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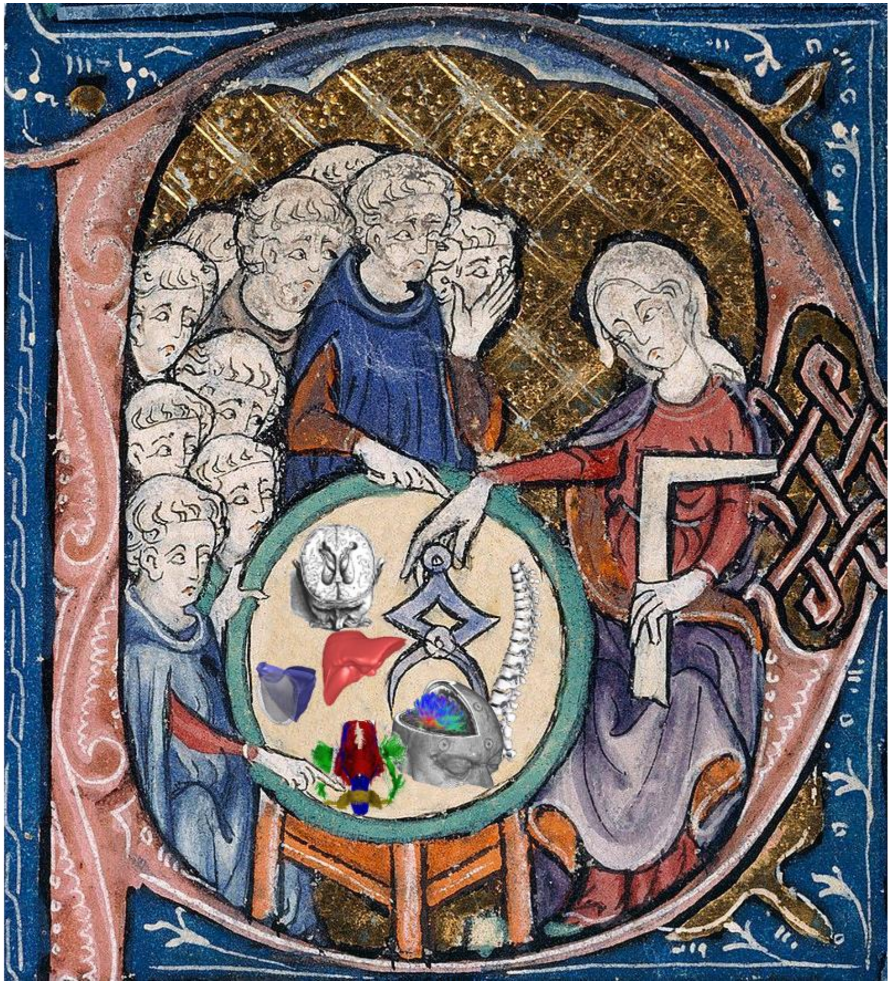
With contributions of Vincent Arsigny, Marco Lorenzi, Christof Seiler, Jonathan Boisvert, Nicholas Ayache, etc

Geometric Statistics for Computational Anatomy



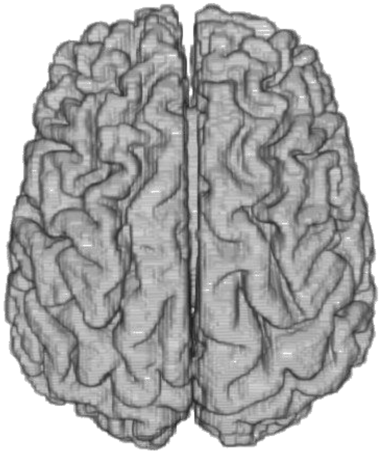
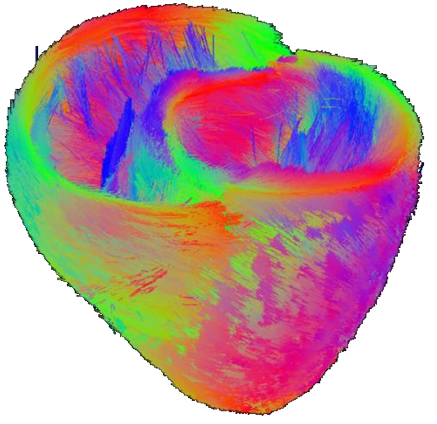
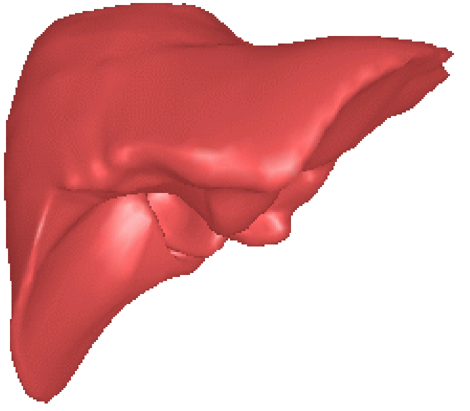
Freely adapted from “Women teaching geometry”, in Adelard of Bath translation of Euclid’s elements, 1310.

