Riemannian and Finslerian geometry

for diffusion weighted magnetic resonance imaging

Luc Florack

Heuristics

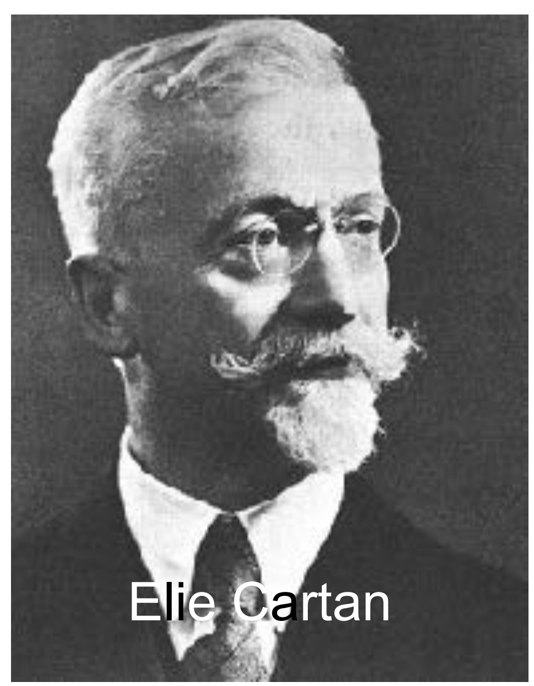
Finsler manifold.

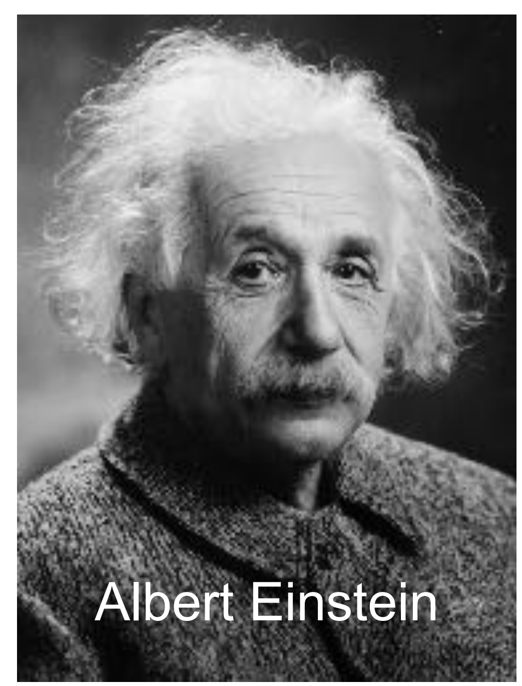
- A Finsler manifold is a space (M,F) of spatial base points $x \in M$, furnished with a notion of a ‘line’ or ‘length element’ $ds \doteq F(x,dx)$.
- The ‘infinitesimal displacement vectors’ dx are ‘infinitely scalable’ into finite ‘tangent’ or ‘velocity vectors’ $\dot{x}$, viz. $dx = \dot{x}\, dt$.
- The collection of all $(x, \dot{x})$ is called the tangent bundle TM over M.
- Integrating the line element along a curve C produces the ‘length’ of that curve:

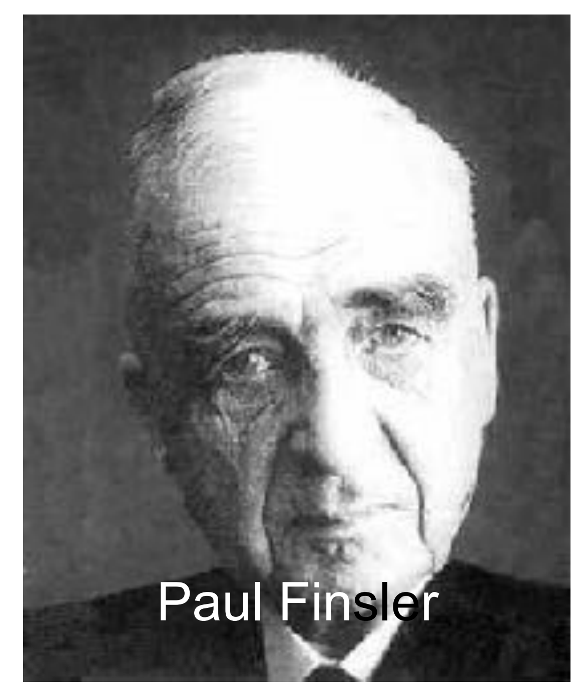
$$\mathscr{L}(C) = \int_C ds = \int_C F(x, dx)$$


Bernhard Riemann


Sophus Lie


Elie Cartan


Albert Einstein


Paul Finsler

Applications

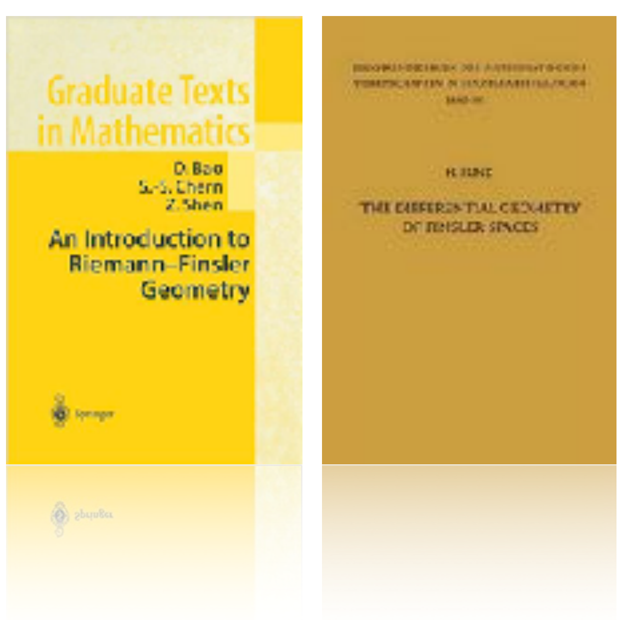
Examples (anisotropic media).

- Mechanics
  - e.g. F(x,dx) := infinitesimal displacement, or infinitesimal travel time, etc.
- Optimal control
  - e.g. F(x,dx) := local cost function for infinitesimal movement of Reeds-Shepp car
- Optics
  - e.g. F(x,dx) := infinitesimal travel time for light propagation
- Seismology
  - e.g. F(x,dx) := infinitesimal travel time for seismic ray propagation
- Ecology
  - e.g. F(x,dx) := infinitesimal energy for coral reef state transition
- Relativity
  - e.g. F(x,dx) := infinitesimal (pseudo-Finslerian) spacetime line element
- Diffusion MRI
  - e.g. F(x,dx) := infinitesimal hydrogen spin diffusion

Axiomatics

Literature.

- © David Bao et al.
- © Hanno Rund et al.



The Riemann-DTI paradigm

DWMRI signal attenuation.

$$E(x,q,\tau) = \exp\left[-\tau D(x,q,\tau)\right]$$

$$\left[q = \gamma \int \mathrm{g}(t)dt \right]$$

q-space variable

Propagator.

$$P(x,\xi,\tau) = \int_{\mathbb{R}^3} e^{2\pi i q\cdot\xi}\, E(x,q,\tau)\, dq$$

diffusion tensor

DTI. $E_{\rm DTI}(x,q,\tau) = \exp\left[-\tau D_{\rm DTI}(x,q,\tau)\right]$

Einstein ∑-convention

$$D_{\rm DTI}(x,q,\tau) = D^{ij}(x) q_i q_j$$

quadratic assumption…

$$P_{\rm DTI}(x,\xi,\tau) = \int_{\mathbb{R}^3} e^{2\pi i q\cdot\xi}\, E_{\rm DTI}(x,q,\tau)\, dq = \frac{1}{\sqrt{4\pi\tau^2}^3} \exp\left[-\frac{1}{4\tau} D_{ij}(x)\xi^i\xi^j\right]$$

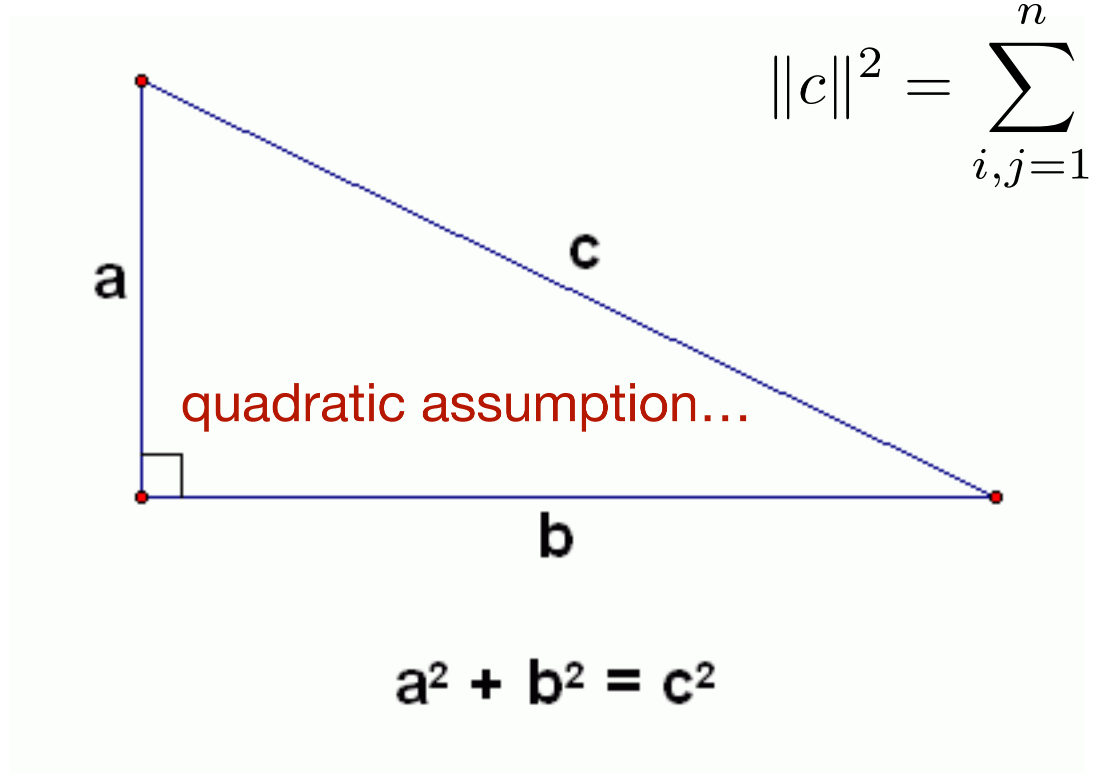
another quadratic form…

$$D_{ik}(x)D^{kj}(x) = \delta^i_j$$

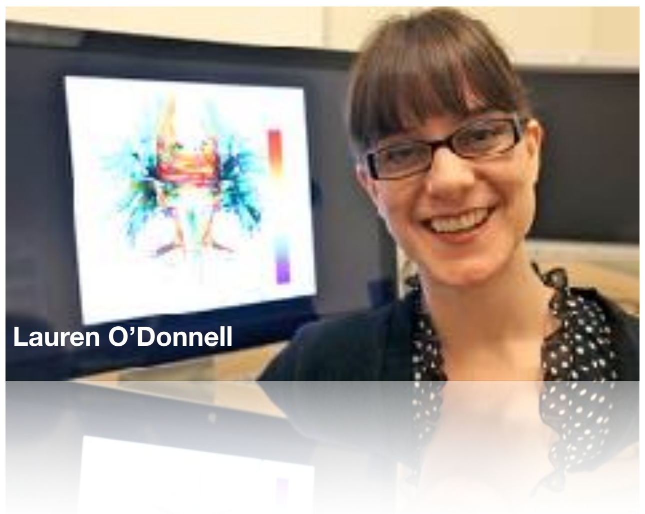
inverse diffusion tensor

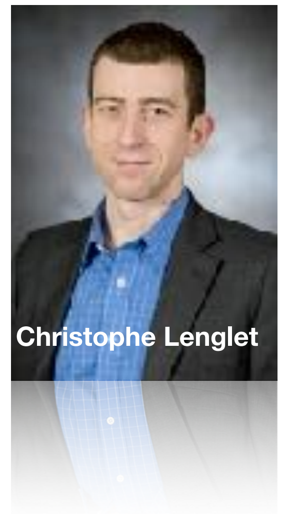
The Riemann-DTI paradigm

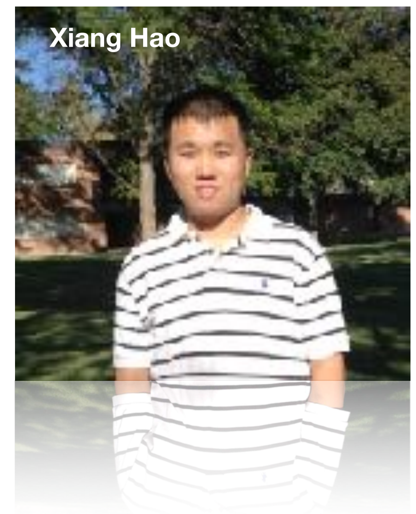
$$\|c\|^2 = \sum_{i,j=1}^{n} g_{ij} c^i c^j$$

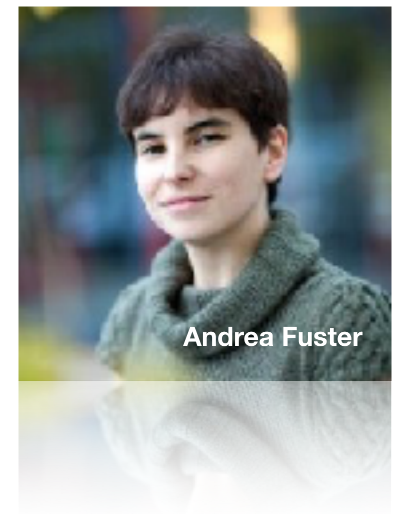


The Riemann-DTI paradigm









Ansatz [1,2].

$$g_{ij}(x) = D_{ij}(x)$$

Riemann metric tensor

$$D_{ik}(x)D^{kj}(x) = \delta^i_j$$

inverse diffusion tensor

Adaptations [3,4].

$$g_{ij}(x) = e^{\alpha(x)}D_{ij}(x)$$

$$g_{ij}(x) = (\mathrm{adj}\, D)_{ij}(x)$$

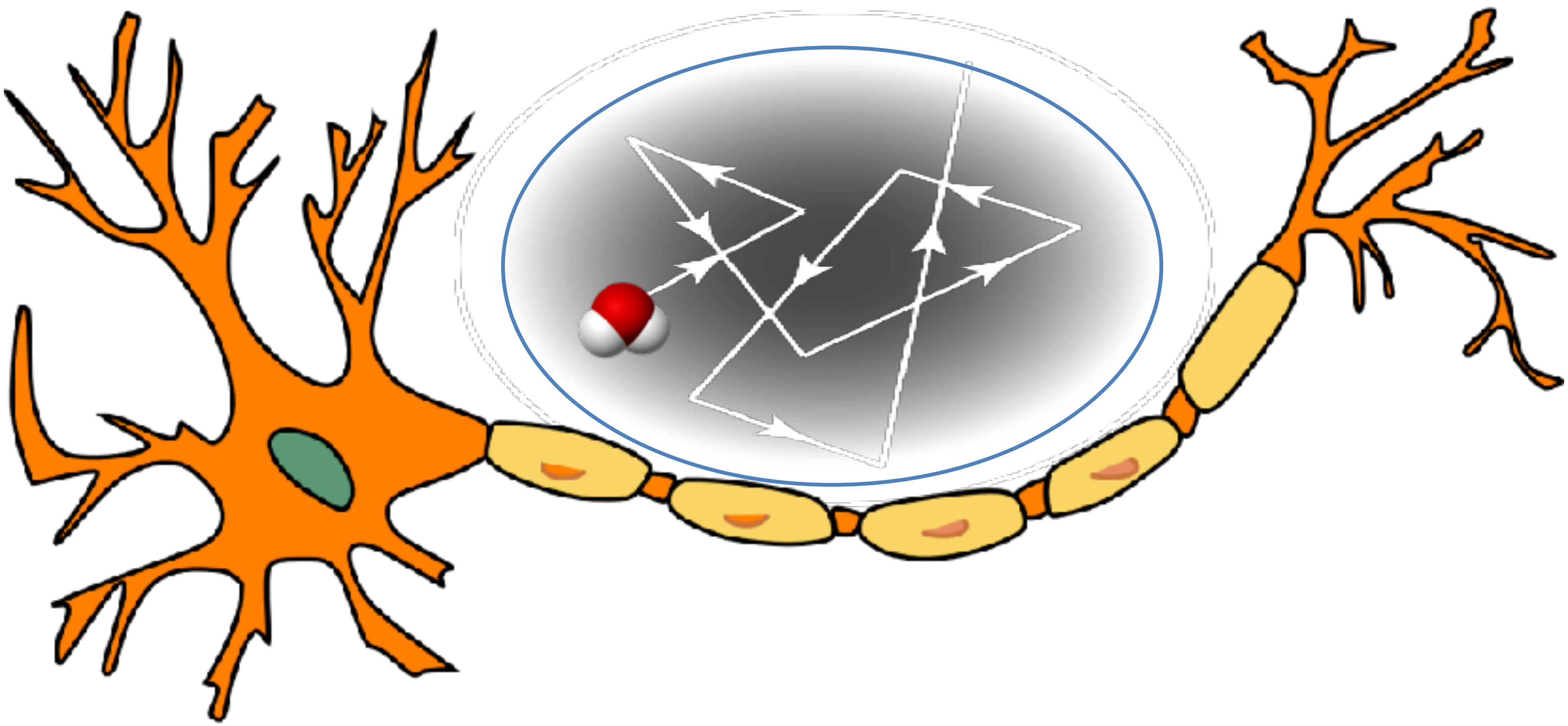
$$(\mathrm{adj}\, D)_{ik}(x)D^{kj}(x) = \delta^j_i \det D^\bullet(x)$$

adjugate diffusion tensor

References.

