gamma_m)$. It admits two variational formulations. Let $M(x,y)$ be the function

$$M(x,y)=\frac{1}{2}\left((y-x)^2+\left(\sqrt{1-x^2}+\sqrt{1-y^2}\right)^2\right) \tag{7}$$

$$L_2g=\frac{1}{\pi}\int_{\Gamma_m}\log\left\{\frac{M(x,y)}{|x-y|}\right\}g(y)dy \tag{8}$$

The first one is:

$$(N_1 g, g^t) = \frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m} \log\left\{\frac{M(x,y)}{|x-y|}\right\} g'(x)\left(g^t(y)\right)' \, dy\, dx = \int_{\Gamma_m} \varphi(x) g^t(x) dx \qquad (9)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$, which gives a first norm on the space $H_*^{1/2}(\Gamma_m)$:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m} \log\left[\frac{M(x,y)}{|x-y|}\right] g'(x)\, g'(y)\, dy\, dx \;\geq\; C\, \|g\|^2_{H_*^{1/2}(\Gamma_m)}\, ; \forall g \in H_*^{1/2}(\Gamma_m) \qquad (10)$$

The second one is

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m} \frac{d^2}{dxdy} \log\left[\frac{M(x,y)}{|x-y|}\right] (g(x)-g(y))\left(g^t(x)-g^t(y)\right) dy dx = \int_{\Gamma_m} \varphi(x) g^t(x) dx \qquad (11)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$,

So we have a second norm on the space $H_*^{1/2}(\Gamma_m)$ which is:

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m} \left\{\frac{1-xy}{w(x)w(y)}\right\} \frac{\left(g(x)-g(y)\right)^2}{(x-y)^2} dy dx \geq C\, \|g\|^2_{H_*^{1/2}(\Gamma_m)}, \forall g \in H_*^{1/2}(\Gamma_m) \qquad (12)$$

where the weight function w is given by

$$w(x) := \sqrt{1 - x^2} \qquad \text{for } x \in (-1, 1). \qquad (13)$$

We can also consider the Neumann problem

$$\begin{cases} -\Delta u \ = \ 0 & \text{for} \quad \mathbf{x} \in \Omega \\ \gamma_m^+ \partial_n u = \gamma_m^- \partial_n u = \varphi & \text{for} \quad \mathbf{x} \in \Gamma_m, \quad \varphi \in H^{-1/2}(\Gamma_m) \end{cases} \tag{14}$$

which can be represent as a double layer potential of harmonic solution in the domain Ω of the form .

$$u(\mathbf{x}) \ = \ \frac{1}{\pi} \int_{\Gamma_m} \frac{x_2}{|\mathbf{x}-\mathbf{y}|^2} \alpha(\mathbf{y}) \mathrm{d}\mathbf{y} \ , \qquad \text{for} \quad \mathbf{x} \in \Omega \ , \tag{15}$$

Then the unknown α is the solution of the hyper singular integral equation:

$$\varphi(x) \ = N_2 \, \alpha = \frac{1}{\pi} \oint_{\Gamma_m} \frac{1}{|x-y|^2} \alpha(y) dy \qquad \text{for} \quad x \in \Gamma \, . \tag{16}$$

where α is also the jump of the Dirichlet trace of the solution of problem (14).

A variational formulation of the integral equation (16) in the space $\widetilde{H}^{1/2}(\Gamma_m)$ is

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\alpha'(t)(\alpha^t(\tau))'\,dtd\tau=\int_{\Gamma_m}\varphi(\tau)\alpha^t(\tau)d\tau,\forall\alpha^t\in\widetilde{H}^{1/2}(\Gamma_m) \tag{17}$$

The associated operator $\widetilde{\mathcal{D}}$ is a bijection from $\widetilde{H}^{1/2}(\Gamma_m)$ to $H^{-1/2}(\Gamma_m)$. Moreover, this bilinear form is coercive, i.e.,

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\alpha'(t)\alpha(\tau)'\,dtd\tau\geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)},\forall\alpha\in\widetilde{H}^{1/2}(\Gamma_m). \tag{18}$$

This operator admits a second variational formulation which is

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))\,(\alpha^t(x)-\alpha^t(y))}{|x-y|^2}dxdy+\frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)\alpha^t(x)}{1-x^2}dx=\int_{\Gamma_m}\varphi(x)\alpha^t(x)dx \tag{19}$$

for all $\alpha^t\in\widetilde{H}^{1/2}(\Gamma_m)$, and the next expression is a norm on $\widetilde{H}^{1/2}(\Gamma_m)$

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy+\frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)^2}{1-x^2}dx\geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)},\forall\alpha\in\widetilde{H}^{1/2}(\Gamma_m) \tag{20}$$

The inverse operator is a bijection of $H^{-1/2}(\Gamma_m)$ onto $\widetilde{H}^{1/2}(\Gamma_m)$. The associated operator is symmetric and coercive in the space $H^{-1/2}(\Gamma_m)$.
It admits the following variational formulation:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi^t(y)dydx=\int_{\Gamma_m}\alpha(x)\varphi^t(x)dx,\ \forall\varphi\in H^{-1/2}(\Gamma_m) \tag{21}$$

and thus the following expression is a norm on the space $H^{-1/2}(\Gamma_m)$

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi(y)dydx\geq C\,\|\varphi\|^2_{H^{-1/2}(\Gamma_m)},\ \forall\varphi\in H^{-1/2}(\Gamma_m) \tag{22}$$

The operators L_1, L_2, N_1, N_2, D, D^* are linked by the identities

$$\begin{aligned}
L_2 \circ N_2 &= -L_2 \circ D^* \circ L_1 \circ D = I\,, && I \in \widetilde{H}^{1/2}(\Gamma_m) \\
L_1 \circ N_1 &= -L_1 \circ D \circ L_2 \circ D^* = I\,, && I \in H_*^{1/2}(\Gamma_m) \\
N_1 \circ L_1 &= -D \circ L_2 \circ D^* \circ L_1 = I\,, && I \in \widetilde{H}_0^{-1/2}(\Gamma_m) \\
N_2 \circ L_2 &= -D^* \circ L_2 \circ D \circ L_1 = I\,, && I \in H^{-1/2}(\Gamma_m)
\end{aligned}$$

$L_1 \circ D$ is continuous and invertible from $\widetilde{H}^{1/2}(\Gamma_m)$ into $H_*^{1/2}(\Gamma_m)$.
$L_2 \circ D^*$ is continuous and invertible from $H_*^{1/2}(\Gamma_m)$ into $\widetilde{H}^{1/2}(\Gamma_m)$.
$D^* \circ L_1$ is continuous and invertible from $\widetilde{H}_0^{-1/2}(\Gamma_m)$ into $H^{-1/2}(\Gamma_m)$.
$D \circ L_2$ is continuous and invertible from $H^{-1/2}(\Gamma_m)$ into $\widetilde{H}_0^{-1/2}(\Gamma_m)$.

The Dirichlet and Neumann Laplacian Δ_D, Δ_N are linked to L_1, L_2 and N_1, N_2:

$$\begin{aligned}
L_1 &= (-\Delta_D)^{-\frac{1}{2}}\,; && -N_1 = (-\Delta_D)^{\frac{1}{2}}\,; \\
L_2 &= (-\Delta_N)^{-\frac{1}{2}}\,; && -N_2 = (-\Delta_N)^{\frac{1}{2}}\,.
\end{aligned}$$

The disc in R^3

We try now to extend these results to the unit disc in $\mathbb{R}^3$.

We Introduce the splitting of the space $\mathbb{R}^3$ into two half-spaces $\pi_\pm := \left\{\mathbf{x} \in \mathbb{R}^3 \ : \ x_3 \gtrless 0\right\}$, by the plane $x_3 = 0$ that will be denote as Γ.
Let $\mathfrak{c}$ be the circle of center at the origin and of radius 1 in the plane Γ.
Let $\mathbb{D}$ be the plane disc delimitated by the circle $\mathfrak{c}$ and $\mathbb{D}$ the associated flat domain in $\mathbb{R}^3$.
Now its complement in $\mathbb{R}^2$, is $\Gamma_f := \Gamma \setminus \bar{\mathbb{D}}$.
Henceforth, the problem domain is denoted by $\Omega := \mathbb{R}^3 \setminus \bar{\mathbb{D}}$.
We also consider the sphere $\mathbb{S}$ of radius 1 and center at the origin in R^3.
The disc $\mathbb{D}$ divide this sphere into two half-sphere that we denote respectively $\mathbb{S}^+$ and $\mathbb{S}^-$.