ASCLEPIOS

CobCom, Nov 2017

Inria

Computational Anatomy

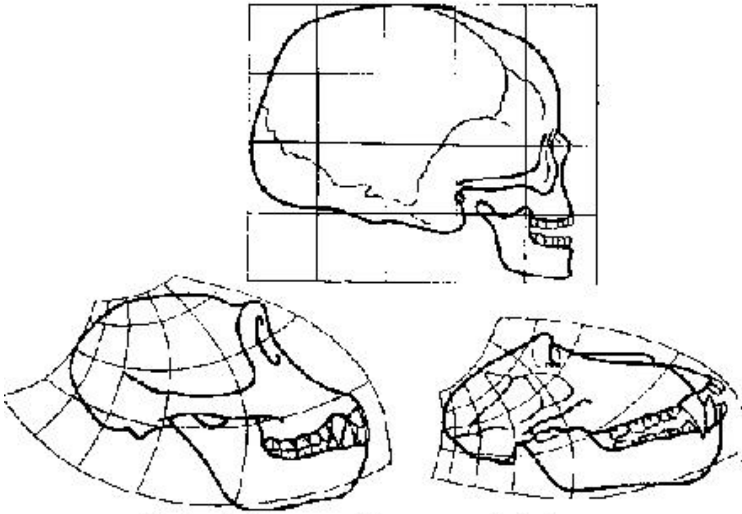
Revolution of medical imaging:

- From dissection to in-vivo in-situ medical imaging (MRI, d-MRI, CT)
- Large number of subjects: from representative individual to population

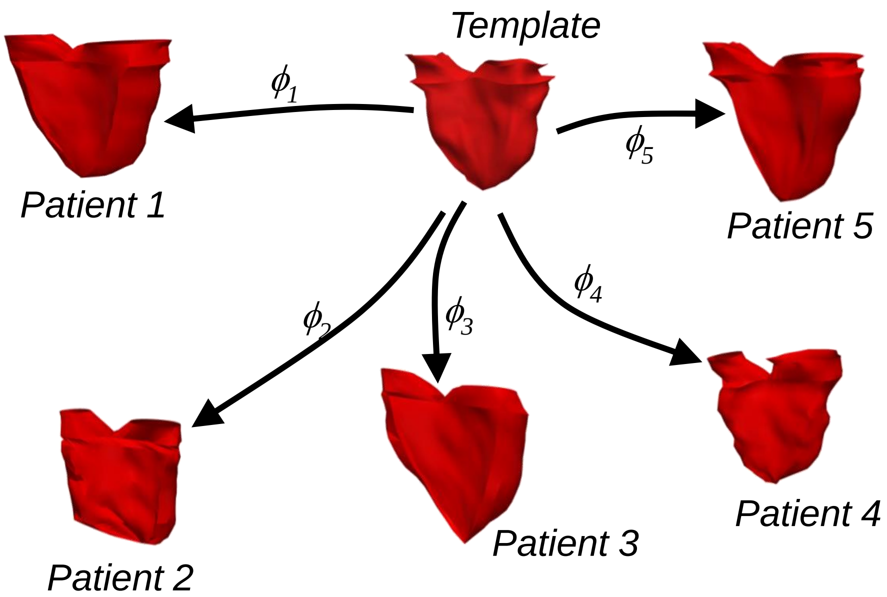
Design mathematical methods and algorithms to model and analyze the anatomy

- Statistics of organ shapes across subjects in species, populations, diseases…
  - Mean shape, Shape variability (Covariance), contrast diseases
- Model organ development across time (heart-beat, growth, ageing, ages…)
  - Predictive (vs descriptive) models of evolution, Correlation with clinical variables