1. Lauren O'Donnell et al, LNCS 2488:459-466 (2002)
2. Christophe Lenglet et al, LNCS 3024: 127-140 (2004)
3. Xiang Hao et al, LNCS 6801:13-24 (2011)
4. Andrea Fuster et al, JMIV 54: 1-14 (2016)

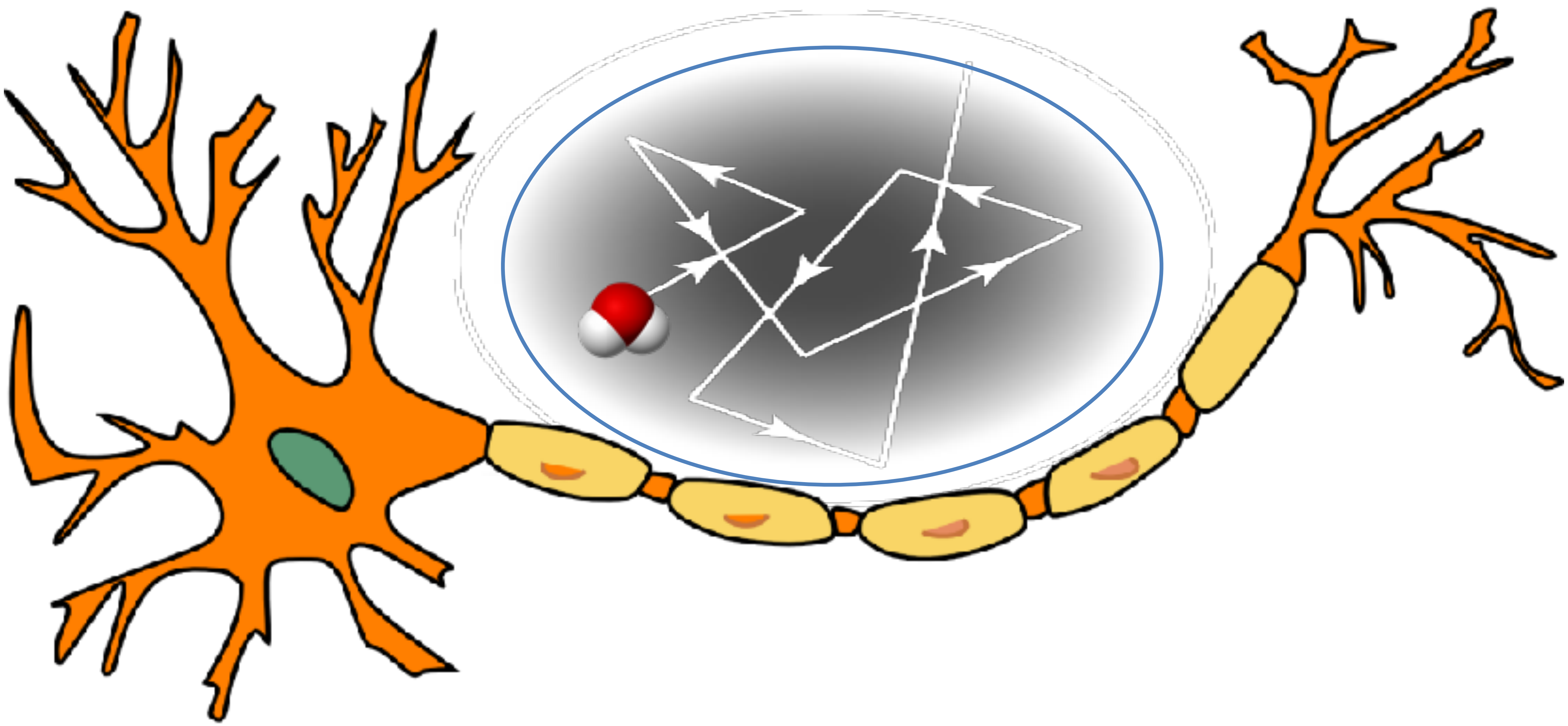
The Riemann-DTI paradigm



Hypothesis.

- Tissue microstructure imparts non-random barriers to water diffusion.

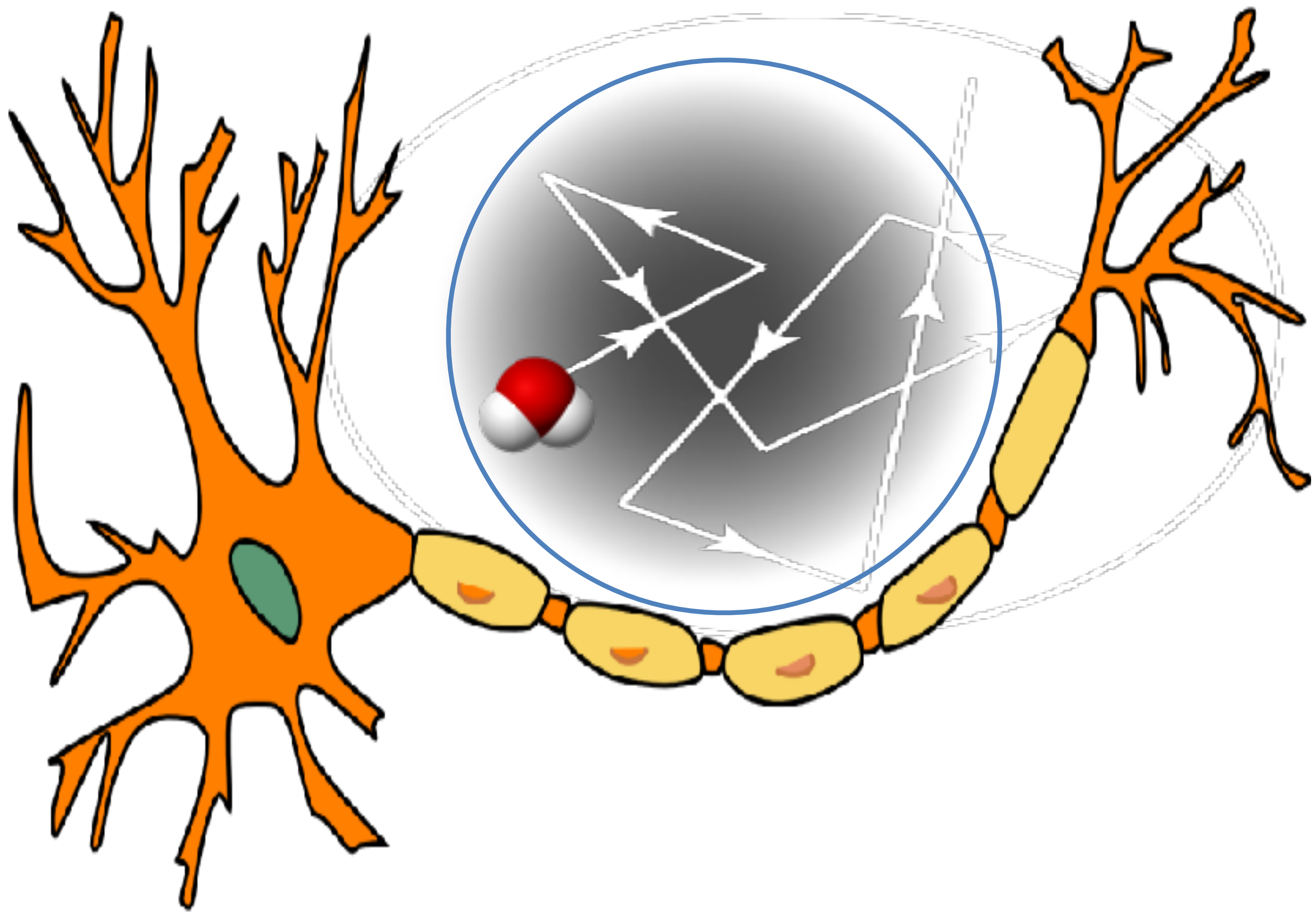
The Riemann-DTI paradigm



Conjecture.

- Extrinsic diffusion on Euclidean space ≈ intrinsic geometry of a Riemannian space

The Riemann-DTI paradigm



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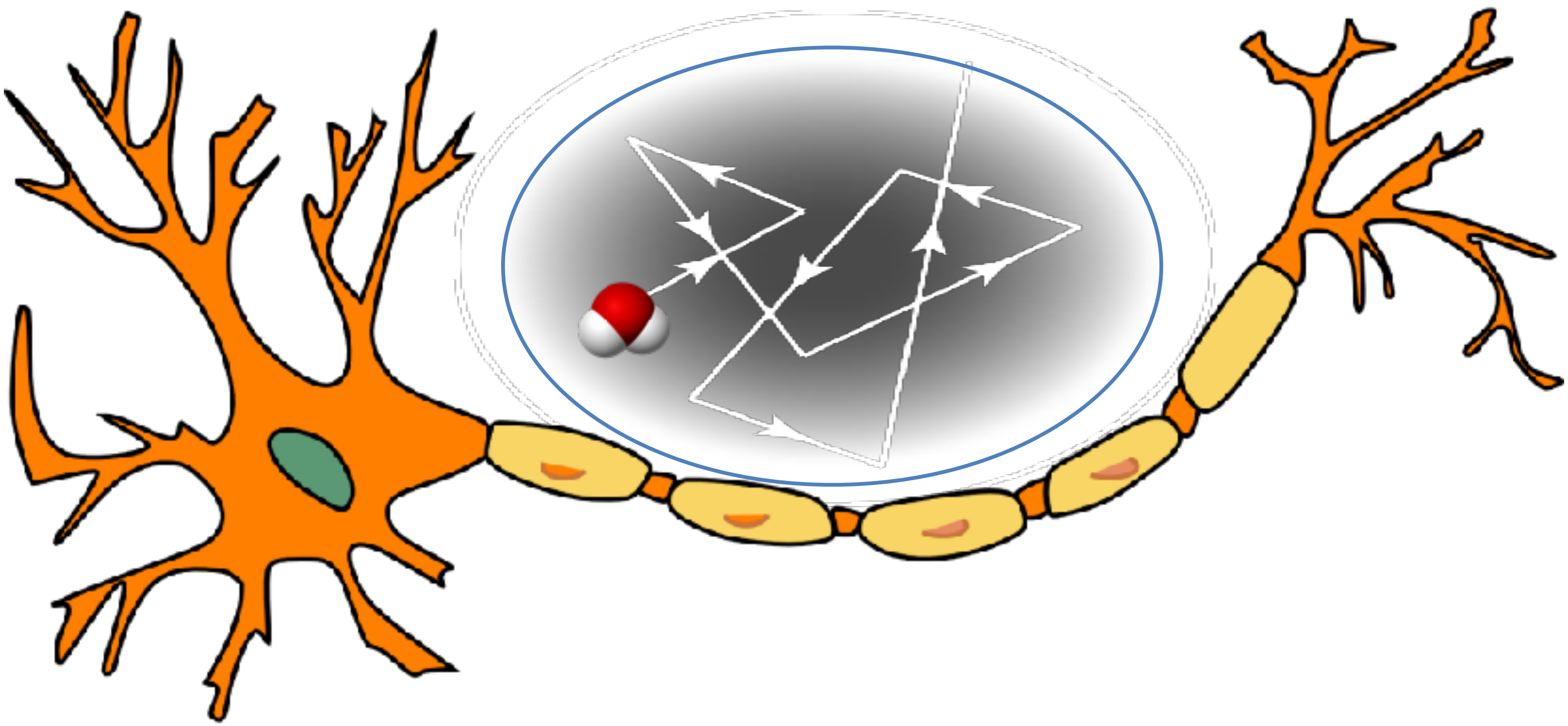

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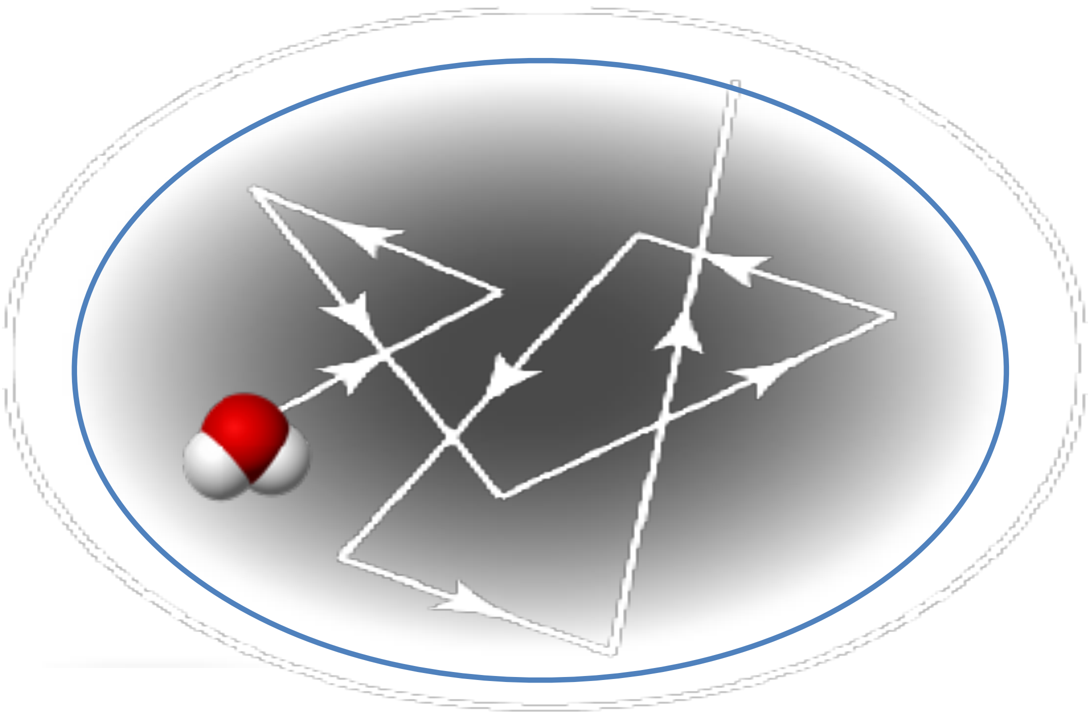
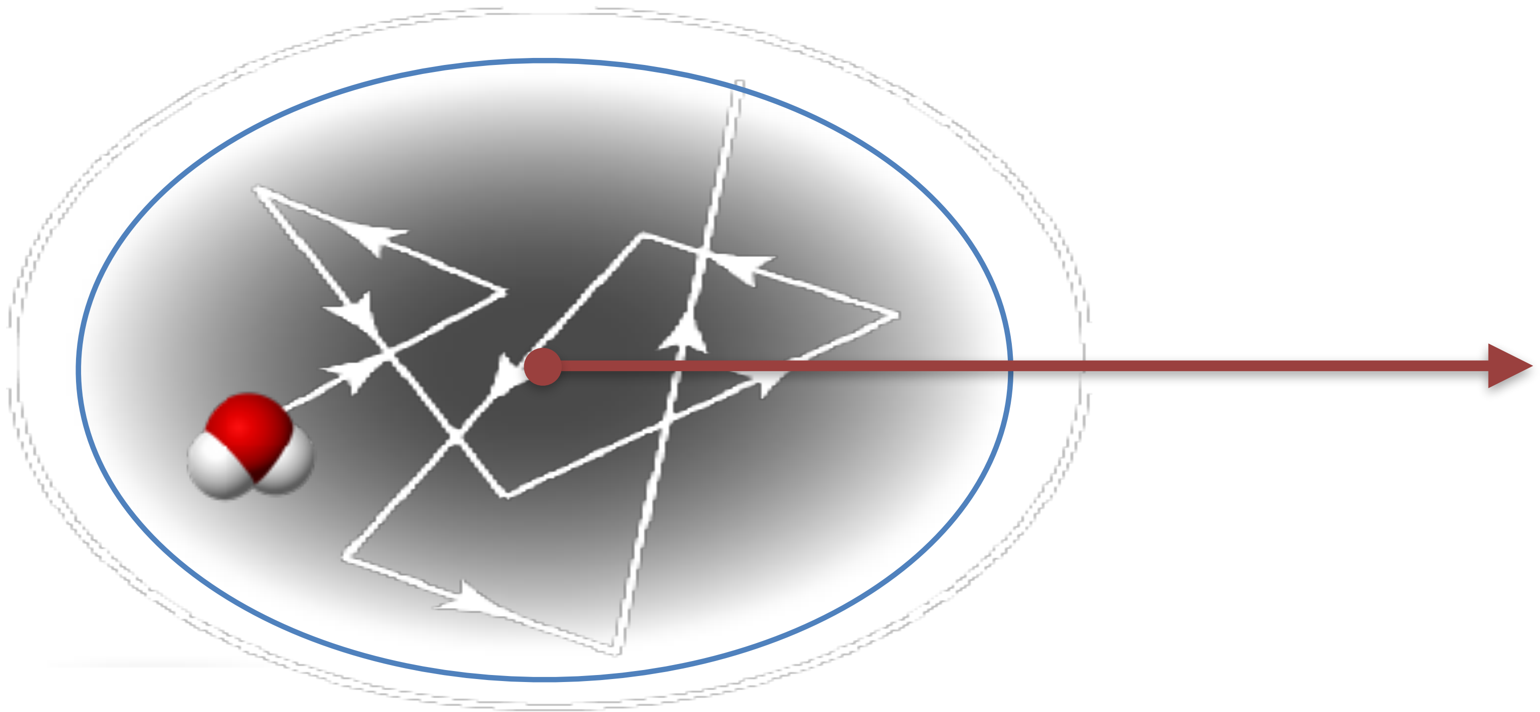