The unit sphere in $\mathbb{R}^3$ and its equatorial disc

We consider the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$ (Fig. 1) and the spherical coordinates: (r, θ, φ), where r is the radius and θ, φ the two Euler angles.

$$\begin{cases} x_1 = r \sin\theta \cos\varphi, \\ x_2 = r \sin\theta \sin\varphi, \\ x_3 = r \cos\theta. \end{cases} \tag{25}$$

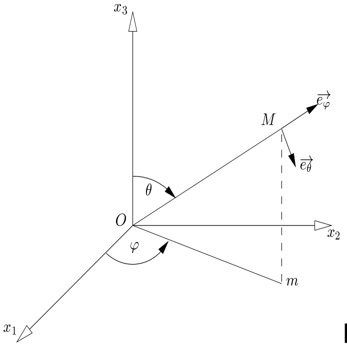

Fig. 1: Spherical coordinates

The vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unitary. The vector $\mathbf{e}_\rho$ directed along Om is unitary.

– A point $\mathbf{x}$ on the circular domain $\mathbb{D}$ will be defined using its coordinates (x_1, x_2) or in circular coordinates by $(0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi)$.

– A point $\mathbf{x}^+$ (resp. $\mathbf{x}^-$) on the half sphere $\mathbb{S}^+$ (resp. $\mathbb{S}^-$) will be defined using $(0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi)$ (resp. $(\frac{\pi}{2} \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$).

–The projection $\mathbf{x}$ of a point $\mathbf{x}^+$ situed on the half sphere $\mathbb{S}^+$ onto the domain $\mathbb{D}$ has for circular coordinates $\mathbf{x}$: $(\rho = \sin(\theta), \varphi)$.

– The projection $\mathbf{x}$ of a point $\mathbf{x}^-$ situed on the half sphere $\mathbb{S}^-$ onto the domain $\mathbb{D}$ has for circular coordinates $\mathbf{x}$: $(\rho = \sin(\theta), \varphi)$.

–To a point $\mathbf{x}$, we associate the points $\mathbf{x}^+$ and $\mathbf{x}^-$ which projections are $\mathbf{x}$.

Notations

Let $\mathcal{O} \subseteq \mathbb{R}^d$, with $d = 1, 2$, be open. We denote by $\mathscr{C}^k(\mathcal{O})$ the space of k-times differentiable continuous functions over $\mathcal{O}$ with $k \in \mathbb{N}_0$. Its subspace of compactly supported functions is $\mathscr{C}_0^k(\mathcal{O})$ and for infinitely differentiable functions we write $\mathscr{D}(\mathcal{O}) \equiv \mathscr{C}_0^\infty(D)$. The space of distributions or linear functionals over $\mathscr{D}(\mathcal{O})$ is $\mathscr{D}'(\mathcal{O})$. Also, let $L^p(\mathcal{O})$ be the standard class of functions with bounded L^p-norm over $\mathcal{O}$. By $\mathbb{S}'(\mathcal{O})$ we denote the Schwartz space of tempered distributions.

Duality products are denoted by angular brackets, $\langle \cdot \,, \cdot \rangle$, with subscripts accounting for the duality pairing. Inner products are denoted by round brackets, $(\cdot \,, \cdot)$, with integration domains specified by subscripts. Furthermore, operators are denoted in mild calligraphic style and complex conjugates by overline. The adjoint of an operator will be specified by an asterisk.

The disk $\mathbb{D}$ is a Lipschitz domain in $\mathbb{R}^2$. For any $s > 0$, $\widetilde{H}^s(\mathbb{D})$ is the space of functions whose extension by zero to Γ belongs to $H^s(\Gamma)$. For $s = 1/2$, we have the four following different spaces

$$\widetilde{H}^{-1/2}(\mathbb{D}) \equiv \left(H^{1/2}(\mathbb{D})\right)' \quad \text{and} \quad H^{-1/2}(\mathbb{D}) \equiv \left(\widetilde{H}^{1/2}(\mathbb{D})\right)', \tag{26}$$

Define restrictions over the half-spaces: $u^\pm := u|_{\pi_\pm}$. We introduce the *trace operators* $\gamma^\pm : \mathscr{D}(\overline{\pi}_\pm) \to \mathscr{D}(\Gamma)$ as $\gamma^\pm u := \lim_{\epsilon \to 0^\pm} u(x_1, x_2, \epsilon) = \gamma^\pm u^\pm$.

Theorem

We denote by $\gamma^\pm_{\Gamma_b}$ the trace operator:

$$\begin{array}{rccl} \gamma^\pm_{\Gamma_b} : \mathscr{D}(\overline{\pi}_\pm) & \longrightarrow & \mathscr{D}(\Gamma_b) \\ u^\pm & \longmapsto & \gamma^\pm_{\Gamma_b} u^\pm = \gamma^\pm u^\pm|_{\Gamma_b}. \end{array} \tag{27}$$

If $s > 1/2$, a unique extension to a bounded linear operator $\gamma^\pm_{\Gamma_b} : H^s_{loc}(\pi_\pm) \to H^{s-1/2}(\Gamma_b)$ can be obtained by density of $\mathscr{D}(\overline{\pi}_\pm)$ in $H^s(\pi_\pm)$.

Let $[\gamma] := \gamma^+ - \gamma^-$ represent the jump operator across Γ. As Γ is not orientable, we set $\mathbf{n}$ pointing along the positive x_3-axis, i.e. $\mathbf{n} = \hat{\mathbf{x}}_3$.

Weighted Sobolev spaces

Since the problem domain Ω is unbounded (cf. Section 11), one usually works in either local Sobolev spaces or in weighted ones such as

$$W^{1,-1}(\Omega) = \left\{ u \in \mathscr{D}'(\Omega) : \frac{u}{(1+r^2)^{1/2}} \in L^2(\Omega), \nabla u \in L^2(\Omega) \right\}, \tag{28}$$

which coincides with the standard $H^1_{\text{loc}}(\Omega)$ for a bounded part of Ω and avoids specifying behaviors at infinity [5]. Furthermore, these weighted spaces are Hilbert whereas local Sobolev spaces are only of Fréchet type. We also define the subspace:

$$W^{1,-1}_0(\Omega) = \left\{ u \in W^{1,-1}(\Omega) : \gamma^{\pm}_{\mathbb{D}} u = 0 \right\} . \tag{29}$$

Lemma ([5], Section 2.5.4)

Define the norm:

$$|u|^2_{1,-1,\Omega} := \int_\Omega |\nabla u(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \,. \tag{30}$$

Then, there exists $c > 0$ such that

$$\|u\|_{W_0^{1,-1}(\Omega)} \leq c\,|u|_{1,-1,\Omega}\,, \quad \forall\, u \in W_0^{1,-1}(\Omega). \tag{31}$$

Moreover, this norm is also a norm on the space $W^{1,-1}(\Omega)$. Specifically, there exists $c > 0$ such that

$$\|u\|_{W^{1,-1}(\Omega)} \leq c\,|u|_{1,-1,\Omega} \quad \forall\, u \in W^{1,-1}(\Omega)\,. \tag{32}$$

Now, traces on Γ for elements in $W^{1,-1}(\Omega)$ lie in the usual $H^{1/2}_{\text{loc}}(\Gamma)$, and their restriction to a bounded Γ_b generates the subspace $H^{1/2}(\Gamma_b)$.

Dirichlet Problems

Instead of directly considering the standard Laplace problems, we start by tackling a slightly different Laplace problem with two different Dirichlet conditions $g^{\pm}$ from above and below on $\mathbb{D}$. These boundary data lie in the Hilbert space:

$$\mathbb{X} := \left\{ \mathbf{g} = (g^+, g^-) \in H^{1/2}(\mathbb{D}) \times H^{1/2}(\mathbb{D}) : g^+ - g^- \in \widetilde{H}^{1/2}(\mathbb{D}) \right\} \qquad (33)$$

with norm

$$\|\mathbf{g}\|^2_{\mathbb{X}} := \|g^+\|^2_{H^{1/2}(\mathbb{D})} + \|g^-\|^2_{H^{1/2}(\mathbb{D})} + \|g^+ - g^-\|^2_{\widetilde{H}^{1/2}(\mathbb{D})} .$$

Equivalently, we define the Hilbert space for Neumann data:

$$\mathbb{Y} := \left\{ \boldsymbol{\varphi} = (\varphi^+, \varphi^-) \in H^{-1/2}(\mathbb{D}) \times H^{-1/2}(\mathbb{D}) : \varphi^+ - \varphi^- \in \widetilde{H}^{-1/2}(\mathbb{D}) \right\} \qquad (34)$$

with similar norm:

$$\|\boldsymbol{\varphi}\|^2_{\mathbb{Y}} := \|\varphi^+\|^2_{H^{-1/2}(\mathbb{D})} + \|\varphi^-\|^2_{H^{-1/2}(\mathbb{D})} + \|\varphi^+ - \varphi^-\|^2_{\widetilde{H}^{-1/2}(\mathbb{D})} .$$