Cross-sectional Deformation-based Morphometry



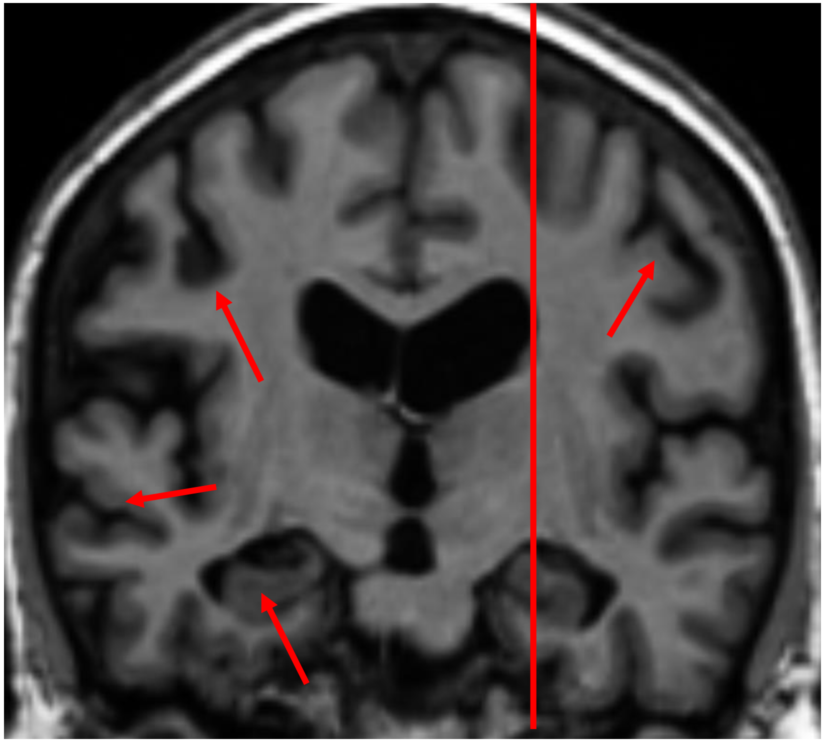
Skulls of a human, a chimpanzee and a baboon and transformations between them



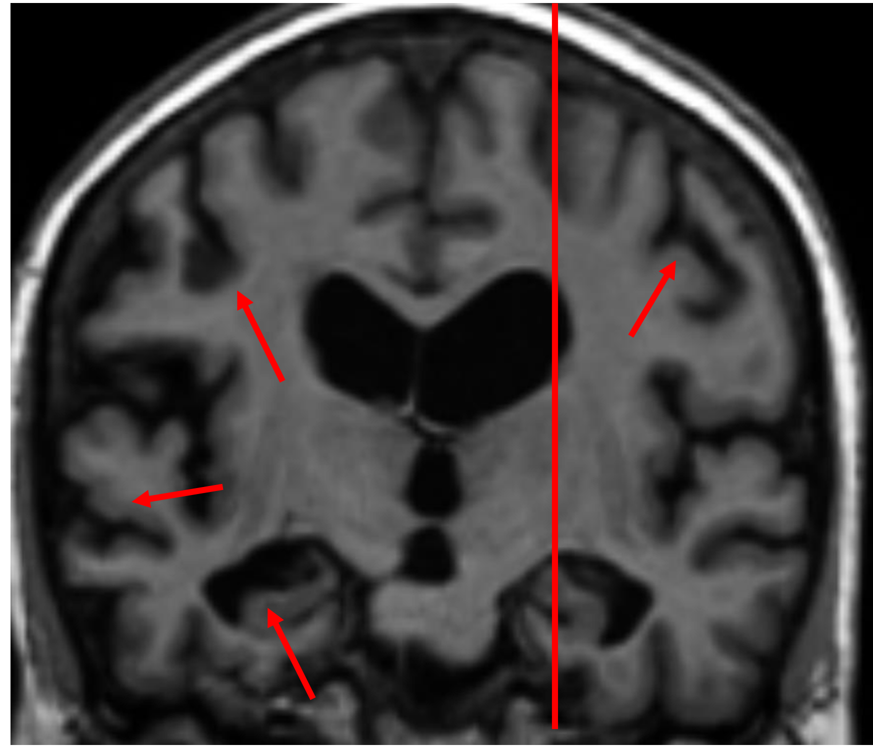
Measure of deformation [D'Arcy Thompson 1917, Grenander & Miller]

- Observation = **random deformation** of a reference template
- Deterministic template = anatomical invariants [Atlas ~ mean]
- Random deformations = geometrical variability [Covariance matrix]