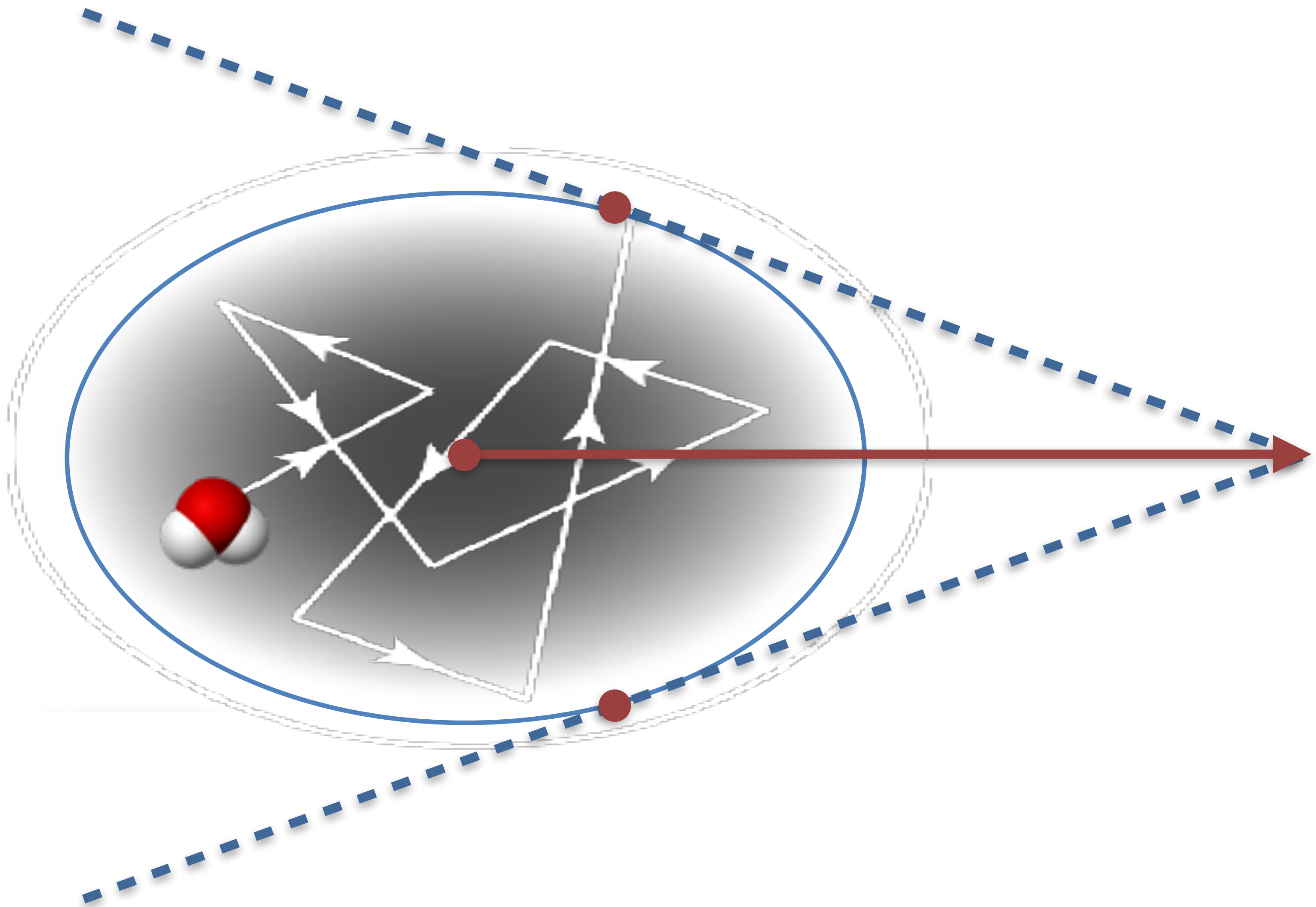
Terminology.