The Dirichlet problem we consider is:

Problem

For $\mathbf{g} \in \mathbb{X}$*, find* $u \in W^{1,-1}(\Omega)$ *such that:*

$$\begin{cases} -\Delta\, u \;=\; 0, & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_{\mathbb{D}}^{+} \\ \gamma_{\mathbb{D}}^{-} \end{pmatrix} u \;=\; \mathbf{g}, & \mathbf{x} \in \mathbb{D}. \end{cases} \tag{35}$$

Theorem

If $\mathbf{g} \in \mathbb{X}$*, then the Problem (35) has a unique solution in* $W^{1,-1}(\Omega)$*.*

The solution to Problem (35) can be split as follows. To any function u in $W^{1,-1}(\Omega)$, one associates restrictions $u^\pm$ on $\pi_\pm$ belonging to $W^{1,-1}(\pi_\pm)$. Denote by $\check{u}^\pm \in W^{1,-1}(\mathbb{R}^d)$ the mirror reflection of $u^\pm$ over $\pi_\mp$. Average and jump solutions defined over $\mathbb{R}^2$ are written as

$$\begin{cases} u_{\mathrm{avg}} := \dfrac{\check{u}^+ + \check{u}^-}{2}, \\ u_{\mathrm{jmp}} := \dfrac{\check{u}^+ - \check{u}^-}{2}, \end{cases} \quad \text{associated to the data} \quad \begin{cases} g_{\mathrm{avg}} := \dfrac{g^+ + g^-}{2}, \\ g_{\mathrm{jmp}} := \dfrac{g^+ - g^-}{2}. \end{cases} \tag{36}$$

Normal traces can also be similarly decomposed. Due to the orientation of the normal, they take the form:

$$\begin{cases} \gamma_{\mathbb{D}}\partial_n u_{\mathrm{avg}} := \frac{1}{2}\hat{\mathbf{x}}_3 \cdot \nabla(\check{u}^+ - \check{u}^-), \\ \gamma_{\mathbb{D}}\partial_n u_{\mathrm{jmp}} := \frac{1}{2}\hat{\mathbf{x}}_3 \cdot \nabla(\check{u}^+ + \check{u}^-), \end{cases} \quad \text{associated to the values} \begin{cases} u_{\mathrm{avg}}, \\ u_{\mathrm{jmp}}, \end{cases} \tag{37}$$

and we have the associated Green's formula (as $\left(\nabla u_{\mathrm{avg}}\,,\,\nabla v_{\mathrm{jmp}}\right)_\Omega = 0$):

$$(\nabla u\,,\,\nabla v)_\Omega = \left\langle \gamma_{\mathbb{D}}\partial_n u_{\mathrm{avg}}\,,\,\gamma_{\mathbb{D}} v_{\mathrm{avg}} \right\rangle_{H^{1/2}(\mathbb{D})} + \left\langle \gamma_{\mathbb{D}}\partial_n u_{\mathrm{jmp}}\,,\,\gamma_{\mathbb{D}} v_{\mathrm{jmp}} \right\rangle_{\widetilde{H}^{1/2}(\mathbb{D})}, \tag{38}$$

for $v \in W^{1,-1}(\mathbb{R}^2)$ split into average and jump parts.

Theorem

The solution of the Dirichlet Problem 3, is such that its Neumann trace at $\mathbb{D}$ belongs to the space $\mathbb{Y}$. There exists a unique Dirichlet-to-Neumann (DtN) map $\mathcal{D} : \mathbb{X} \to \mathbb{Y}$ satisfying

$$\langle \mathcal{D}\,\mathbf{g}\,,\,\mathbf{g}\rangle_{\mathbb{X}} \;\geq\; C\,\|\mathbf{g}\|^2_{\mathbb{X}}\,, \tag{39}$$

for $\mathbf{g}$ in $\mathbb{X}$, and where the vector duality product is given by:

$$\langle \mathcal{D}\,\mathbf{g}\,,\,\mathbf{g}\rangle_{\mathbb{X}} = \big\langle \mathcal{D}\,\mathbf{g}_{avg}\,,\,\mathbf{g}_{avg}\big\rangle_{H^{1/2}(\mathbb{D})} + \big\langle \mathcal{D}\,\mathbf{g}_{jmp}\,,\,\mathbf{g}_{jmp}\big\rangle_{\widetilde{H}^{1/2}(\mathbb{D})}\,. \tag{40}$$

Corollary

For $g^{\pm} =: g \in H^{1/2}(\mathbb{D})$, the corresponding solution of Problem (35) in Ω is symmetric with respect to Γ. Moreover, there exists a unique DtN operator $\mathcal{D}_s : H^{1/2}(\mathbb{D}) \to \widetilde{H}^{-1/2}(\mathbb{D})$ satisfying

$$\langle \mathcal{D}_s\,g\,,\,g\rangle_{H^{1/2}(\mathbb{D})} \;\geq\; C_s\,\|g\|^2_{H^{1/2}(\mathbb{D})}\,. \tag{41}$$

Corollary

For $g^{\pm} = \pm g \in \widetilde{H}^{1/2}(\mathbb{D})$, the associated solution of Problem (35) is antisymmetric with respect to Γ and there exists a unique DtN operator $\mathcal{D}_{as} : \widetilde{H}^{1/2}(\mathbb{D}) \to H^{-1/2}(\mathbb{D})$. Moreover, the energy inequality holds

$$\langle \mathcal{D}_{as}\, g \,,\, g \rangle_{\widetilde{H}^{1/2}(\mathbb{D})} \;\geq\; C_{as}\, \|g\|^2_{\widetilde{H}^{1/2}(\mathbb{D})} \,. \tag{42}$$

Neumann Problems

As in the Dirichlet case, we now define the general problem:

Problem

Find $u \in W^{1,-1}(\mathbb{R}^3)$ such that

$$\begin{cases} -\Delta\, u \;=\; 0, & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_{\mathbb{D}}^{+} \partial_n u \\ \gamma_{\mathbb{D}}^{-} \partial_n u \end{pmatrix} = \varphi, & \mathbf{x} \in \mathbb{D}, \end{cases} \tag{43}$$

where φ belongs to the space $\mathbb{Y}$.

Theorem

The Neumann Problem (43) has a unique solution in the space $W^{1,-1}(\mathbb{R}^3)$ if and only if $\varphi \in \mathbb{Y}$.

Theorem

The solution of the Neumann Problem (43), is such that its Dirichlet trace at $\mathbb{D}$ belongs to the space $\mathbb{X}$. There exists a unique Neumann-to-Dirichlet (NtD) map $\mathcal{N} : \mathbb{Y} \to \mathbb{X}$ satisfying

$$\langle \mathcal{N}\varphi\,,\,\varphi\rangle_{\mathbb{Y}} \;\geq\; C\,\|\varphi\|^2_{\mathbb{Y}}\,, \tag{44}$$

for φ in $\mathbb{Y}$, and where the vector duality product is given by:

$$\langle \mathcal{N}\varphi\,,\,\varphi\rangle_{\mathbb{Y}} = \Big\langle \mathcal{N}\varphi_{avg}\,,\,\varphi_{avg}\Big\rangle_{\widetilde{H}^{-1/2}(\Gamma_c)} + \Big\langle \mathcal{N}\varphi_{jmp}\,,\,\varphi_{jmp}\Big\rangle_{H^{-1/2}(\Gamma_c)}\,. \tag{45}$$

Symmetric (antisymmetric) Neumann problems can be stated as follows: Find $u_s, u_{as} \in W^{1,-1}(\mathbb{R}^3)$ such that

$$\begin{cases} -\Delta\, u_s \;=\; 0, & \mathbf{x} \in \Omega,\\ [\gamma_{\mathbb{D}}\partial_n u_s] \;=\; \varphi, & \mathbf{x} \in \mathbb{D}, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\, u_{as} \;=\; 0, & \mathbf{x} \in \Omega,\\ \gamma^{\pm}_{\mathbb{D}}\partial_n u_{as} = \varphi, & \mathbf{x} \in \mathbb{D}, \end{cases} \tag{46}$$

for data φ in the space $\widetilde{H}^{-1/2}(\mathbb{D})$ and φ in $H^{-1/2}(\mathbb{D})$ respectively.

Corollary

The symmetric Neumann Problem (46) has a unique solution in $W^{1,-1}(\mathbb{R}^3)$ if and only if $\varphi \in \widetilde{H}^{-1/2}(\mathbb{D}$. Thus, there exists a unique continuous and invertible NtD, denoted $\mathcal{N}_s : \widetilde{H}^{-1/2}(\mathbb{D}) \to H^{1/2}(\mathbb{D})$. Moreover, the energy inequality holds

$$\langle \mathcal{N}_s \varphi \, , \, \varphi \rangle_{\mathbb{D}} \;\geq\; C \, \|\varphi\|^2_{\widetilde{H}^{-1/2}(\mathbb{D})} \, . \tag{47}$$

The inverse of this application is the operator $\mathcal{D}_s$ defined in Corollary 6.