Longitudinal structural damage in Alzheimer's Disease



baseline

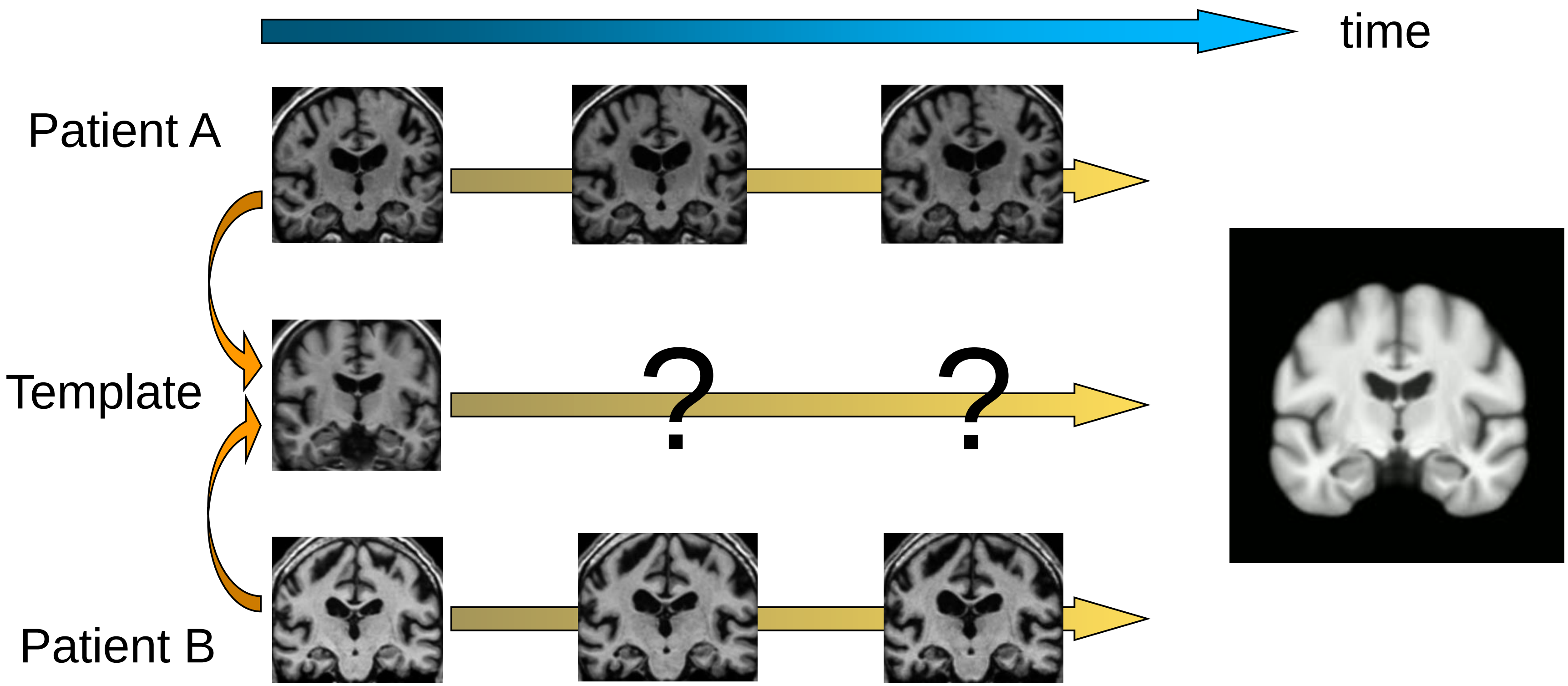


2 years follow-up

Widespread cortical thinning

Longitudinal deformation analysis

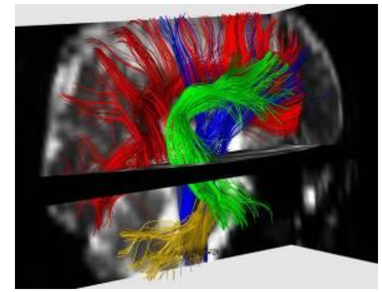
Deformation trajectories in different reference spaces



Mean longitudinal deformation across subjects?

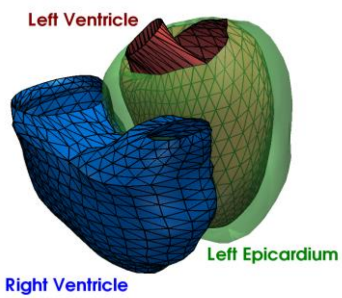
Convenient mathematical settings for transformations?