- gauge figure = unit sphere = indicatrix = Riemannian metric = inner product

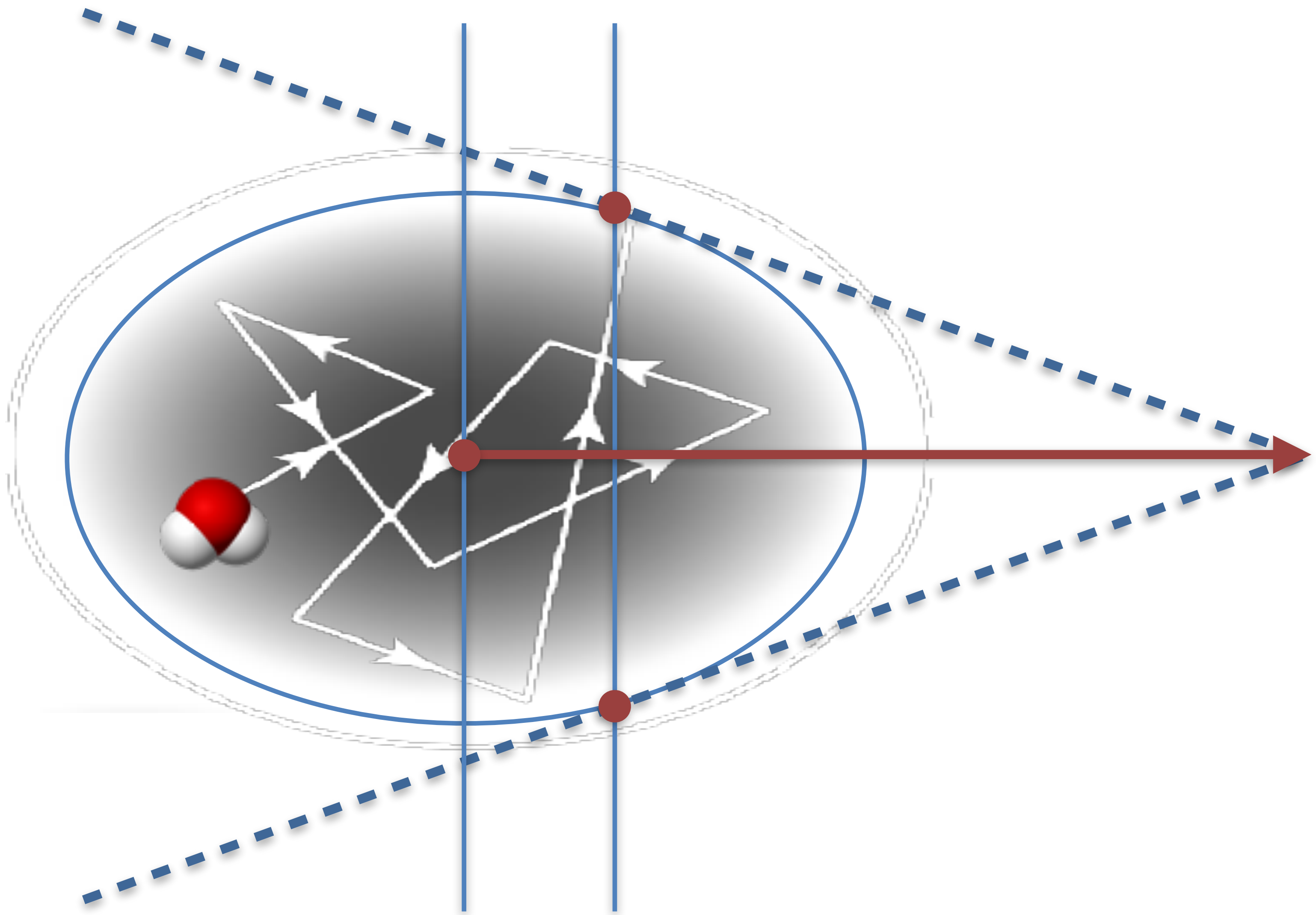
The Riemann-DTI paradigm



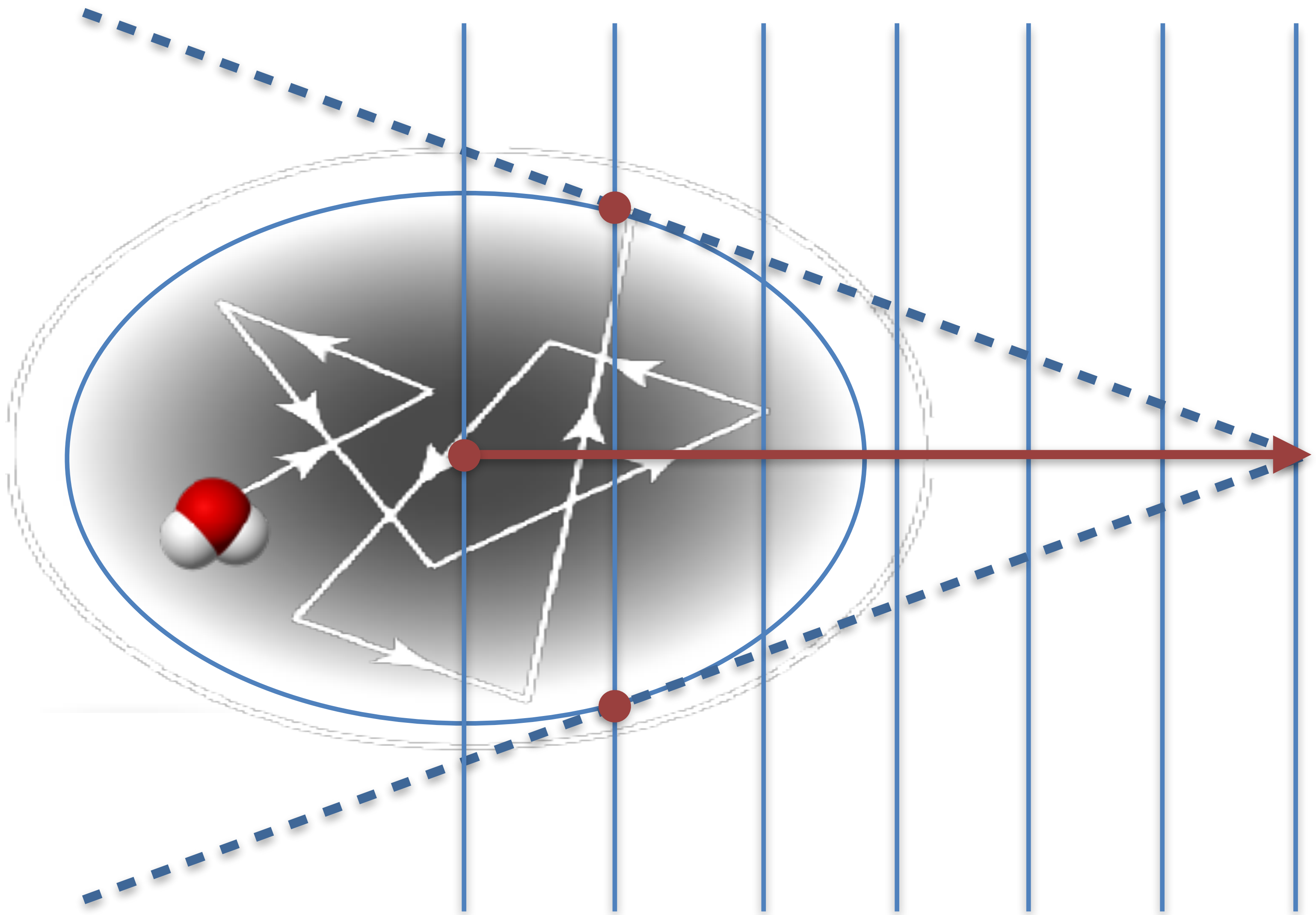
The Riemann-DTI paradigm



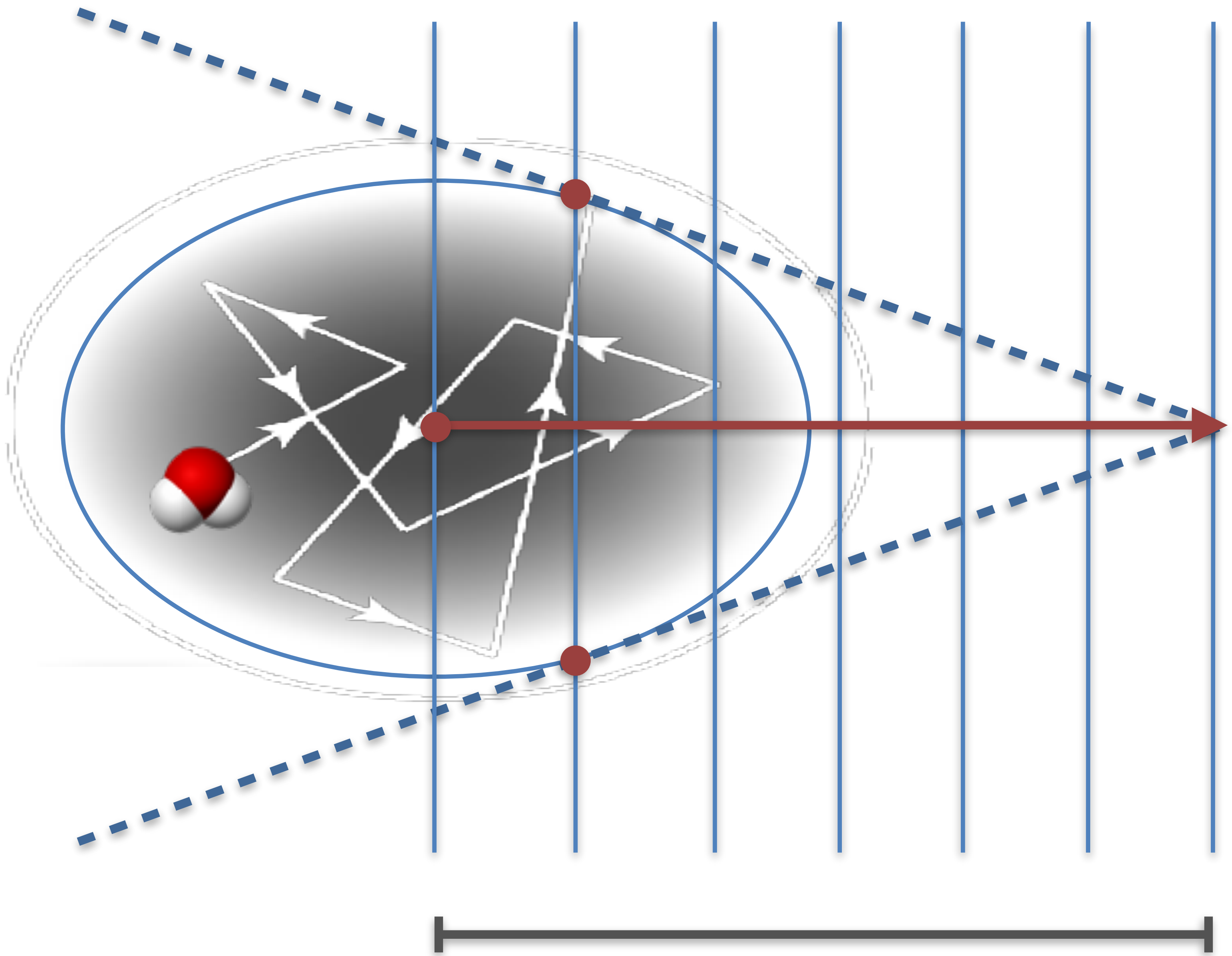
The Riemann-DTI paradigm



The Riemann-DTI paradigm



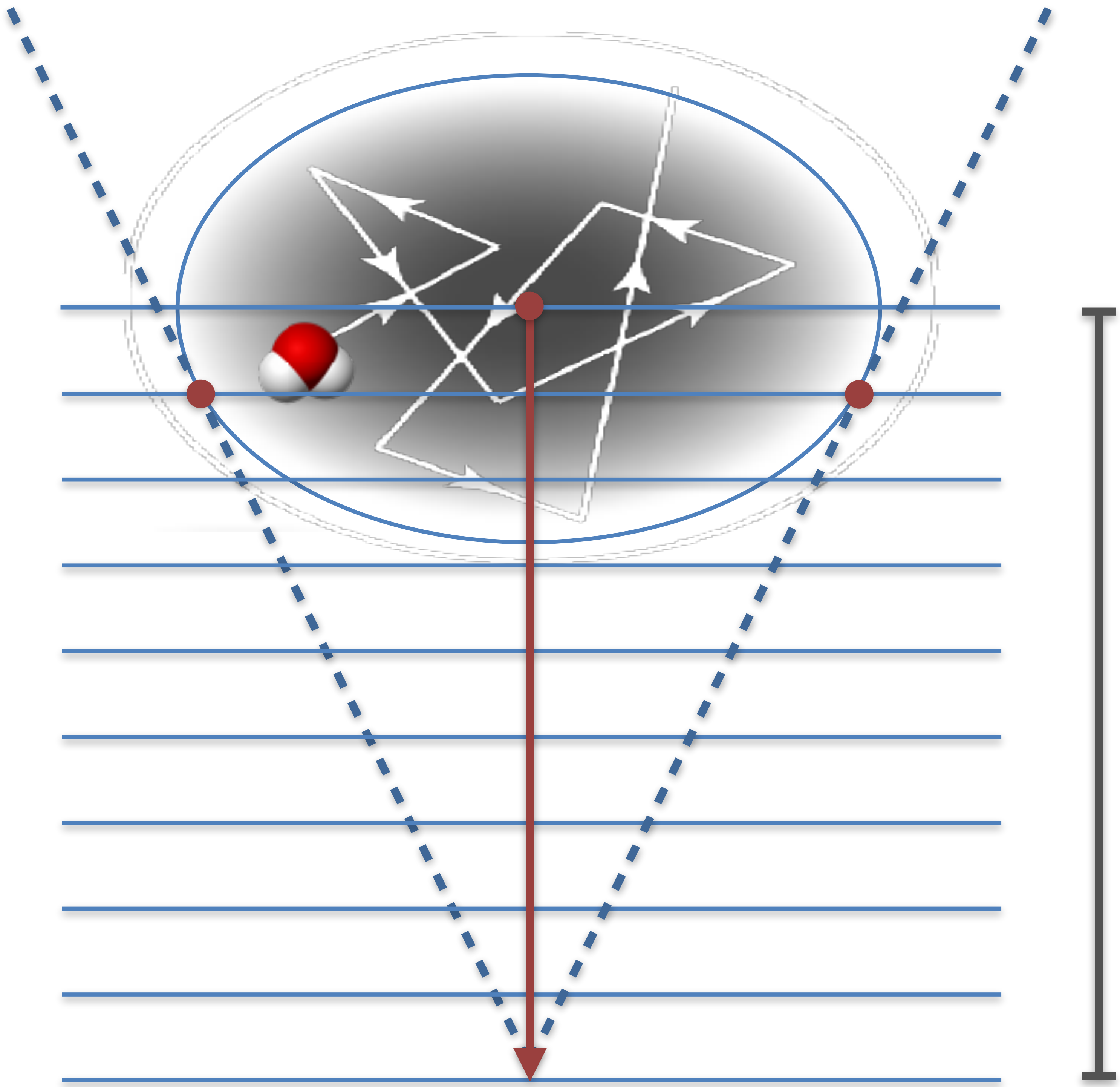
The Riemann-DTI paradigm



length² = 6

$$\|c\|^2 = g_{ij}c^i c^j$$

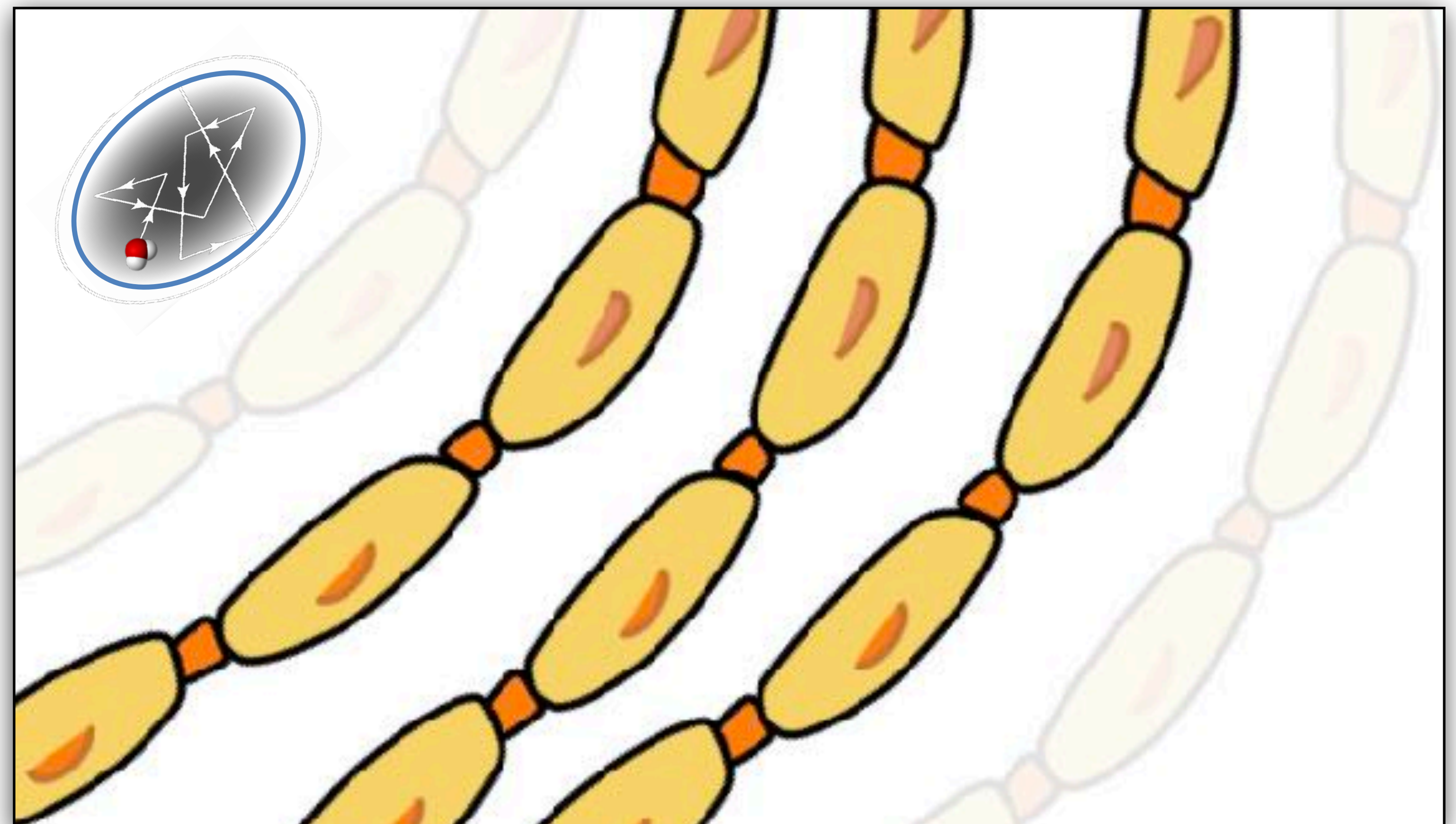
The Riemann-DTI paradigm



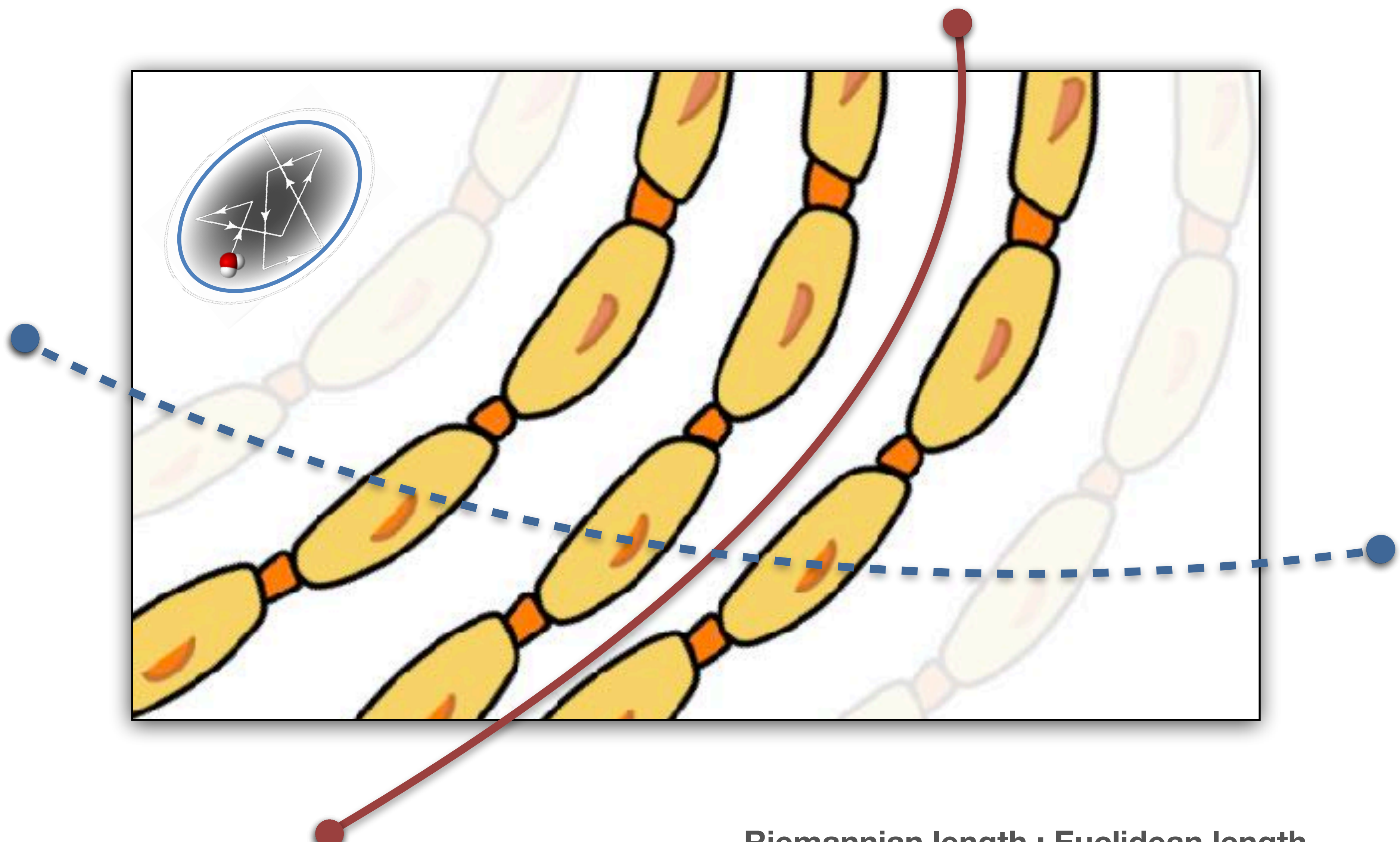
length² = 9

$$\|c\|^2 = g_{ij}c^i c^j$$

The Riemann-DTI paradigm & geodesic tractography



The Riemann-DTI paradigm & geodesic tractography



		Riemannian length : Euclidean length
‘short’ geodesic	(solid line)	5.0 : 6.0 < 1
‘long’ geodesic	(dashed line)	7.5 : 6.0 > 1

The Riemann-DTI paradigm & geodesic tractography

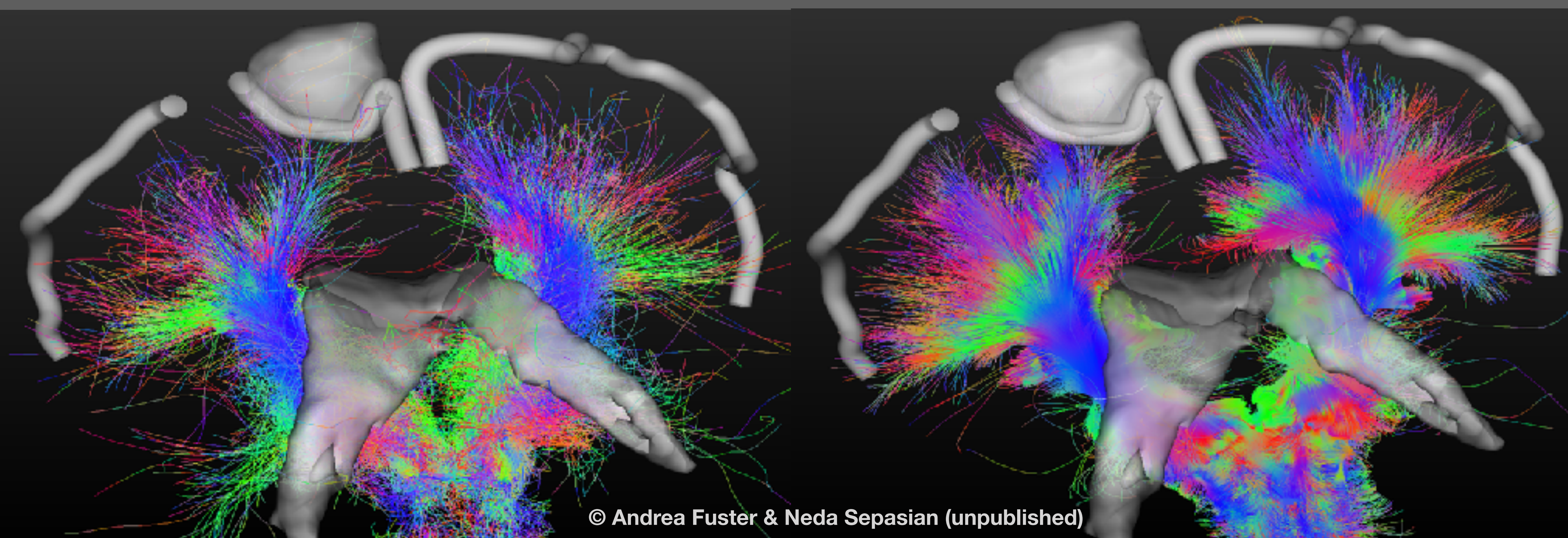
$$g_{ij}(x) = D_{ij}(x)$$

$$g_{ij}(x) = (\mathrm{adj}\, D)_{ij}(x)$$

tumour infiltration

irregular fibres

cf. Pujol et al. The DTI Challenge, MICCAI 2015



ventricle infiltration

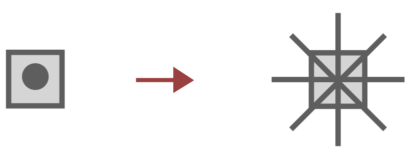
smooth fibres

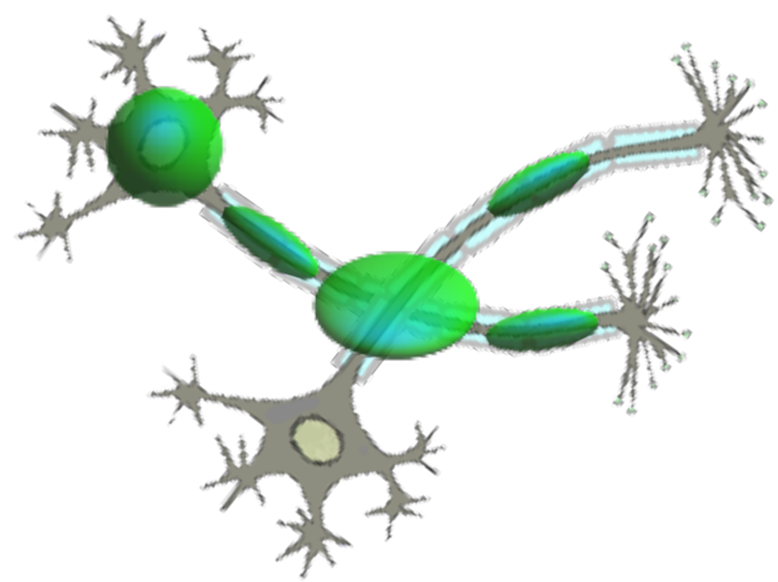
The Riemann-DTI paradigm & geodesic tractography