Corollary

The antisymmetric Neumann problem (46)has a unique solution in $W^{1,-1}(\mathbb{R}^3)$ if and only if $\phi \in H^{-1/2}(\mathbb{D})$. Hence, there exists a unique continuous and invertible $\mathcal{N}_{as} : H^{-1/2}(\mathbb{D}) \to \widetilde{H}^{1/2}(\mathbb{D})$ satisfying

$$\langle \mathcal{N}_{as} \varphi \, , \, \varphi \rangle_{\mathbb{D}} \;\geq\; C \, \|\varphi\|^2_{H^{-1/2}(\mathbb{D})} \, . \tag{48}$$

The inverse of this application is the operator $\mathcal{D}_{as}$ defined in Corollary 7.

We now present the main results of this work: explicit variational forms or regularizations for the weakly- and hyper-singular operators over the disk $\mathbb{D}$ and their inverses as well as associated Calderón-type identities. In fact, we will show that there exist two equivalent forms for the inverse of the weakly singular operator and two equivalent representations for the hypersingular operator. Moreover, we study the mapping properties of the underlying operators and derive useful identities for numerical applications.

Symmetric problem and weakly singular operator

The solution of the symmetric Dirichlet and Neumann solutions are given via the **simple layer potential** . For the symmetric Neumann problem, one just simply introduces the data φ in the potential LS_s and then the simple layer potential gives the solution in R^3.

$$u(\mathbf{y}) \;=\; \frac{1}{4\pi}\int_{\mathbb{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\varphi(\mathbf{x})d_{\mathbb{D}}(\mathbf{x}) \qquad \mathbf{y}\in R^3. \tag{49}$$

The solution of the Dirichlet problem is obtained via solving the following integral equation on $\mathbb{D}$: find φ such that

$$\frac{1}{4\pi}\int_{\mathbb{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\varphi(\mathbf{x})d_{\mathbb{D}}(\mathbf{x}) \;=\; g(\mathbf{y})\,, \qquad \mathbf{y}\in\mathbb{D}. \tag{50}$$

and then the simple layer potential (49) gives the solution in R^3.

Theorem

The symmetric variational formulation of the integral equation (50) *in the space* $\widetilde{H}^{-1/2}(\mathbb{D})$ *is*

$$\left\langle LS_s\varphi\,,\,\varphi^t\right\rangle_{\mathbb{D}} = \left\langle g\,,\,\varphi^t\right\rangle_{\mathbb{D}}\,, \qquad \forall\,\varphi^t \in \widetilde{H}^{-1/2}(\mathbb{D}), \tag{51}$$

which is coercive, i.e.

$$\langle LS_s\varphi\,,\,\varphi\rangle_{\mathbb{D}} \geq C\,\|\varphi\|^2_{\widetilde{H}^{-1/2}(\mathbb{D})}\,, \qquad \forall\,\varphi \in \widetilde{H}^{-1/2}(\mathbb{D}). \tag{52}$$

The associated operator, $\mathcal{N}_s$ *(cf. Corollary 11), is a bijection between* $\widetilde{H}^{-1/2}(\mathbb{D})$ *and* $H^{1/2}(\mathbb{D})$.

Theorem

We denote by LN_s the integral operator which is the inverse of LS_s and is associated to $\mathcal{D}_s$ (cf. Corollary 6). Its kernel is denote by LKN_s. It is symmetric and coercive in $H^{1/2}(\mathbb{D})$. It admits two variational formulations:

$$\left\langle LS_{as}\,\mathbf{curl}_{\mathbb{D}}\, g\,,\,\mathbf{curl}_{\mathbb{D}}\, g^t\right\rangle_{\mathbb{D}} = \left\langle \varphi\,,\,g^t\right\rangle_{\mathbb{D}}, \qquad \forall\, g^t \in H^{1/2}(\mathbb{D}), \tag{53}$$

$$\begin{cases} -\dfrac{1}{2}\displaystyle\int_{\mathbb{D}\times\mathbb{D}} LNK_s(\mathbf{x},\mathbf{y})\,(g(\mathbf{x})-g(\mathbf{y}))\,(g^t(\mathbf{x})-g^t(\mathbf{y}))\,d_{\mathbb{D}}(\mathbf{y})d_{\mathbb{D}}(\mathbf{x}) \\ \qquad +\dfrac{4}{\pi}\displaystyle\int_{\mathbb{D}} \dfrac{g(\mathbf{x})g^t(\mathbf{x})}{\sqrt{(1-\rho(\mathbf{x})^2)}}\,d_{\mathbb{D}}(\mathbf{x}) = \left\langle \varphi\,,\,g^t\right\rangle_{\mathbb{D}}, \qquad \forall\, g^t \in H^{1/2}(\mathbb{D}). \end{cases} \tag{54}$$

These formulations in turn yield two expressions for the $H^{1/2}(\mathbb{D})$-norm:

Theorem

$$\langle LS_{as}\,\mathbf{curl}_{\mathbb{D}}\, g\,,\,\mathbf{curl}_{\mathbb{D}}\, g\rangle_{\mathbb{D}} \;\geq\; C\,\|g\|^2_{H^{1/2}(\mathbb{D})}\,, \qquad \forall\, g\in H^{1/2}(\mathbb{D}). \tag{55}$$

$$\begin{cases} -\dfrac{1}{2}\displaystyle\int_{\mathbb{D}\times\mathbb{D}} LNK_s(\mathbf{x},\mathbf{y})\,(g(\mathbf{x})-g(\mathbf{y}))^2 d_{\mathbb{D}}(\mathbf{y}) d_{\mathbb{D}}(\mathbf{x}) \\ +\dfrac{4}{\pi}\displaystyle\int_{\mathbb{D}} \dfrac{(g(\mathbf{x}))^2}{\sqrt{(1-\rho(\mathbf{x})^2)}} d_{\mathbb{D}}(\mathbf{x}) \;\;\geq\; C\,\|g\|^2_{H^{1/2}(\mathbb{D})}\,. \end{cases} \tag{56}$$

Antisymmetric problem and hypersingular operator

The solution of the antisymmetric Dirichlet and Neumann solutions are given via the **double layer potential** given by

$$LD_{as}(u(\mathbf{y})) = -\frac{1}{4\pi}\int_{\mathbb{D}} \frac{y_3}{\|\mathbf{x}-\mathbf{y}\|^3} g(\mathbf{x})\, d_{\mathbb{D}}(\mathbf{x}) \tag{57}$$

The solution of the antisymmetric Dirichlet problem is retrieved using the double layer potential (57) with the data g which also give the solution in R^3.

$$u(\mathbf{y}) = -\frac{1}{4\pi}\int_{\mathbb{D}} \frac{y_3}{\|\mathbf{x}-\mathbf{y}\|^3} g(\mathbf{x})\, d_{\mathbb{D}}(\mathbf{x}), \qquad \mathbf{y} \in R^3. \tag{58}$$

The solution of the Neumann problem is obtained via first solving the following hypersingular integral equation on $\mathbb{D}$: find φ such that

$$-\frac{1}{4\pi}\oint_{\mathbb{D}} \frac{1}{\|\mathbf{x}-\mathbf{y}\|^3} g(\mathbf{x}) d_{\mathbb{D}}(\mathbf{x}) = \varphi(\mathbf{y}), \qquad \text{for} \quad \mathbf{y} \in \mathbb{D}, \tag{59}$$

where the modified integral is understood as either a finite part integral for sufficiently regular g or in a weak sense for functions in Sobolev spaces. Then the double layer potential (58) gives the solution in R^3.

We denote by LN_{as} the hyper singular integral operator associated to the equation (59) and by LKN_{as} its kernel. We denote by LS_{as} the integral operator which is the inverse of LN_{as} and by LKS_{as} its kernel.

Theorem

A symmetric variational formulation for (59) in the Hilbert space $\widetilde{H}^{1/2}(\mathbb{D})$ is

$$\big\langle LS_s\,\mathbf{curl}_{\mathbb{D}}\,g\,,\,\mathbf{curl}_{\mathbb{D}}\,g^t\big\rangle_{\mathbb{D}} \;=\; \big\langle \varphi\,,\,g^t\big\rangle_{\mathbb{D}}\,,\qquad \forall\, g^t \in \widetilde{H}^{1/2}(\mathbb{D}). \tag{60}$$

Moreover, this bilinear form is coercive, i.e.

$$\langle LS_s\,\mathbf{curl}_{\mathbb{D}}\,g\,,\,\mathbf{curl}_{\mathbb{D}}\,g\rangle_{\mathbb{D}}\,;\geq\; C\,\|g\|^2_{\widetilde{H}^{1/2}(\mathbb{D})}\,,\qquad \forall\, g \in \widetilde{H}^{1/2}(\mathbb{D}). \tag{61}$$

The associated operator, $\mathcal{D}_{as}$ (Corollary 7), is a bijection from the space $\widetilde{H}^{1/2}(\mathbb{D})$ to $H^{-1/2}(\mathbb{D})$.