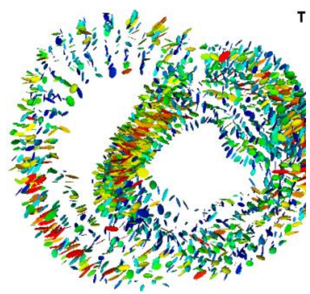
Geometric features in Computational Anatomy

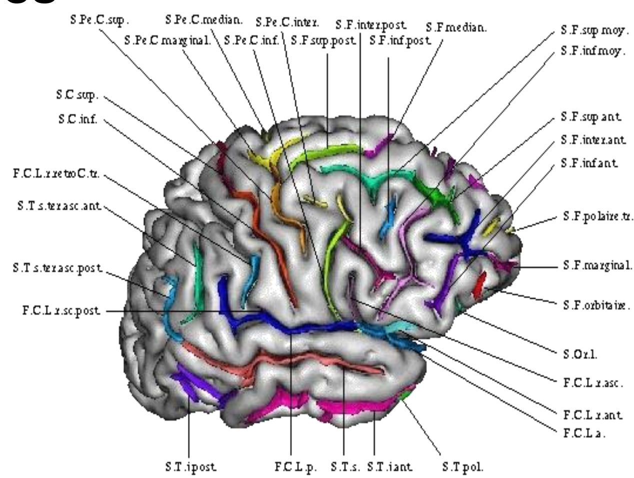


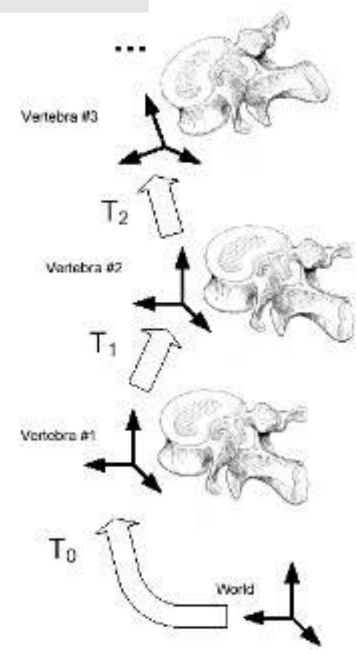
Noisy geometric features

- Tensors, covariance matrices
- Curves, fiber tracts, surfaces
- Shapes & quotient spaces









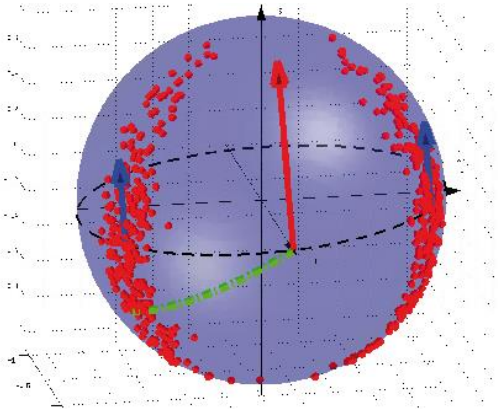
- **Transformations**
  - Rigid, affine, locally affine, diffeomorphisms

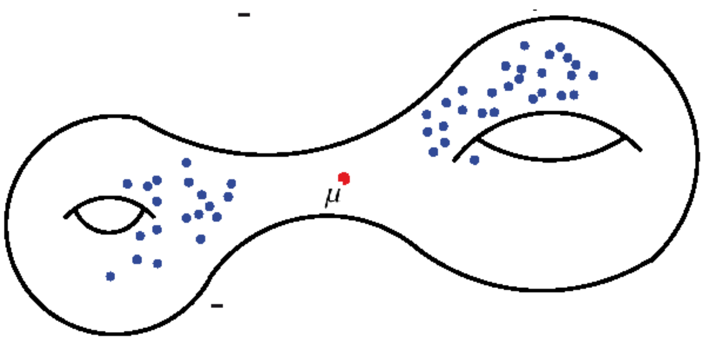
Goal: statistical modeling at the population level

- Deal with noise consistently on these **non-Euclidean manifolds**
- **A consistent computing framework for simple statistics**

Simple statistics… but of geometric quantities

Mean unit vector on the sphere? On a double torus?





Means of 3D rotations?