'100 % false positives'

geodesic completeness
=
redundant connections

pro: pixels → geodesic congruences



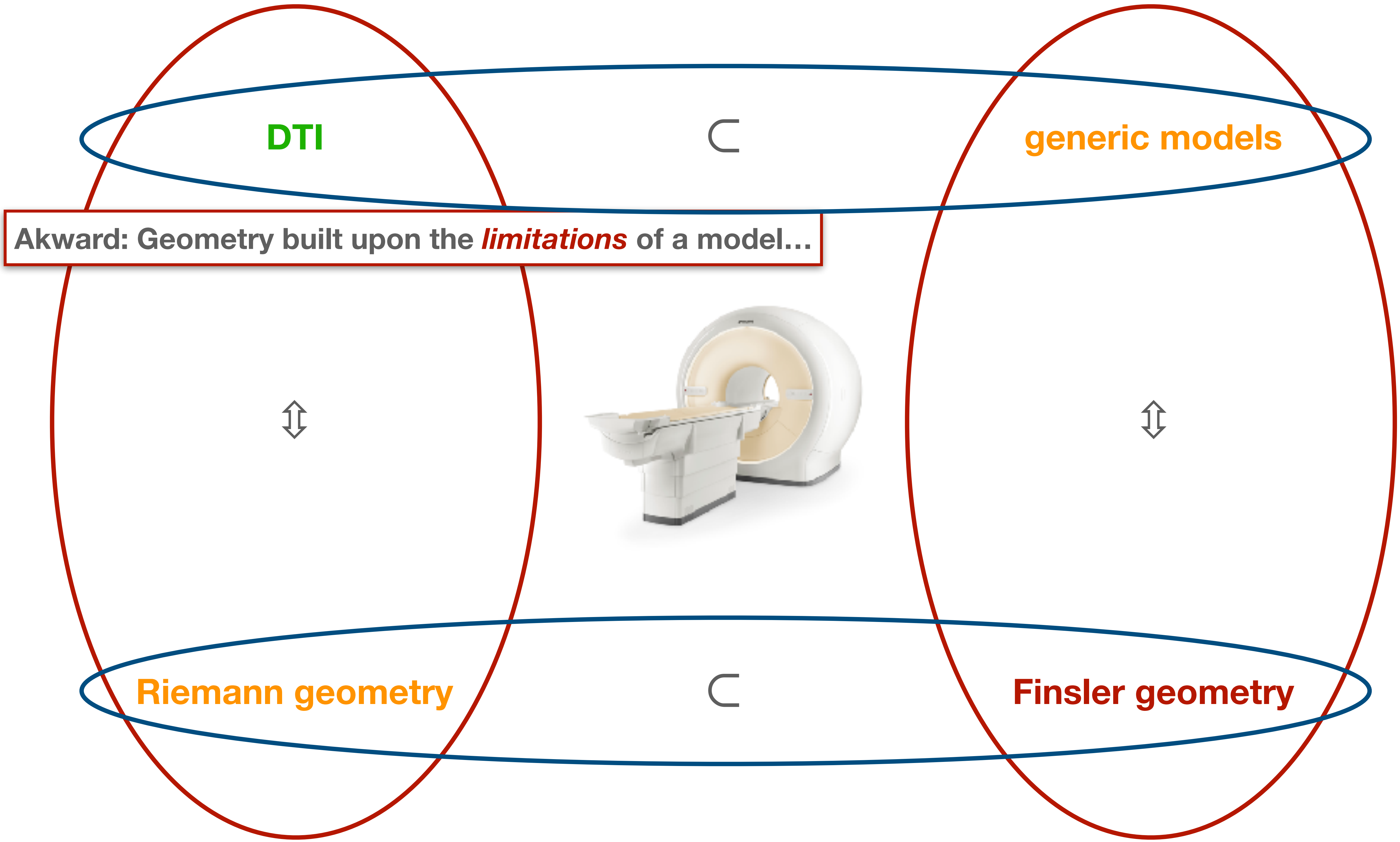


ellipsoidal gauge figure
=
poor angular resolution

con: destructive interference of orientation preferences

Riemannian and Finslerian geometry

for diffusion weighted magnetic resonance imaging



Heuristics

Literature.

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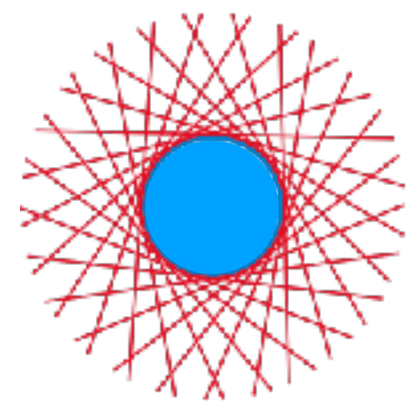
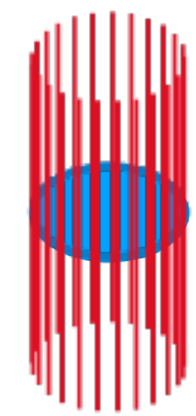
- © J. Melonakos et al, “Finsler Active Countours”. PAMI 30:3 (2008)
- © De Boer et al., “Statistical Analysis of Minimum Cost Path based Structural Brain Connectivity”. NeuroImage 55:2 (2011)
- © Astola, “Multi-Scale Riemann-Finsler Geometry: Applications to Diffusion Tensor Imaging and High Angular Resolution Diffusion Imaging”. PhD Thesis (2010)
- © Astola & Florack, “Finsler Geometry on Higher Order Tensor Fields and Applications to High Angular Resolution Diffusion
- © Astola et al., “Finsler Streamline Tracking with Single Tensor Orientation Distribution Function for High Angular Resolution Diffusion Imaging”. JMIV 41:3 (2011)
- © Sepasian et al., “Riemann-Finsler Multi-Valued Geodesic Tractography for HARDI”. In: “Visualisation and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data”, Westin et al. (Eds.), Springer (2014)
- © Fuster & Pabst, “Finsler pp-Waves”. Phys. Rev. D 94:10 (2016)
- © Florack et al., “Riemann-Finsler Geometry for Diffusion Weighted Magnetics Resonance Imaging”. In: “Visualisation and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data”, Westin et al. (Eds.), Springer (2014)
- © Florack et al., “Direction-Controlled DTI Interpolation”. In: “Visualisation and Processing of Higher Order Descriptors for Multi-Valued Data”, Hotz et al. (Eds.), Springer (2015)
- © Dela Haije et al., “Structural Connectivity Analysis using Finsler Geometry” (submitted)

Terminology

Tangent bundle.



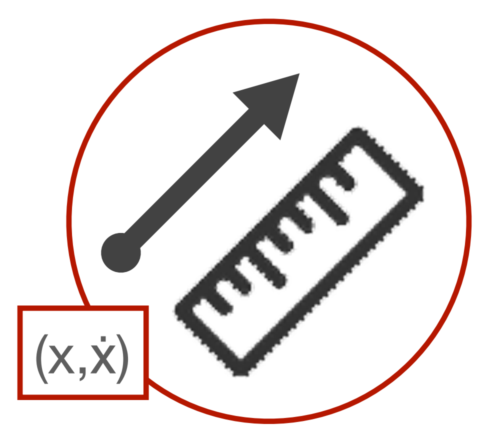


- $TM = \{ (x,\dot{x}) \mid x \in M,\ \dot{x} \in T_xM \}$

Slit tangent bundle.

- $TM \backslash 0 = \{ (x,\dot{x}) \in TM \mid \dot{x} \neq 0 \}$

Sphere bundle.



- $SM = \{ (x,\dot{x}) \in TM \mid F(x,\dot{x}) = 1 \}$

Projectivized tangent bundle.

- $PTM = \{ (x,\dot{x}) \in TM \mid F(x,\dot{x}) = 1 ,\ \ \dot{x} \sim (-\dot{x}) \}$

Finsler function

Finsler function.

$(\lambda \in \mathbb{R}\,,\, \dot{x} \neq 0\,,\, \xi \neq 0)$

$$F(x, \lambda \dot{x}) = |\lambda| F(x, \dot{x}) \qquad \text{(homogeneity)}$$

$$F(x, \dot{x}) > 0 \qquad \text{(positivity)}$$

$$\frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \xi^i \xi^j > 0 \qquad \text{(convexity)}$$

Notes.

- The Finsler function 'lives' on the 2n-dimensional tangent bundle TM.
- A Finsler function defines a (smoothly varying) local norm $||\dot{x}||_x = F(x,\dot{x})$ for a vector $\dot{x}$ at anchor point x.
- The line integral (⊛) is independent of curve parametrisation:

$$\mathscr{L}(C) = \int_C ds = \int_C F(x, dx) = \int_{t_-}^{t_+} F(x(t), \dot{x}(t))\, dt \qquad (\circledast)$$

Finsler metric

Finsler metric. $$g_{ij}(x,\dot{x}) \doteq \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j} \iff F(x,\dot{x}) = \sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}$$

Notes.

- The Finsler metric is a second order symmetric positive definite covariant tensor.
- The Finsler metric is homogeneous of degree 0.
- The Finsler metric ‘lives’ on the (2n-1)-dimensional projectivized tangent bundle PTM.

Riemann metric

position only

Riemann metric. $$g_{ij}(x) = \frac{1}{2}\frac{\partial^2 F_{\mathrm{R}}^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j} \quad \Longleftrightarrow \quad F_{\mathrm{R}}(x,\dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$$

Pythagorean rule

Notes.

- A Riemann metric defines an inner product induced norm ('Pythagorean rule').
- Finsler geometry is 'just' Riemannian geometry without the quadratic assumption.

The Finsler-DTI paradigm

DWMRI signal attenuation and propagator.

$$E(x,q,\tau) = \exp\left[-\tau D(x,q,\tau)\right] \qquad P(x,\xi,\tau) = \int_{\mathbb{R}^3} e^{2\pi i q\cdot\xi}\, E(x,q,\tau)\, dq$$

Dual Finsler function.

$$\frac{1}{2}H^2(x,q) = \sup_{\dot{x}\in \mathrm{TM}_x}\left[\langle q|\dot{x}\rangle - \frac{1}{2}F^2(x,\dot{x})\right]$$

$$H(x,q) = F(x,\dot{x}) \qquad \dot{x}^i \doteq g^{ij}(x,q)q_j$$

© Dela Haije, “Finsler Geometry and Diffusion MRI”. PhD Thesis (2017)

Notes.

(i) Riemann-DTI paradigm ~ central limit theorem: $H^2(x,q) \propto \sum_{ij} D^{ij}(x) q_i q_j$

(ii) Finsler-DTI paradigm: cf. PhD thesis Tom Dela Haije

The Finsler-DTI paradigm

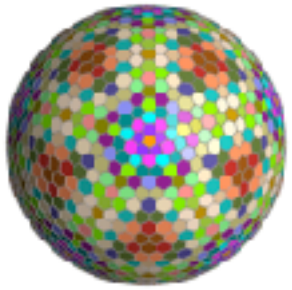
Finsler metric & dual Finsler metric.

$$g^{ij}(x,q) = \frac{1}{2}\frac{\partial^2 H^2(x,q)}{\partial q_i \partial q_j} \qquad g_{ij}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}$$

Figuratrix & indicatrix.

$$H^2(x,q) = g^{ij}(x,q) q_i q_j = 1 \qquad F^2(x,\dot{x}) = g_{ij}(x,\dot{x})\dot{x}^i \dot{x}^j = 1$$

Osculating figuratrices & osculating indicatrices.

$\boldsymbol{\vartheta} \in$ 

$$g^{ij}(x,\vartheta) q_i q_j = 1 \qquad g_{ij}(x,\vartheta)\dot{x}^i \dot{x}^j = 1$$

Note.

$$\int_{\mathrm{T^*M}_x} g^{ij}(x,\vartheta)\delta(\vartheta - q) q_i q_j \, d\vartheta = g^{ij}(x,q) q_i q_j \qquad \int_{\mathrm{TM}_x} g_{ij}(x,\vartheta)\delta(\vartheta - \dot{x})\dot{x}^i \dot{x}^j \, d\vartheta = g_{ij}(x,\dot{x})\dot{x}^i \dot{x}^j$$

The Finsler-DTI paradigm

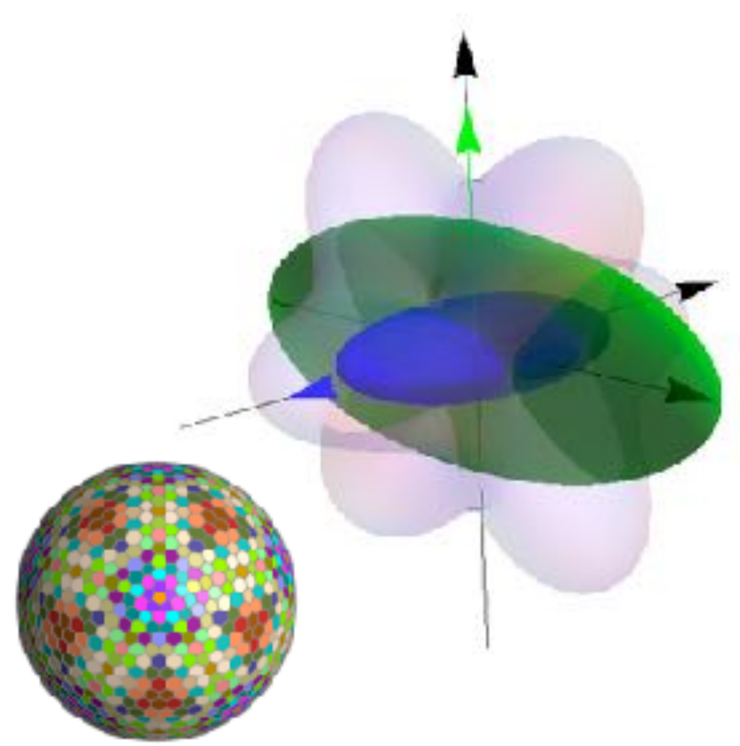
Note.

$$\int_{\mathrm{T^*M}_x} g^{ij}(x,\vartheta)\delta(\vartheta - q)q_i q_j \, d\vartheta = g^{ij}(x,q)q_i q_j \qquad \int_{\mathrm{TM}_x} g_{ij}(x,\vartheta)\delta(\vartheta - \dot{x})\dot{x}^i \dot{x}^j \, d\vartheta = g_{ij}(x,\dot{x})\dot{x}^i \dot{x}^j$$

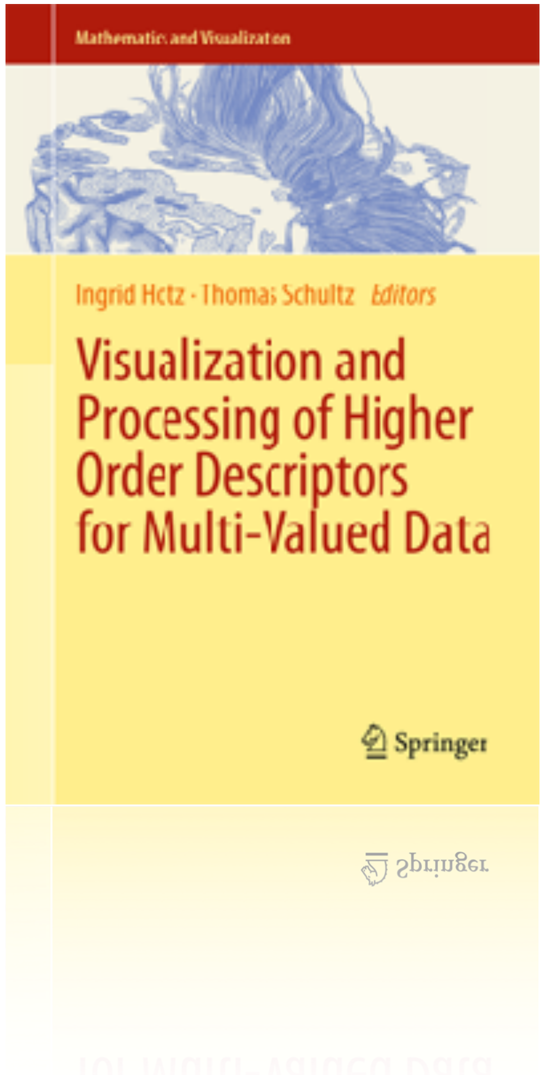
Interpretation.