Theorem

This operator admits an alternative variational formulation:

$$\begin{cases} \dfrac{1}{8\pi}\displaystyle\int_{\mathbb{D}\times\mathbb{D}} \dfrac{(g(\mathbf{x})-g(\mathbf{y}))\left(g^t(\mathbf{x})-g^t(\mathbf{y})\right)}{\|\mathbf{x}-\mathbf{y}\|^3} d_{\mathbb{D}}(\mathbf{x}) d_{\mathbb{D}}(\mathbf{y}) \\ +\dfrac{1}{\pi}\displaystyle\int_{\mathbb{D}} \dfrac{E(\rho(\mathbf{x}))g(\mathbf{x})g^t(\mathbf{x})}{(1-\rho(\mathbf{x})^2)} d_{\mathbb{D}}(\mathbf{x}); = \ \left\langle \varphi\, ,\, g^t \right\rangle_{\mathbb{D}}, \forall g^t \in \widetilde{H}^{1/2}(\mathbb{D}). \end{cases} \tag{62}$$

$$\begin{cases} \dfrac{1}{8\pi}\displaystyle\int_{\mathbb{D}\times\mathbb{D}} \dfrac{(g(\mathbf{x})-g(\mathbf{y}))^2}{\|\mathbf{x}-\mathbf{y}\|^3} d_{\mathbb{D}}(\mathbf{x}) d_{\mathbb{D}}(\mathbf{y}) \\ +\dfrac{1}{\pi}\displaystyle\int_{\mathbb{D}} \dfrac{E(\rho(\mathbf{x}))(g(\mathbf{x}))^2}{(1-\rho(\mathbf{x})^2)} d_{\mathbb{D}}(\mathbf{x}) \ \geq C\, \|g\|^2_{\widetilde{H}^{1/2}(\mathbb{D})}, \quad \forall\, g \in \widetilde{H}^{1/2}(\mathbb{D}). \end{cases} \tag{63}$$

where the **elliptic function** $E(\rho)$ *is given by*

$$E(\rho) = \int_0^{\frac{\pi}{2}} \sqrt{1-\rho^2 sin^2(\alpha)} d\alpha. \tag{64}$$

Theorem

The operator LS_{as} is symmetric and coercive in $H^{1/2}(\mathbb{D})$. It is associated to the operator $\mathcal{N}_{as} = \mathcal{D}_{as}^{-1}$ (cf. Corollary 12) and is a bijection of $H^{-1/2}(\mathbb{D})$ onto $\widetilde{H}^{1/2}(\mathbb{D})$, symmetric and coercive. It admits the following variational formulation:

$$\left\langle LS_{as}\varphi\,,\,\varphi^t\right\rangle_{\mathbb{D}} \;=\; \left\langle g\,,\,\varphi^t\right\rangle_{\mathbb{D}}\,, \qquad \forall\,\phi \in H^{-1/2}(\mathbb{D}), \tag{65}$$

and thus, provides a norm on the space $H^{-1/2}(\mathbb{D})$

$$\langle LS_{as}\varphi\,,\,\varphi\rangle \;\geq\; C\,\|\varphi\|^2_{H^{-1/2}(\mathbb{D})}\,, \qquad \forall\,\varphi \in H^{-1/2}(\mathbb{D}). \tag{66}$$

Decomposition on basis functions

We have introduced the four symmetric integral operators $LS_s, LS_{as}, LN_s, LN_{as}$ related to the Laplace equation on the disc $\mathbb{D}$, such that $LN_s \circ LS_s = I$, $LS_{as} \circ LN_{as} = I$. We denote the associated kernels by $LK_s, LK_{as}, LNK_s, LNK_{as}$.
The two kernels LK_s, LNK_{as} are known and only depends on $\mathbf{x} - \mathbf{y}$.
The two others kernels LK_{as}, LNK_s, associated to the inverse of the operators LN_{as}, LS_s, are not the restriction on $\mathbb{D}$ of kernels defined in the space R^3.
They depends symmetrically on the variables $\mathbf{x}$ and $\mathbf{y}$, but not only on $\mathbf{x} - \mathbf{y}$.
The kernel of the operator LS_s which is $\dfrac{1}{4\pi}\dfrac{1}{\|\mathbf{x}-\mathbf{y}\|}$, is related to the kernel associated to the operator LN_{as} which is $-\dfrac{1}{4\pi}\dfrac{1}{\|\mathbf{x}-\mathbf{y}\|^3}$, via the identity

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|^3} = \Delta_{\mathbb{D}}\Big(\frac{1}{\|\mathbf{x}-\mathbf{y}\|}\Big). \tag{67}$$

Spherical Harmonics, Associated Legendre functions

In order to obtain some explicit expressions of these kernels and also some links between them, we introduce some basis functions related to the well known spherical harmonics. These spherical harmonics functions, define on the sphere S of radius one associated to the disk $\mathbb{D}$ as an equatorial plan. The spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator also define on the sphere S. We introduce here the spherical harmonics and the associated kinetic moments.

The Rodrigues formula gives the expression of the Legendre polynomial $\mathbb{P}_l$:

$$\mathbb{P}_l(x) = \frac{(-1)^l}{2^l l!}\left(\frac{d}{dx}\right)^l (1-x^2)^l. \tag{68}$$

The Spherical Harmonics are the functions $Y_l^m(x,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(x)$, solutions with separate variables of the differential equation ($x = x_3$)

$$\frac{1}{1-x^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial}{\partial x}u\right) + l(l+1)u = 0. \tag{69}$$

$$Y_l^m(x,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(x) \tag{70}$$

The functions $\mathbb{P}_l^m(x)$, called the Associated Legendre functions, are the solutions of the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\mathbb{P}_l^m\right)+l(l+1)\mathbb{P}_l^m-\frac{m^2}{1-x^2}\mathbb{P}_l^m=0. \tag{71}$$

For $m=0$, Y_l^0 is the Legendre polynomial $\mathbb{P}_l$.
In order to describe the functions Y_l^m, we introduce the kinetic moments L_+, L_-, L_3, express in the angles $(\theta,\varphi),(x_3=\cos(\theta))$

$$L_3 u=\frac{1}{i}\frac{\partial}{\partial\varphi}u. \tag{72}$$

$$L_+ u=e^{i\varphi}\left(\frac{\partial}{\partial\theta}u+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{73}$$

$$L_- u=e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}u+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{74}$$

The kinetic moments L_+, L_-, L_3, satisfy the relations of commutation:

$$[L_+,L_-]=2L_3, \qquad \text{where} \quad [A,B]=AB-BA. \tag{75}$$

$$[L_3,L_+]=L_+, \qquad [L_3,L_-]=-L_-, \tag{76}$$

The Laplace-Beltrami operator Δ_S is then $\Delta_S = -\frac{1}{2}(L_+L_- + L_-L_+) - (L_3)^2$ and the following relation of commutation hold:

$$[\Delta_S, L_+] = [\Delta_S, L_-] = [\Delta_{\mathbb{S}}, L_3] = 0. \tag{77}$$

This relations of commutation (77) show that each eigenspace of the operator Δ_S is invariant by the action of the operators L_+, L_- and L_3. So the spherical harmonics of order l are the $2l+1$ solutions of the equation (69) of the form

$$Y_l^m(\theta,\varphi) = \left[\frac{(l+1/2)}{2\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} e^{im\varphi}\mathbb{P}_l^m(\cos\theta). \tag{78}$$

The associated Legendre $\mathbb{P}_l^m(\cos\theta)$ are define using the Legendre functions

$$\begin{cases} \mathbb{P}_l^m(\cos\theta) = (\sin\theta)^m \left(\dfrac{d}{dx}\right)^m \mathbb{P}_l(\cos\theta);\ \text{if}\ \ 0 \le m \le l, \\ \mathbb{P}_l^{-m}(x) = (-1)^m \dfrac{(l-m)!}{(l+m)!}\mathbb{P}_l^m(x),\ \text{if}\ -l \le m \le l, \\ \mathbb{P}_l^m(\cos\theta) = \dfrac{(-1)^{l+m}}{2^l l!}\dfrac{(l+m)!}{(l-m)!}(\sin\theta)^{-m}\left(\dfrac{d}{dx}\right)^{l-m}(1-x^2)^l. \\ \mathbb{P}_l^m(x) = \dfrac{(-1)^{l+m}}{2^l l!}(1-x^2)^{m/2}\left(\dfrac{d}{dx}\right)^{l+m}(1-x^2)^l. \end{cases} \tag{79}$$