- Rotation matrix or unit quaternion: mean is not a rotation

$$\underline{\mathbf{R}} = \frac{1}{n}\sum_i \mathbf{R}_i \qquad \underline{q} = \frac{1}{n}\sum_i q_i \qquad \underline{r} = \frac{1}{n}\sum_i r_i$$

- Euler angles: mean depend on the order

Outline

Statistical computing on Riemannian manifolds

- Computing on Riemannian manifolds
- Simple statistics on manifolds
- Dimension reduction

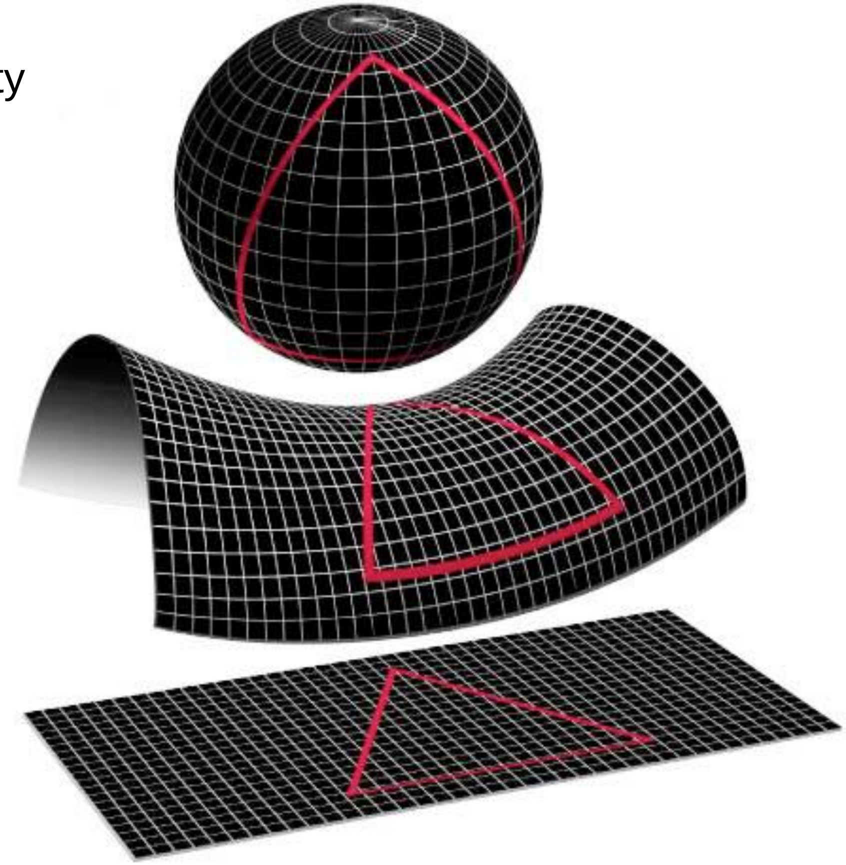
An affine setting for Lie groups

Conclusions

Differentiable manifolds

Définition:

- Locally Euclidean Topological space which can be globally curved
  - Same dimension + differential regularity