- The dual Finsler metric represents an orientation-parametrized *family* of DTI tensors of the kind considered in the Riemann-DTI paradigm.
- In the Riemannian limit all members of this family coincide.

$\vartheta \in$

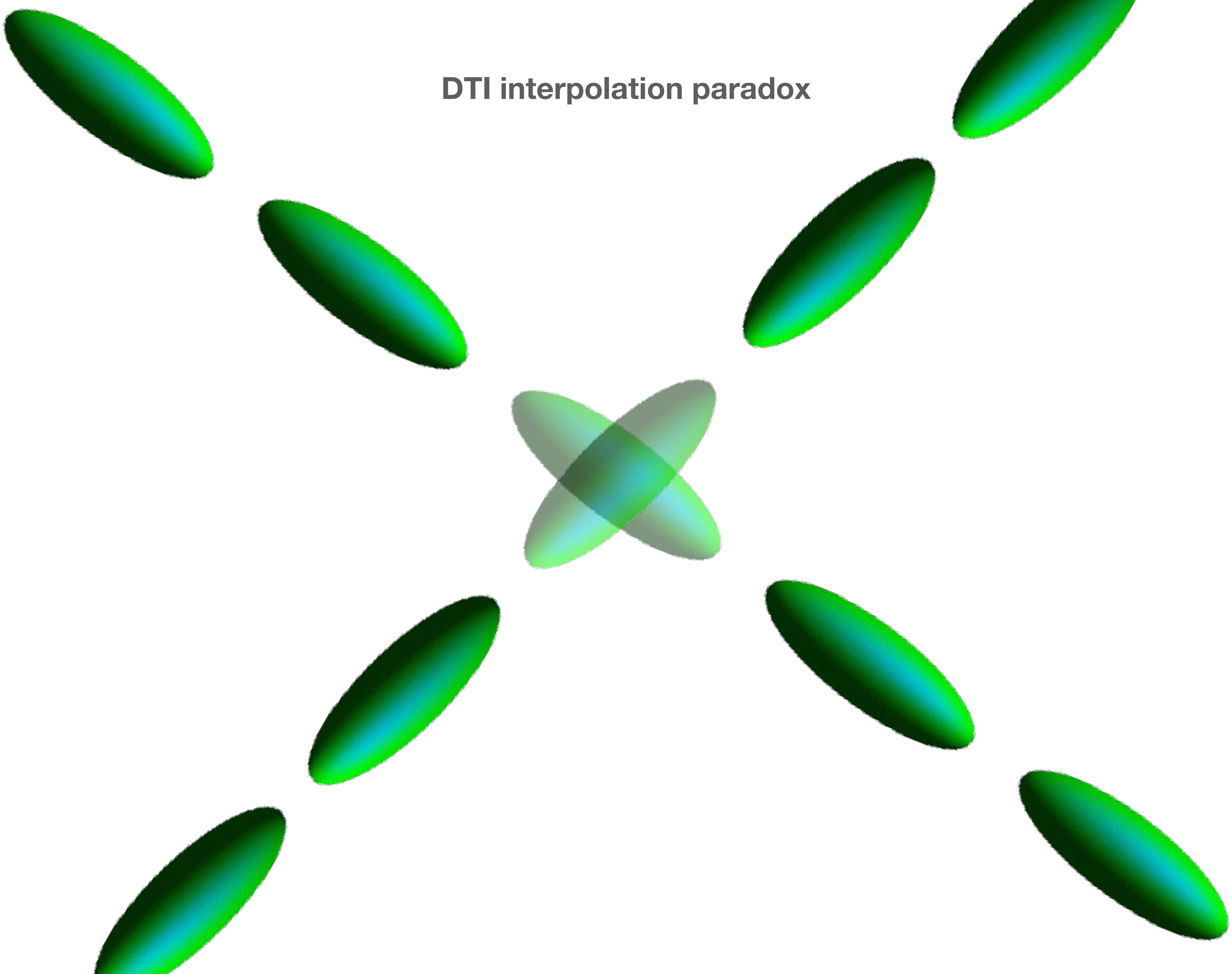


Application: DTI interpolation

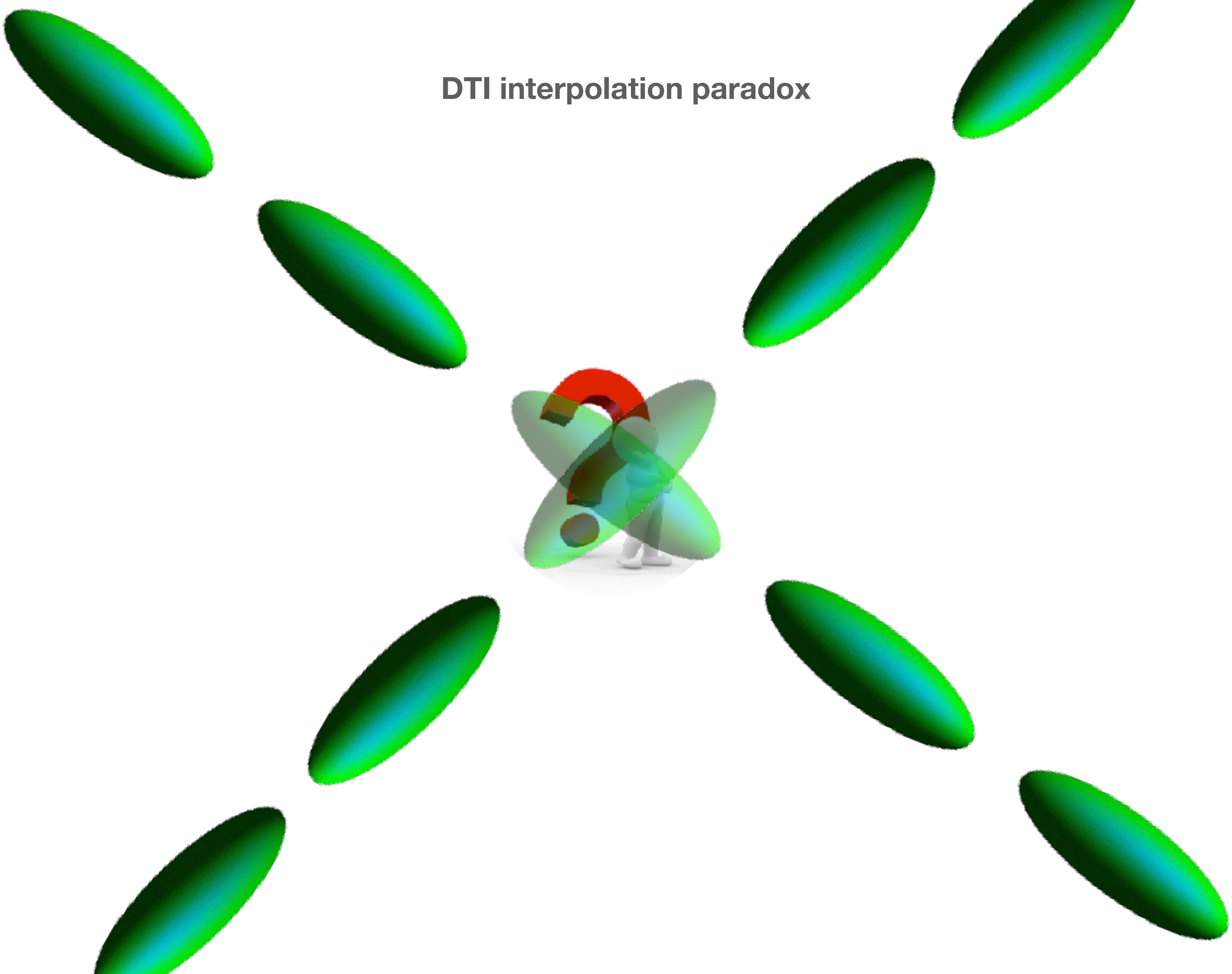


© Florack, Dela Haije & Fuster. in: Hotz & Schultz, Springer 2015

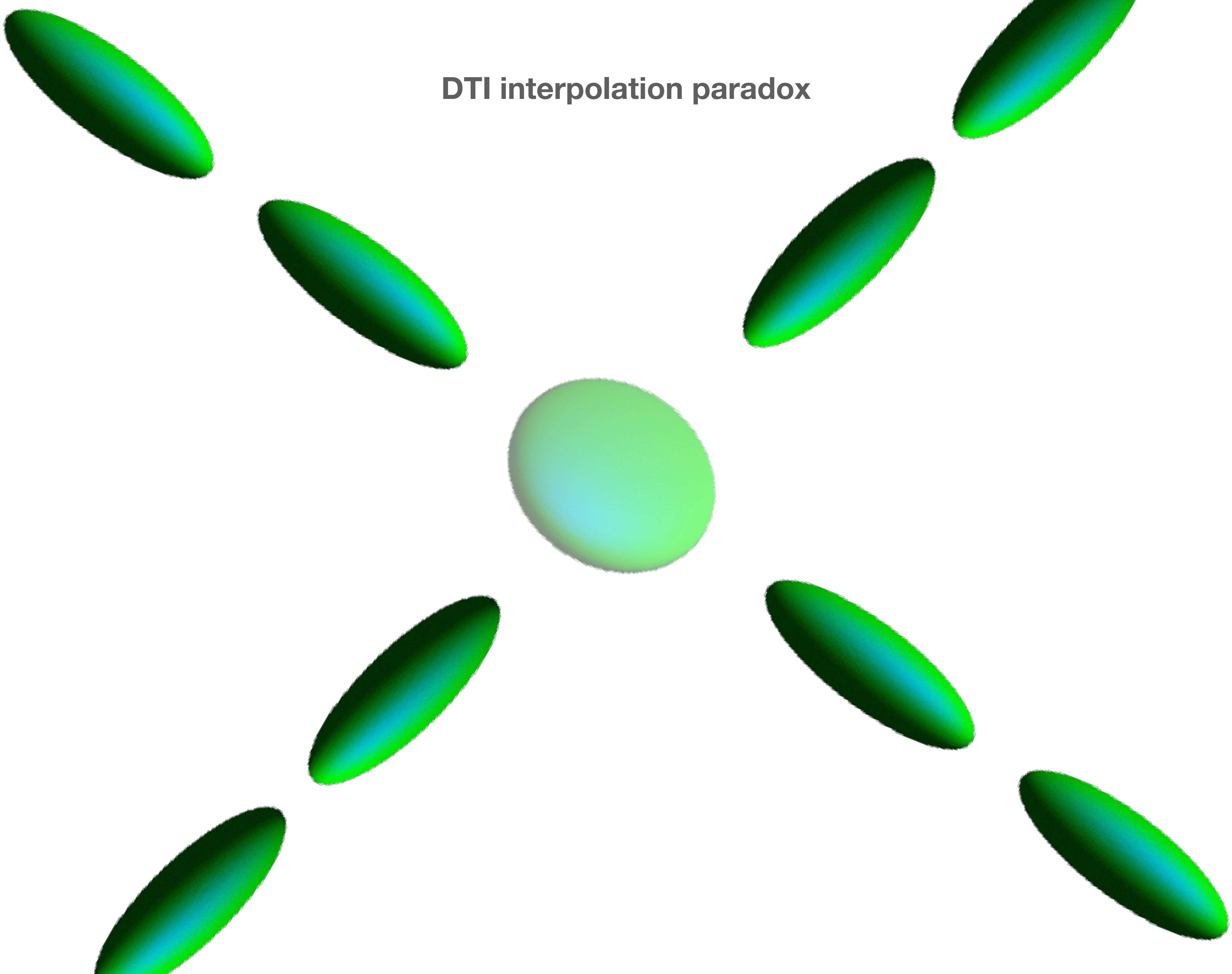
DTI interpolation paradox



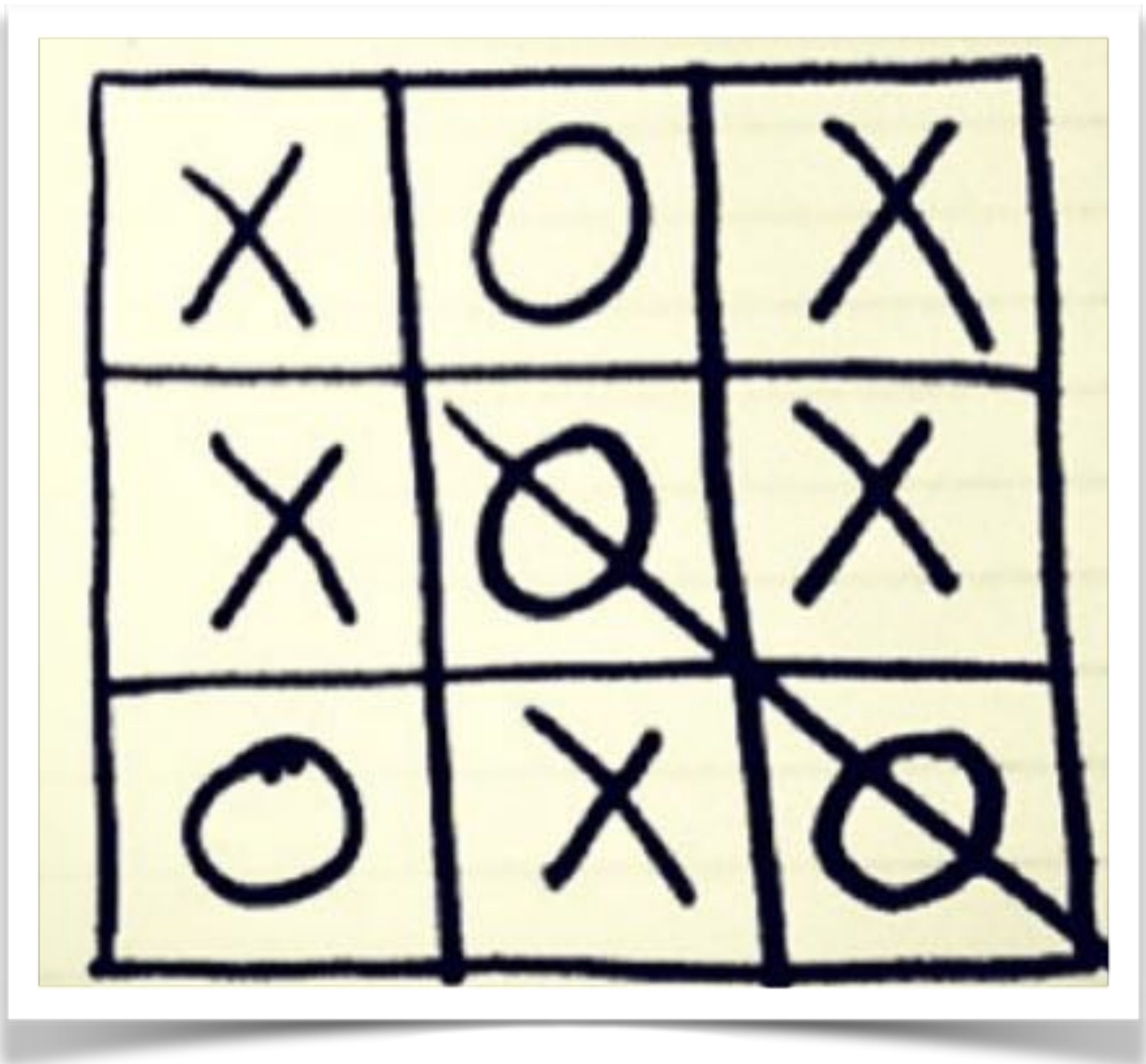
DTI interpolation paradox



DTI interpolation paradox



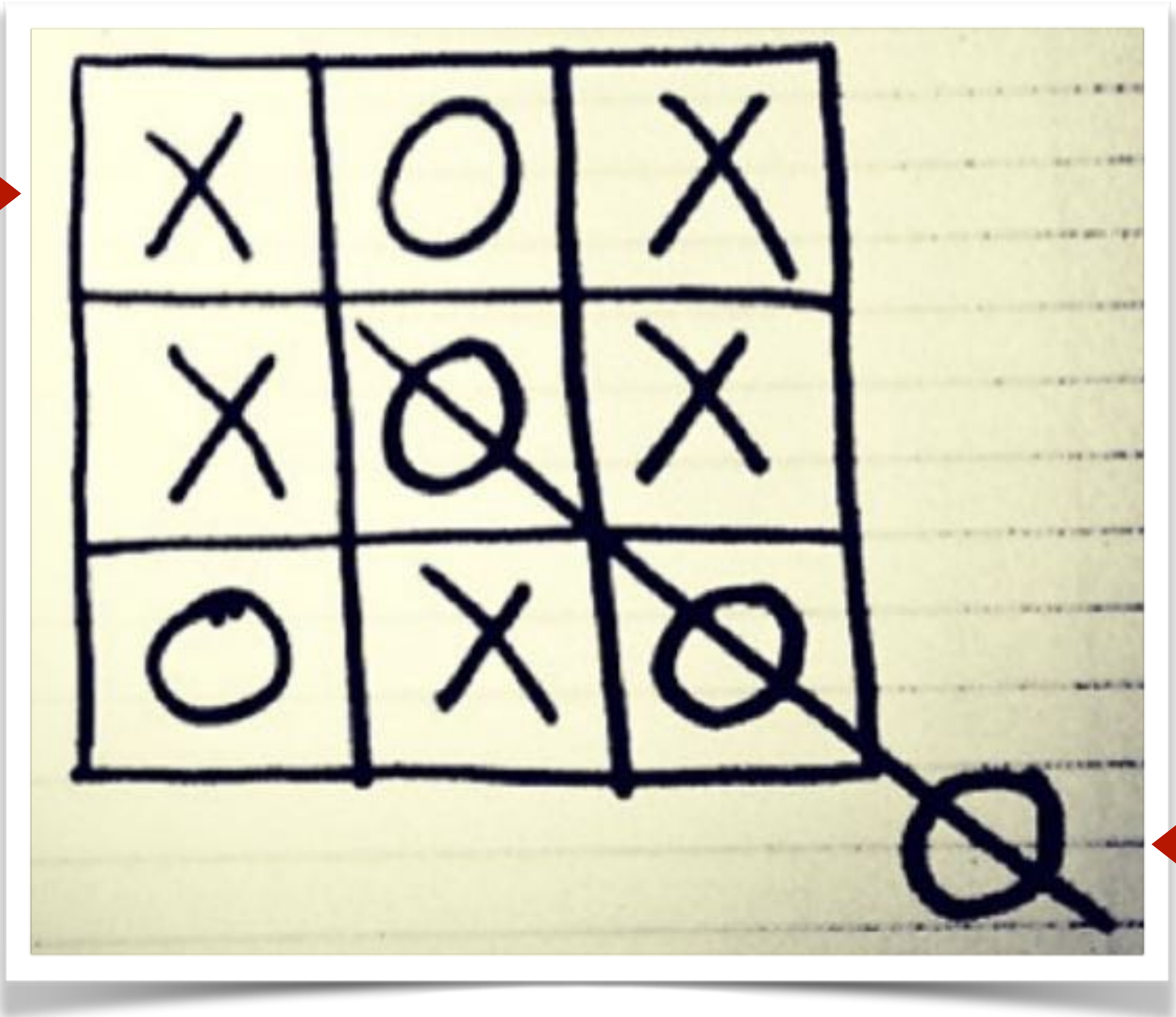
DTI interpolation paradox



DTI interpolation paradox

think out of the box…

Riemannian frame



Finslerian extension

Riemann metric weighted averaging

Finsler manifold

Definition. $(0 \leq \alpha \leq 1)$

(i) $F_g^2(x, \dot{x}) = g_{ij}(x)\dot{x}^i\dot{x}^j$

(ii) $F_h^2(x, \dot{x}) = h_{ij}(x)\dot{x}^i\dot{x}^j$

input: two 3D-DTI tensors

(iii) $F^2(x, \dot{x}) = F_g^{2\alpha}(x, \dot{x})\, F_h^{2(1-\alpha)}(x, \dot{x})$

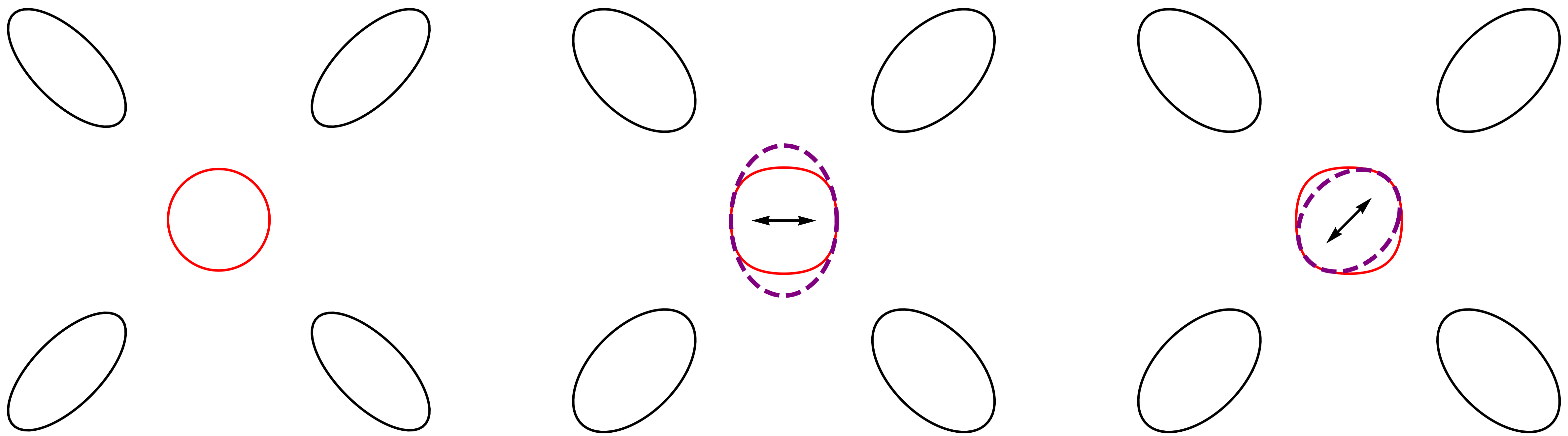
(iv) $g_{ij}(x, \dot{x}) = \dfrac{1}{2}\dfrac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}$ (⁎)

output: one 5D-DTI tensor

Claim.

(i) The tensor (⁎) is a Finsler metric.

(ii) An analytical, closed-form solution exists.



Cartan tensor

Cartan tensor.

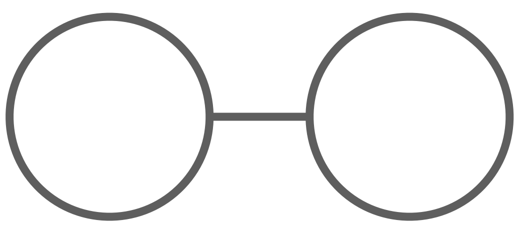