They satisfies

$$\begin{cases} \displaystyle\int_{-1}^{+1} (\mathbb{P}_l(x))^2\, dx = \frac{1}{l+1/2}; \\ \mathbb{P}_l(1) = 1,\ l \geq 0; \quad \mathbb{P}_l(0) = 0,\ l \text{ odd}; \quad \mathbb{P}_{2l}(0) = \dfrac{(-1)^l}{2^l l!},\ l \geq 0. \end{cases} \tag{80}$$

Their parity is $l + m$. They satisfy the following orthogonality relations

$$\int_{-1}^{+1} \mathbb{P}_{l_1}^m(x)\mathbb{P}_{l_2}^m(x)dx = 0, \quad \text{if } l_1 \neq l_2, \tag{81}$$

$$\int_{-1}^{+1} \frac{\mathbb{P}_l^{m_1}(x)\mathbb{P}_l^{m_2}(x)}{1-x^2}dx = 0, \quad \text{if } m_1 \neq m_2 \text{ and } m_1 \neq -m_2. \tag{82}$$

$$\int_S \frac{Y_l^{m_1}(\theta,\varphi)\overline{Y_l^{m_2}}(\theta,\varphi)}{\sin\theta}d\theta d\varphi = 0, \quad \text{if } m_1 \neq -m_2. \tag{83}$$

The functions Y_l^m are the eigenvalues of the Laplace-Beltrami operator $-\Delta_S$ defined on S. They satisfy the following orthogonality relations:

$$\int_S Y_{l_1}^{m_1}(\theta,\varphi)\overline{Y_{l_2}^{m_2}}(\theta,\varphi)\sin\theta d\theta d\varphi = \delta_{l_1}^{l_2}\delta_{m_1}^{m_2}. \tag{84}$$

$$\int_S (\overrightarrow{\mathbf{grad}}_{\mathbb{S}} Y_{l_1}^{m_1}(\theta,\varphi)\cdot\overrightarrow{\mathbf{grad}}_{\mathbb{S}}\overline{Y_{l_2}^{m_2}}(\theta,\varphi))\sin\theta d\theta d\varphi = 0,\ m_1\neq m_2,\ l_1\neq l_2. \tag{85}$$

$$-\Delta_S Y_l^m = l(l+1)Y_l^m,\quad L_3 Y_l^m = mY_l^m, \tag{86}$$

$$L_+ Y_l^m = \sqrt{(l-m)(l+m+1)}\,Y_l^{m+1},\ L_- Y_l^m = \sqrt{(l+m)(l-m+1)}\,Y_l^{m-1}. \tag{87}$$

$$\begin{cases} (2l+1)\xi P_l^m = (l-m+1)\ P_{l+1}^m(\xi) + (l+m)\ P_{l-1}^m(\xi); \\ (1-\xi^2)\dfrac{\partial P_l^m}{\partial\xi} = \dfrac{1}{2l+1}\Big((l+1)(l+m)\ P_{l-1}^m(\xi) - l(l-m+1)\ P_{l+1}^m(\xi)\Big); \\ \sqrt{(1-\xi^2)}\dfrac{\partial P_l^m}{\partial\xi} = \dfrac{1}{2}\Big((l-m+1)(l+m)\ P_l^{m-1}(\xi) - P_l^{m+1}(\xi)\Big); \end{cases} \tag{88}$$

Operators on the disc

We associate to the functions $U_s(\mathbf{x}^+)$ and $U_{as}(\mathbf{x}^+)$), defined on the sphere $\mathbb{S}^+$ (variables: θ, φ), the functions $u_s(\mathbf{x})$ and $u_{as}(\mathbf{x})$ defined on the disc $\mathbb{D}$ (variables: $\rho = \sin(\theta)$, φ, $0 \leq \theta \leq \frac{\pi}{2}$), where $\mathbf{x}$ is the projection on the disc of the vector $\mathbf{x}^+$. We define the following vectors $\overrightarrow{\mathbf{grad}}_{\mathbb{D}}$ and $\overrightarrow{\mathbf{curl}}_{\mathbb{D}}$ as

$$\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}) = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi \tag{89}$$

$$\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}) = -\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\rho + \frac{\partial u}{\partial \rho}\mathbf{e}_\varphi \tag{90}$$

We define the operators $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_3$ of derivation on the disc

$$\begin{cases} \mathcal{L}_+ u = e^{i\varphi}\Big(\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\[2ex] \mathcal{L}_- u = e^{-i\varphi}\Big(-\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\[2ex] \mathcal{L}_3 u = \dfrac{1}{i}\dfrac{\partial u}{\partial \varphi} \end{cases} \tag{91}$$

They trivially satisfy

$$\overline{\mathcal{L}_+ u} = -\mathcal{L}_- \overline{u}; \qquad \overline{\mathcal{L}_- u} = -\mathcal{L}_+ \overline{u}; \qquad \overline{\mathcal{L}_3 u} = -\mathcal{L}_3 \overline{u} \tag{92}$$

When $u = 0$ or $v = 0$ on the circle $\mathfrak{c}$, an integration by part give the result

$$\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial u}{\partial\rho} + i\frac{1}{\rho}\frac{\partial u}{\partial\varphi}\Big) v\,\rho d\rho d\varphi = -\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial v}{\partial\rho} + i\frac{1}{\rho}\frac{\partial v}{\partial\varphi}\Big) u\,\rho d\rho d\varphi \tag{93}$$

which means that the operators $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_3$ are formally anti-adjoint with respect to the duality in $L^2(\mathbb{D})$.

$$\Delta_{\mathbb{D}} = -\frac{1}{2}\Big(\mathcal{L}_+ \mathcal{L}_- + \mathcal{L}_- \mathcal{L}_+\Big) = \Big(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\Big) \tag{94}$$

We have

$$\left\{\begin{array}{l} \Big(\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{curl}}_{\mathbb{D}} v(\mathbf{y})\Big) = \Big(\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{grad}}_{\mathbb{D}} v(\mathbf{y})\Big) \\ = -\dfrac{1}{2}\Big(\mathcal{L}_+ u(\mathbf{x})\,\mathcal{L}_- v(\mathbf{y}) + \mathcal{L}_- u(\mathbf{x})\,\mathcal{L}_+ v(\mathbf{y})\Big) \end{array}\right. \tag{95}$$

Images of the Spherical Harmonics

The parity of the Spherical Harmonics Y_l^m with respect to the variable $x = \cos(\theta)$ is the parity of $l+m$. Thus the vectorial space $\mathbb{Y}$ generated by the Spherical Harmonics $Y_l^m; 0 \leq l; -l \leq m \leq l,$ can be split into two **subspaces** $\mathbb{Y}_s$ **and** $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$ which are respectively :

$$\mathbb{Y}_s = \{Y_l^m; 0 \leq l; \quad -l \leq m \leq l; \quad l+m \quad \text{even}\}$$
$$\mathbb{Y}_{as} = \{Y_l^m; 1 \leq l; \quad -l+1 \leq m \leq l-1; \quad l+m \quad \text{odd}\}$$

The Spherical Harmonics functions $Y_{l_1}^{m_1}$ are an orthogonal basis and thus

$$\int_{\mathbb{S}^+} \Big(Y_{l_1}^{m_1}(\mathbf{x}) \overline{Y}_{l_2}^{m_2}(\mathbf{x}) \Big) \sin(\theta) d\theta d\varphi = \frac{1}{2} \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{96}$$

$$\int_{\mathbb{S}^+} \Big(\overrightarrow{\mathbf{grad}}_{\mathbb{S}} Y_{l_1}^{m_1}(\mathbf{x}) . \overrightarrow{\mathbf{grad}}_{\mathbb{S}} \overline{Y}_{l_2}^{m_2}(\mathbf{x}) \Big) \sin(\theta) d\theta d\varphi = \frac{1}{2} l(l+1) \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{97}$$

$$\int_{\mathbb{S}^+} \Big(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} Y_{l_1}^{m_1}(\mathbf{x}) . \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{Y}_{l_1}^{m_1}(\mathbf{x}) \Big) \sin(\theta) d\theta d\varphi = \frac{1}{2} l(l+1) \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{98}$$