Simple Examples

- Sphere
- Saddle (hyperbolic space)
- Surface in 3D space

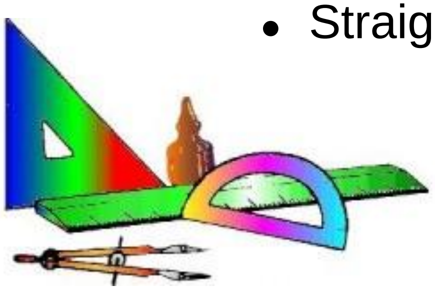
And less simple ones

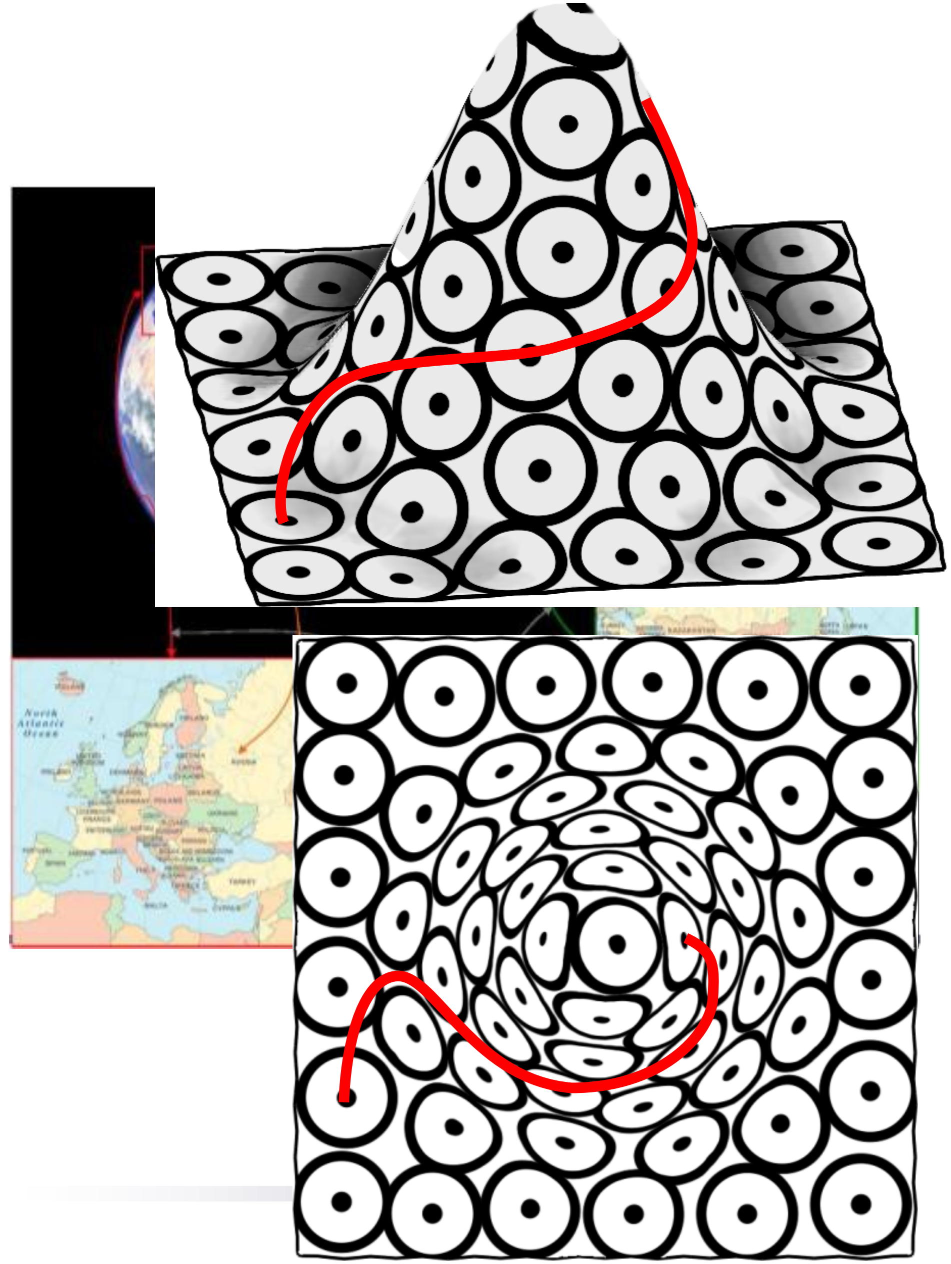
- Projective spaces
- 3D Rotations: $SO_3 \sim P_3$
- Rigid, affine Transformation
- Diffeomorphisms

Differentiable manifolds

Computing in a a manifold

- Extrinsic
  - Embedding in $\mathbb{R}^n$
- Intrinsic
  - Coordinates : charts
  - Atlas = consistent set of charts
- Measuring?
  - Volumes (surfaces)
  - Lengths
  - Straight lines