$$C_{ijk}(x,\dot{x}) \doteq \frac{1}{4}\frac{\partial^3 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j\partial\dot{x}^k}$$

Notes.

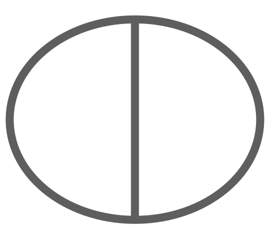
- The Cartan tensor **C** is a symmetric third order covariant tensor on the slit tangent bundle (sphere bundle / projectivized tangent bundle).
- The Cartan tensor is the ẋ-gradient of the metric tensor: $C_{ijk}(x,\dot{x}) = \partial_{\dot{x}^k} g_{ij}(x,\dot{x})$
- Deicke's theorem: Space is Riemannian iff the Cartan tensor vanishes identically.

Cartan scalar maps

5D-DTI



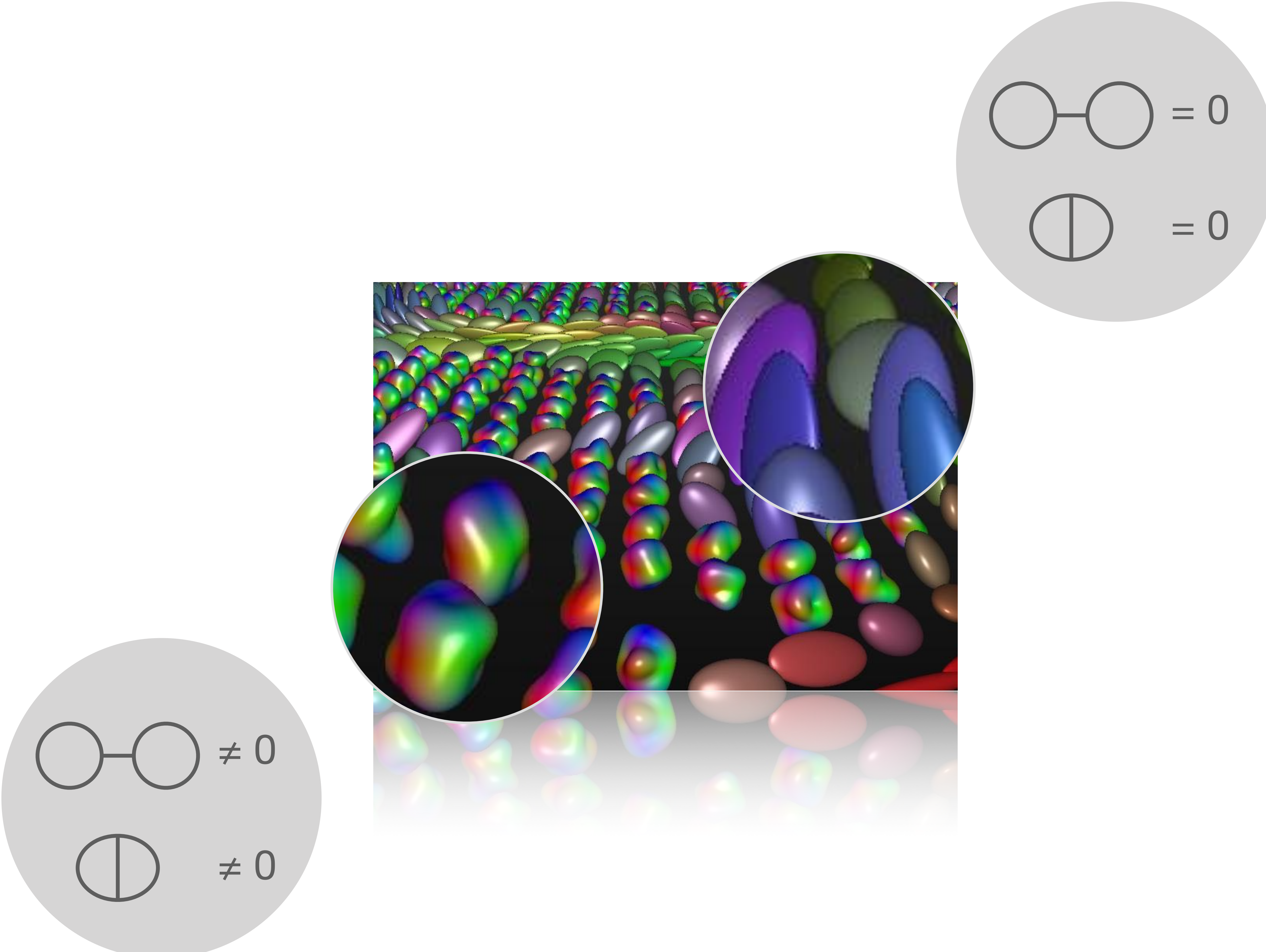
$$= g^{ij}(x,\dot{x})C_{ijk}(x,\dot{x})g^{k\ell}(x,\dot{x})C_{\ell mn}(x,\dot{x})g^{mn}(x,\dot{x})$$



$$= C_{ijk}(x,\dot{x})(x,\dot{x})g^{i\ell}(x,\dot{x})g^{jm}(x,\dot{x})g^{kn}(x,\dot{x})C_{\ell mn}(x,\dot{x})$$

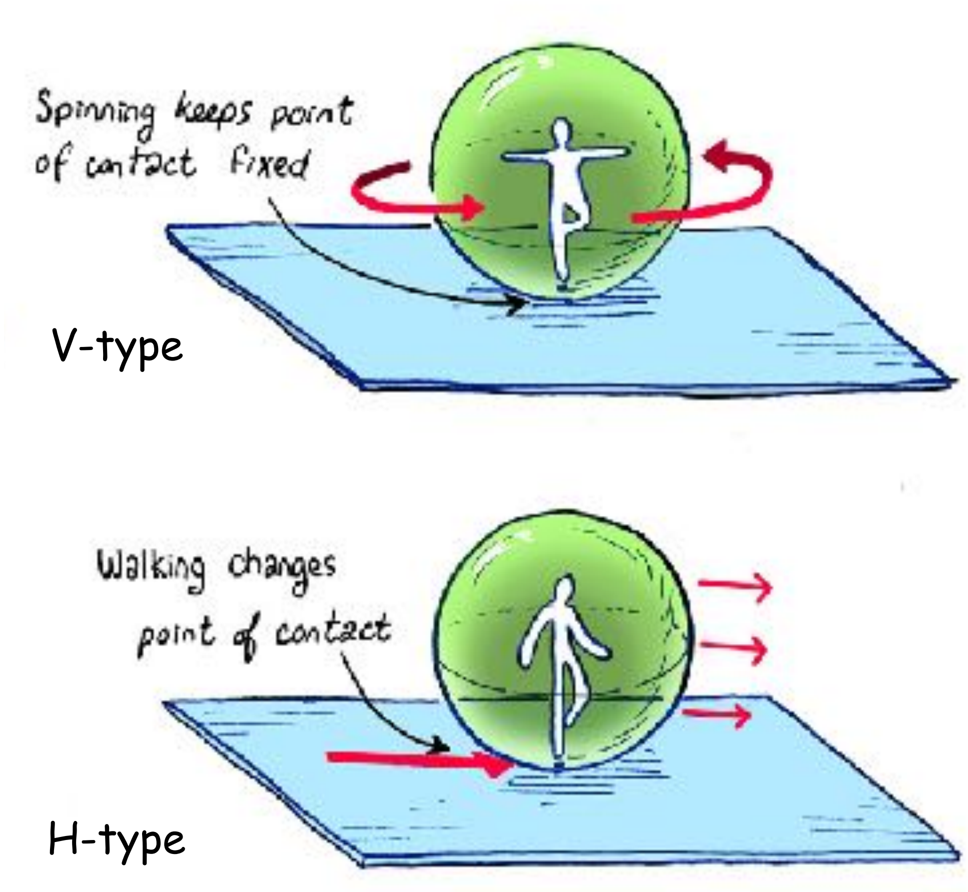
Notes.

- These scalars live on the slit tangent bundle (sphere bundle / projectivized tangent bundle).
- They can be projected in various ways onto the spatial base manifold.
- They can be used as invariant local or tractometric features.
- They are (locally) nontrivial iff the Riemann-DTI model (locally) fails (Deicke's theorem).


= 0
= 0
≠ 0
≠ 0

Tractography from 5D???

HV-splitting



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Horizontal versus vertical transport:

- V-type: Spinning without walking.
- H-type: Spinning-walking preserving a forward gaze.