We introduce now the functions y_l^m defined on the disc $\mathbb{D}$, images of the Spherical Harmonics, which are

$$\begin{cases} y_l^m(\rho,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(\sqrt{(1-\rho^2)}); \quad y_l^{-m}(\rho,\varphi) = (-1)^m\overline{y_l^m(\rho,\varphi)}, \\ \rho^2 = x_1^2 + x_2^2, \quad \gamma_l^m = \left[\dfrac{(l+1/2)}{2\pi}\dfrac{(l-m)!}{(l+m)!}\right]^{1/2}, \quad \xi = \sqrt{1-\rho^2}, \\ y_l^m(x_1,x_2) = C_l^m(x_1+ix_2)^m(\dfrac{d}{d\xi})^{l+m}(1-\xi^2)^l, \quad \xi = \sqrt{1-(x_1^2+x_2^2)}, \\ C_l^m = (-1)^m\left(\dfrac{(l+1/2)}{2\pi}\dfrac{(l-m)!}{(l+m)!}\right)^{1/2}\dfrac{(-1)^l}{2^l l!}. \end{cases} \tag{99}$$

We associated to the two subspaces $\mathbb{Y}_s$ and $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$, the corresponding subspaces on the disc $\mathbb{D}$:

$$\mathcal{Y}_s = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_s\}$$

$$\mathcal{Y}_{as} = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_{as}\}$$

$$\begin{cases} y_0^0(\rho,,\varphi)=\sqrt{\dfrac{1}{4\pi}};\ y_1^1(\rho,,\varphi)=-\sqrt{\dfrac{3}{8\pi}}e^{i\varphi}\rho;\ y_1^{-1}(\rho,,\varphi)=\sqrt{\dfrac{3}{8\pi}}e^{-i\varphi}\rho; \\ y_1^0(\rho,,\varphi)=\sqrt{\dfrac{3}{4\pi}}\sqrt{(1-\rho^2)};\quad y_2^0(\rho,,\varphi)=\sqrt{\dfrac{5}{16\pi}}(2-3\rho^2) \\ y_2^1(\rho,,\varphi)=-\sqrt{\dfrac{15}{8\pi}}\rho\sqrt{(1-\rho^2)}e^{i\varphi};\ y_2^{-1}(\rho,,\varphi)=\sqrt{\dfrac{15}{8\pi}}\rho\sqrt{(1-\rho^2)}e^{-i\varphi}. \end{cases} \tag{100}$$

Using (96), we obtain, in each subspace $\mathcal{Y}_s$ and $\mathcal{Y}_{as}$, the orthogonal identity

$$\int_{\mathbb{D}} \frac{y_{l_1}^{m_1}(\mathbf{x})\overline{y}_{l_2}^{m_2}(\mathbf{x})}{\sqrt{(1-\rho^2)}}\rho d\rho d\varphi = \frac{1}{2}\delta_{l_1}^{l_2}\delta_{m_1}^{m_2}, \tag{101}$$

Remark

The two subspaces of spherical harmonics $\mathbb{Y}_s$ and $\mathbb{Y}_{as}$ are mutually orthogonal on the sphere $\mathbb{S}$, but this is not the case on the half sphere $\mathbb{S}^+$. Thus the two subspaces $y_l^m \in \mathcal{Y}_s$ and $y_l^m \in \mathcal{Y}_{as}$ are not mutually orthogonal on the disk $\mathbb{D}$.

Decomposition on basis functions

Let define the space: $L^2_{\frac{1}{w}}(\mathbb{D}) = \{u(\mathbf{x}), \frac{u^2}{w} \in L^1(\mathbb{D})\}$, associated to the weight $w(\mathbf{x}) = \sqrt{1-\rho(\mathbf{x})^2}$. Then both sets $\{y_l^m \in \mathcal{Y}_s\}$ and $\{y_l^m \in \mathcal{Y}_{as}\}$ are an orthogonal basis in the space $L^2_{\frac{1}{w}}(\mathbb{D})$.

Due to the properties of the associated Legendre functions, the functions in the space $\mathcal{Y}_s$ have a bounded non zero value and a bounded normal derivative closed to the circle $\mathfrak{c}$.

The functions in the space $\mathcal{Y}_{as}$ have closed to the circle $\mathfrak{c}$, a value which goes to zero as $\sqrt{(1-\rho^2}$ and a normal derivative which explodes as $\dfrac{1}{\sqrt{(1-\rho^2}}$.

A function u_{as} in the space $\mathcal{Y}_{as}$ can be extended on the basis $\{y_l^m\}$ which is an orthogonal basis in the weighted space $L^2_{\frac{1}{w}}(\mathbb{D})$ and a basis in the space $H^1_0(\mathbb{D})$.

$$u_{as} = \sum_{1\le l} \sum_m u_l^m y_l^m; \quad -l+1 \le m \le l-1; \quad l+m \quad \text{odd} \tag{102}$$

A function u_s in the space $\mathcal{Y}_s$ can be extended on the basis $\{y_l^m\}$ which is an orthogonal basis in the weighted space $L^2_{\frac{1}{w}}(\mathbb{D})$ and a basis in the space $H^1(\mathbb{D})$.

$$u_s = \sum_{1\leq l}\sum_{m} u_l^m y_l^m; \quad -l \leq m \leq l; \quad l+m \quad \text{even} \tag{103}$$

We consider the associated weighted space: $L^2_w(\mathbb{D}) = \{u(\mathbf{x}), wu^2 \in L^1(\mathbb{D})\}$.
Then both sets $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_s\}$ and $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_{as}\}$ are an orthogonal basis in the space $L^2_w(\mathbb{D})$. A function u can be extended on the basis $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_s\}$ which is an orthogonal basis in the weighted space $L^2_w(\mathbb{D})$

$$u = \sum_{0\leq l}\sum_{m} u_l^m \frac{y_l^m}{w}; \quad -l \leq m \leq l; \quad l+m \quad \text{even} \tag{104}$$

A function u can be also extended on the basis $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_{as}\}$ which is an orthogonal basis in the weighted space $L^2_w(\mathbb{D})$ and a basis in $L^2(\mathbb{D})$.

$$u = \sum_{1\leq l}\sum_{m} u_l^m \frac{y_l^m}{w}; \quad -l+1 \leq m \leq l-1; \quad l+m \quad \text{odd} \tag{105}$$

Operators associated to the Laplace equation

We denote as $\mathcal{L}\mathcal{S}_s, \mathcal{L}\mathcal{S}_{as}, \mathcal{L}\mathcal{N}_s, \mathcal{L}\mathcal{N}_{as}$ the integral operators associated to the Laplace equation in the exterior of the disc $\mathbb{D}$.

The kernel associated to the operator $\mathcal{L}\mathcal{S}_{as}$ is $\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}$ while the kernel associated to the operator $\mathcal{L}\mathcal{N}_{as}$ is $\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|^3} = -\frac{1}{4\pi}\Delta_{\mathbb{D}}(\frac{1}{|\mathbf{x}-\mathbf{y}|})$.

So in order to feet with the above variational formulations (53) and (60) and the properties of the kinetic moments, we defined the operator LN_s as

$$LN_s = -\frac{1}{2}\Big(\mathcal{L}_- \circ LS_{as} \circ \mathcal{L}_+ + \mathcal{L}_+ \circ LS_{as} \circ \mathcal{L}_-\Big) \tag{106}$$

and the operator LS_s as the solution of the equation

$$LN_{as} = -\frac{1}{2}\Big(\mathcal{L}_- \circ LS_s \circ \mathcal{L}_+ + \mathcal{L}_+ \circ LS_s \circ \mathcal{L}_-\Big) \tag{107}$$

Expression of the kernels

S. Krenk and P. A. Martin [2] have showed that the operator LN_{as} satisfies

$$\begin{cases} LN_{as} y_l^m = -\alpha_l^m \dfrac{y_l^m}{w(\mathbf{x})}; \quad -l+1 \leq m \leq l-1;\ l+m \quad \text{odd}; \quad l \geq 1; \\ \alpha_l^m = \dfrac{\Gamma(\frac{l+m+2}{2})\,\Gamma(\frac{l-m+2}{2})}{(\frac{l+m-1}{2})!\,(\frac{l-m-1}{2})!} = (\dfrac{(l+1)^2 - m^2}{4})\dfrac{\Gamma(\frac{l+m+2}{2})\,\Gamma(\frac{l-m+2}{2})}{(\frac{l+m+1}{2})!\,(\frac{l-m+1}{2})!}; \end{cases} \tag{108}$$

The function Gamma (denoted as Γ) of the complex variable, is

$$\begin{cases} \Gamma(z+1) = z\Gamma(z); \quad \Gamma(1-z)\Gamma(z) = \dfrac{\pi}{\sin(\pi z)}; \quad \Gamma(\dfrac{1}{2}) = \sqrt{\pi}: \\ \Gamma(n+\dfrac{1}{2}) = \sqrt{\pi}\,\dfrac{(2n)!}{2^{2n} n!} = \sqrt{\pi}\,\dfrac{(2n-1)!}{2^{2n-1}(n-1)!} = \dfrac{\sqrt{\pi}}{2^n}\Big(\Pi_{i=1}^{n-1}(2i+1)\Big) \end{cases} \tag{109}$$

We will give now an exact expression of these kernels, using an adequate expansion on the basis functions associated to the spaces $\mathcal{Y}_s$ and $\mathcal{Y}_{as}$.