H/V-generators:

- V-type: $\dfrac{\partial}{\partial \dot{x}^j}$
- H-type: $\dfrac{\delta}{\delta x^i} \doteq \dfrac{\partial}{\partial x^i} - N_i^j(x,\dot{x})\dfrac{\partial}{\partial \dot{x}^j}$

HV-splitting

Nonlinear connection.

- A 'nonlinear connection' is needed to ensure a geometrically meaningful HV-splitting.

- Riemannian limit (linear connection): $N^j_i(x,\dot{x}) = \Gamma^j_{ik}(x)\dot{x}^k$

- 'Christoffel symbols of the 2nd kind':

$$\Gamma^j_{ik}(x) \doteq \frac{1}{2} g^{j\ell}(x) \left[\frac{\partial g_{\ell k}(x)}{\partial x^i} + \frac{\partial g_{i\ell}(x)}{\partial x^k} - \frac{\partial g_{ik}(x)}{\partial x^\ell} \right]$$

HV-splitting

Nonlinear connection.

- Finslerian case: nonlinear correction terms, involving the Cartan tensor:

$$N^i_j(x,\dot{x}) = \gamma^i_{jk}(x,\dot{x})\dot{x}^k - C^i_{jk}(x,\dot{x})\gamma^k_{\ell m}(x,\dot{x})\dot{x}^\ell \dot{x}^m$$

formal Christoffel symbols of the 2nd kind

$$N^i_j(x,\dot{x}) = \frac{\partial G^i(x,\dot{x})}{\partial \dot{x}^j} \iff G^i(x,\dot{x}) = \frac{1}{2}\gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k = \frac{1}{2}N^i_j(x,\dot{x})\dot{x}^j \qquad (*)$$

geodesic spray coefficients

HV-splitting

Rate of change along a curve in TM\0**.**

$$\frac{d}{ds} f(x(t+s), y(t+s))\Big|_{s=0} = \dot{x}^i(t) \frac{\partial}{\partial x^i} f(x(t), y(t)) + \dot{y}^i(t) \frac{\partial}{\partial y^i} f(x(t), y(t)) \qquad \left[\; \dot{} \overset{\text{def}}{=} \frac{d}{dt} \; \right]$$

$$= \dot{x}^i(t) \frac{\delta}{\delta x^i} f(x(t), y(t)) + \left[\dot{y}^i(t) + N^i_j(x(t), y(t)) \dot{x}^j(t) \right] \frac{\partial}{\partial y^i} f(x(t), y(t))$$

$$\frac{\partial}{\partial x^i}$$

HV-splitting

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H-component ↑ V-component ↑

H/V curves.

- Vertical curve: $\dot{x}^i(t) = 0$
- Horizontal curve: $\dot{y}^i(t) + N^i_j(x(t), y(t))\dot{x}^j(t) = 0$
- Constant Finslerian speed geodesic (y=ẋ): $\ddot{x}^i(t) + N^i_j(x(t), \dot{x}(t))\dot{x}^j(t) = 0$

$$\ddot{x}^i(t) + 2G^i(x(t), \dot{x}(t)) = 0$$

↑ Finslerian ‘pseudo-force’

Geodesics

Geodesic (global definition).

- A geodesic is a 'locally' shortest path:

$$\mathscr{L}(C) = \int_C F(x, dx) \longrightarrow \min$$

$$\ddot{x}^i + 2G^i(x, \dot{x}) = \frac{d \ln F(x, \dot{x})}{dt} \dot{x}^i$$

Recall (local definition).

- Constant Finslerian speed geodesics: $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$
- Always possibly by slick choice of parametrization (e.g. 'arclength', i.e. such that $F(x,\dot{x})=1$).

The Riemann-DTI paradigm

Conjecture.

- neural fiber bundles correspond to relatively short geodesics in a Riemannian ‘brain space’
- the Riemannian structure can be inferred from ‘3D-DTI’

The Finsler-DTI paradigm

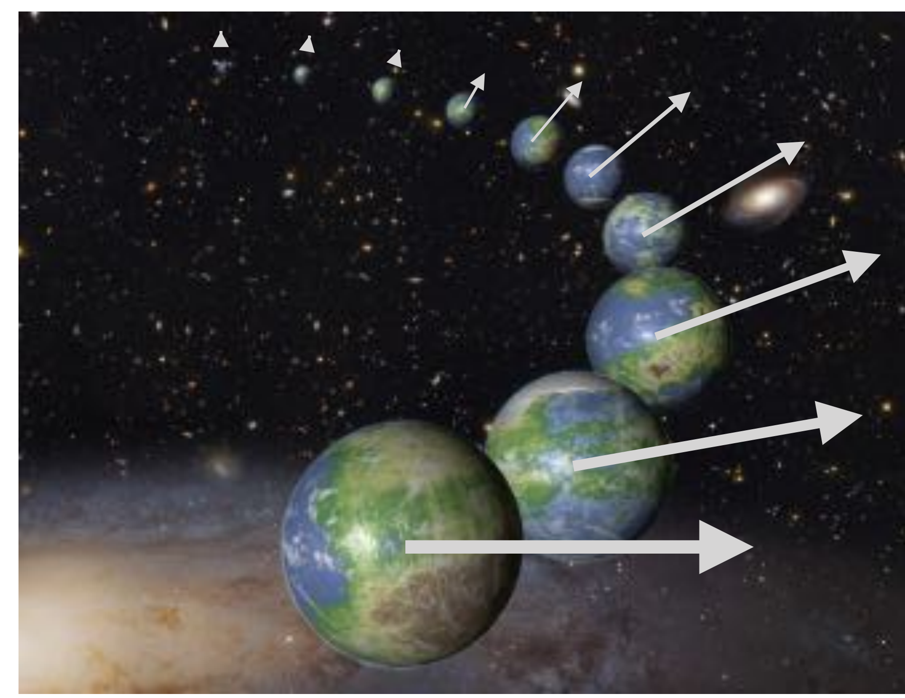
Conjecture.

- neural fiber bundles correspond to relatively short geodesics in a Finslerian ‘brain space’
- the Finslerian structure can be inferred from ‘5D-DTI’

THE END OF THE WORLD AS WE KNOW IT

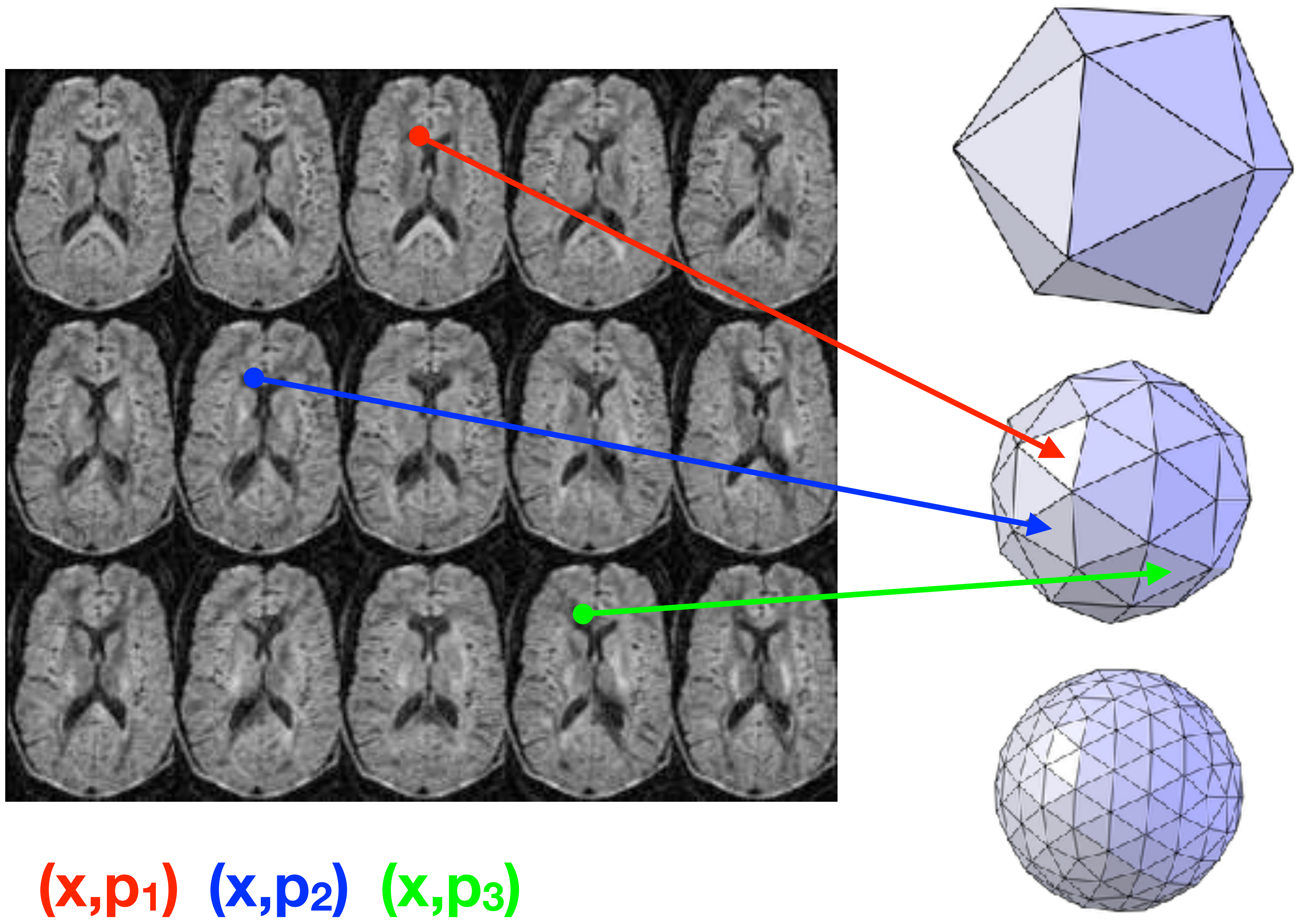
Euclidean

Riemannian



Finslerian

Diffusion Weighted Magnetic Resonance Imaging



The Riemann-DTI paradigm & geodesic tractography

$$G(v,v) = \|v\|^2$$

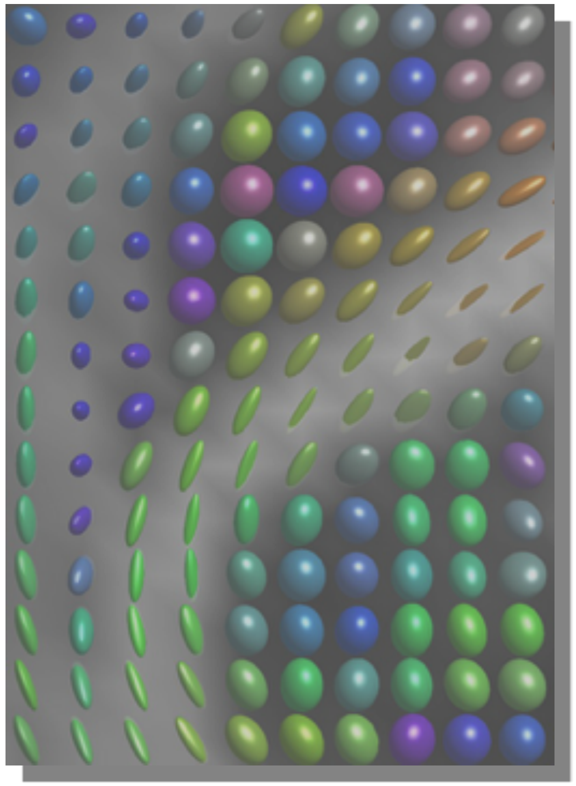
Riemann metric: lengths & angles

$$\nabla_{\dot{x}} \dot{x} = 0$$

Levi-Civita connection: parallel transport

$$\ddot{x}^i + \sum_{jk} \Gamma^i_{jk}(x)\, \dot{x}^j \dot{x}^k = 0$$

Christoffel symbols: "pseudo-forces" (relative to local coordinate frames)



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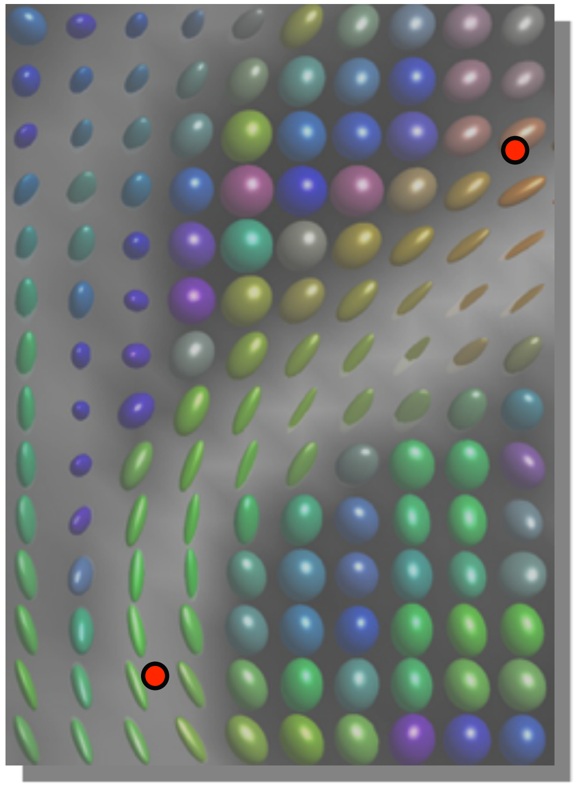

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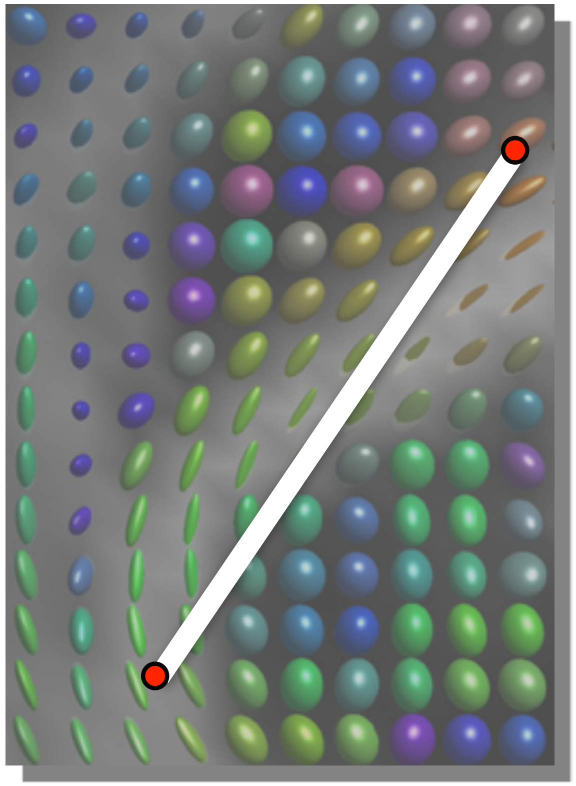

Euclidean geodesic

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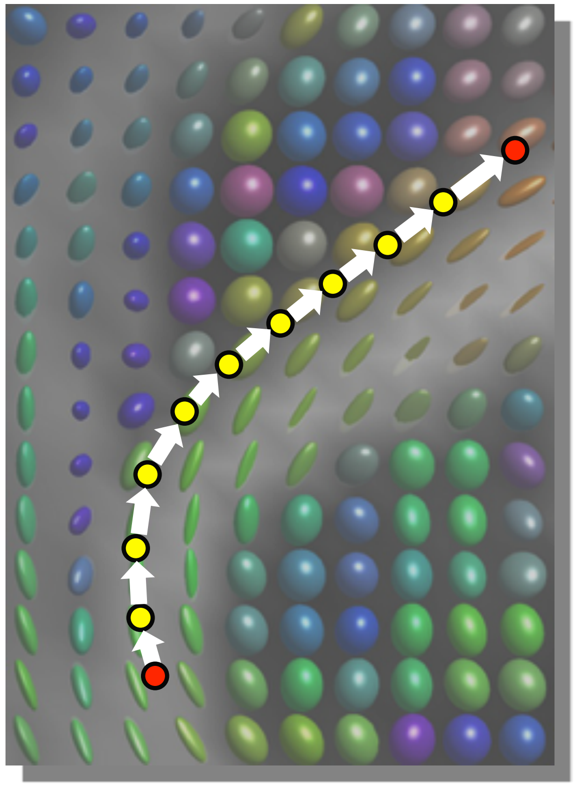

Riemannian geodesic

Riemann metric weighted averaging

Finsler manifold

$$\Delta_{ij}(x, \dot{x}) =$$