Theorem

For all $\mathbf{x}, \mathbf{y} \in D \times D$, *(* $\mathbf{x} \neq \mathbf{y}$*),* LK_s, LK_{as}, LNK_s, LNK_{as} *admits the expansions:*

$$LK_s(\mathbf{x},\mathbf{y}) = \sum_{0\le l}\sum_{m} \frac{1}{\beta_l^m}\left(y_l^m(\mathbf{x})\overline{y_l^m}(\mathbf{y}) + \overline{y_l^m}(\mathbf{x})y_l^m(\mathbf{y})\right); -l \le m \le l;\ l+m \text{ even}. \quad (110)$$

$$LK_{as}(\mathbf{x},\mathbf{y}) = \sum_{0\le l}\sum_{m} \frac{1}{\alpha_l^m}\left(y_l^m(\mathbf{x})\overline{y_l^m}(\mathbf{y}) + \overline{y_l^m}(\mathbf{x})y_l^m(\mathbf{y})\right); -l+1 \le m \le l-1; l+m \text{ odd}. \quad (111)$$

$$LNK_s(\mathbf{x},\mathbf{y}) = -\sum_{0\le l}\sum_{m} \beta_l^m\left(\frac{y_l^m(\mathbf{x})}{w(\mathbf{x})}\frac{\overline{y_l^m}(\mathbf{y})}{w(\mathbf{y})} + \frac{\overline{y_l^m}(\mathbf{x})}{w(\mathbf{x})}\frac{y_l^m(\mathbf{y})}{w(\mathbf{y})}\right); -l \le m \le l;\ l+m \text{ even}. \quad (112)$$

$$LNK_{as}(\mathbf{x},\mathbf{y}) = -\sum_{0\le l}\sum_{m} \alpha_l^m\left(\frac{y_l^m(\mathbf{x})}{w(\mathbf{x})}\frac{\overline{y_l^m}(\mathbf{y})}{w(\mathbf{y})} + \frac{\overline{y_l^m}(\mathbf{x})}{w(\mathbf{x})}\frac{y_l^m(\mathbf{y})}{w(\mathbf{y})}\right); -l+1 \le m \le l-1;\ l+m \text{ odd}. \quad (113)$$

Using (87) we obtain the identities (114) which associated to (106) , and to

$$\begin{cases} \mathcal{L}_+ \, y_l^m = \sqrt{(l-m)(l+m+1)}\,\dfrac{y_l^{m+1}}{\sqrt{(1-\rho^2)}}; \\ \mathcal{L}_- \, y_l^m = \sqrt{(l+m)(l-m+1)}\,\dfrac{y_l^{m-1}}{\sqrt{(1-\rho^2)}}; \end{cases} \tag{114}$$

$\alpha_1^0 = \dfrac{\pi}{4}$, $\beta_1^1 = \beta_1^{-1} = \dfrac{4}{\pi}$, $\beta_1^1 \alpha_1^0 = 1$ leads to the links between α_l^m and β_l^m:

$$\begin{cases} \beta_l^m = \dfrac{1}{2}\Big(\dfrac{(l+m)(l-m+1)}{\alpha_l^{m-1}} + \dfrac{(l-m)(l+m+1)}{\alpha_l^{m+1}}\Big); \quad l+m \text{ even} \\ \alpha_l^m = \dfrac{1}{2}\Big(\dfrac{(l+m)(l-m+1)}{\beta_l^{m-1}} + \dfrac{(l-m)(l+m+1)}{\beta_l^{m+1}}\Big); \quad l+m \text{ odd} \end{cases} \tag{115}$$

Using (115), we obtain others expressions for α_l^m and β_l^m :

$$\begin{cases} \alpha_l^m = \dfrac{\pi}{4}\Big(\Pi_{i=1}^{(l+m-1)/2}\big(\dfrac{2i+1}{2i}\big)\Big)\Big(\Pi_{i=1}^{(l-m-1)/2}\big(\dfrac{2i+1}{2i}\big)\Big); \quad l+m \text{ odd} \\ \beta_l^m = \dfrac{4}{\pi}\Big(\Pi_{i=1}^{(l+m)/2}\big(\dfrac{2i}{2i-1}\big)\Big)\Big(\Pi_{i=1}^{(l-m)/2}\big(\dfrac{2i}{2i-1}\big)\Big); \quad l+m \text{ even} \end{cases} \tag{116}$$

Let the operator of x and y: $\Delta^*_{\mathbb{D}} = \frac{1}{2}(\mathcal{L}_-(\mathbf{x})\,\mathcal{L}_+(\mathbf{y}) + \mathcal{L}_+(\mathbf{x})\,\mathcal{L}_-(\mathbf{y}))$
The kernels LK_s and LNK_{as} and the kernels LK_{as} and LNK_s are linked by

$$-\Delta_{\mathbb{D}} LK_s = LNK_{as}; \qquad \Delta^*_{\mathbb{D}} LK_{as} = LNK_s \tag{117}$$

The kernels LK_s, LK_{as}, LNK_s and LNK_{as} satisfies the identity:

$$\begin{cases} \int_{\mathbb{D}} LK_s(\mathbf{x},\mathbf{y}) d_{\mathbb{D}}(\mathbf{x}) = \dfrac{1}{3\pi} \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-\rho(\mathbf{y})^2 sin^2(\alpha)}d\alpha; \\ \int_{\mathbb{D}} LK_{as}(\mathbf{x},\mathbf{y}) d_{\mathbb{D}}(\mathbf{x}) = \dfrac{4}{\pi}\sqrt{(1-\rho(\mathbf{y})^2)}. \\ \oint_{\mathbb{D}} LNK_s(\mathbf{x},\mathbf{y}) d_{\mathbb{D}}(\mathbf{x}) = -\dfrac{4}{\pi\sqrt{(1-\rho(\mathbf{y})^2)}} \\ \oint_{\mathbb{D}} LNK_{as}(\mathbf{x},\mathbf{y}) d_{\mathbb{D}}(\mathbf{x}) = -\dfrac{1}{\pi(1-\rho(\mathbf{y})^2)} \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1-\rho(\mathbf{y})^2 sin^2(\alpha)}d\alpha \end{cases} \tag{118}$$

Lemma

The kernel $LK_{as}(\mathbf{x},\mathbf{y})$ has the following value

$$LK_{as}(\mathbf{x},\mathbf{y}) = \frac{2}{\pi^2\|\mathbf{x}-\mathbf{y}\|}\arctan\Big(\frac{\sqrt{(1-\rho(\mathbf{x})^2)}\sqrt{(1-\rho(\mathbf{y})^2)}}{\|\mathbf{x}-\mathbf{y}\|}\Big). \tag{119}$$

while the kernel $LNK_s(\mathbf{x},\mathbf{y})$ has the following value

$$\begin{cases} LNK_s(\mathbf{x},\mathbf{y}) = \dfrac{2}{\pi^2\,\|\mathbf{x}-\mathbf{y}\|^3}\arctan\Big(\dfrac{\sqrt{(1-\rho(\mathbf{x})^2)}\sqrt{(1-\rho(\mathbf{y})^2)}}{\|\mathbf{x}-\mathbf{y}\|}\Big) \\ +\dfrac{\Big(1-2\rho(\mathbf{x})\rho(\mathbf{y})\cos(\Phi)+\rho(\mathbf{x})^2\rho(\mathbf{y})^2\cos(2\Phi)\Big)\sqrt{(1-\rho(\mathbf{y})^2)}\sqrt{(1-\rho(\mathbf{x})^2)}}{\pi^2\,\|\mathbf{x}-\mathbf{y}\|^2\,\Big(\|\mathbf{x}-\mathbf{y}\|^2+(1-\rho(\mathbf{x})^2)(1-\rho(\mathbf{y})^2)\Big)^2} \\ -\dfrac{\Big(1-2\rho(\mathbf{x})\rho(\mathbf{y})\cos(\Phi)+\rho(\mathbf{x})^2\rho(\mathbf{y})^2\cos(2\Phi)\Big)}{\pi^2\sqrt{(1-\rho(\mathbf{y})^2)}\sqrt{(1-\rho(\mathbf{x})^2)}\Big(\|\mathbf{x}-\mathbf{y}\|^2+(1-\rho(\mathbf{x})^2)(1-\rho(\mathbf{y})^2)\Big)^2}. \end{cases} \tag{120}$$

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