Measuring extrinsic distances

Basic tool: the scalar product

$$< v, w >= v^t w$$

- Norm of a vector

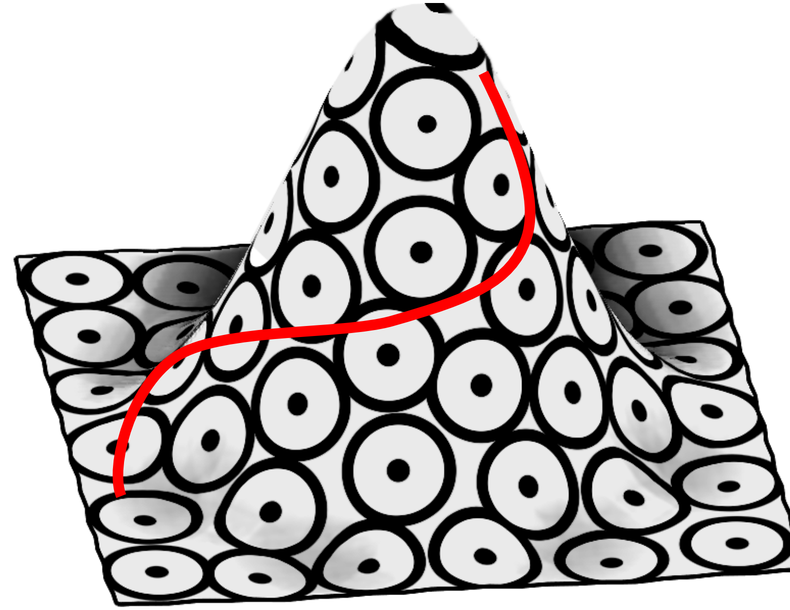
$$\|v\| = \sqrt{< v, v >}$$

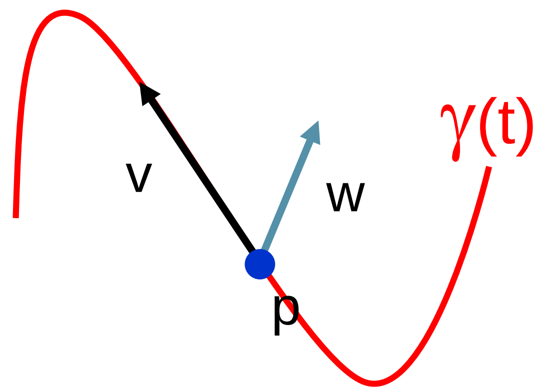
- Angle between vectors

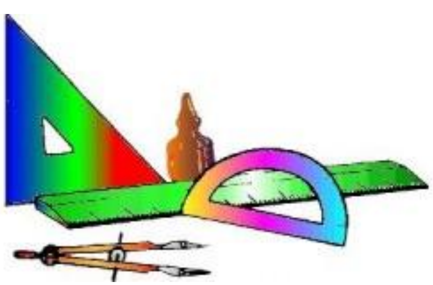
$$< v, w >= \cos(\alpha)\,\|v\|\,\|w\|$$

- Length of a curve

$$L(\gamma) = \int \| \dot{\gamma}(t) \| dt$$







Measuring extrinsic distances

Basic tool: the scalar product

Bernhard Riemann
1826-1866

$$< v, w >_p = {}^t v\, G(p)\, w$$

- Norm of a vector

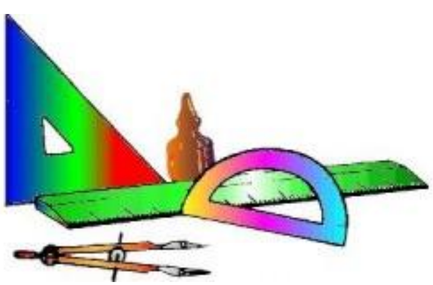
$$\|v\|_p = \sqrt{< v, v >_p}$$

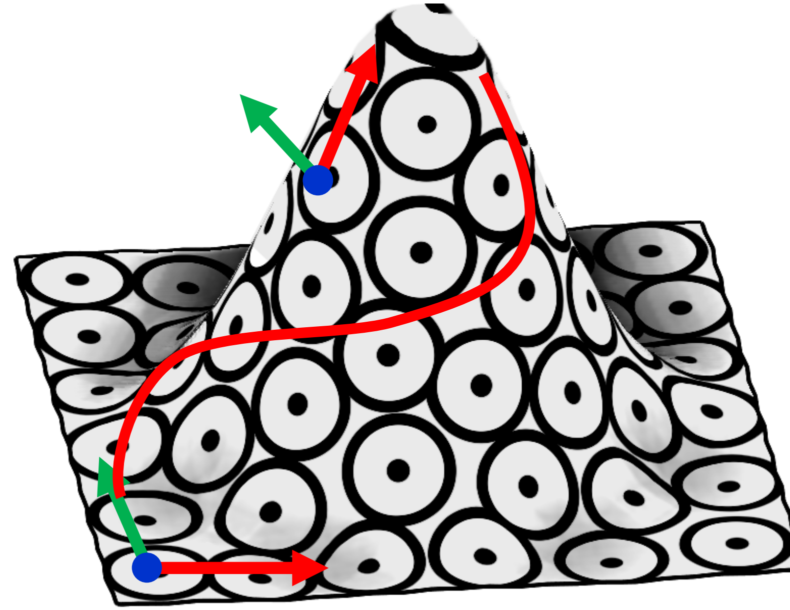
- Angle between vectors

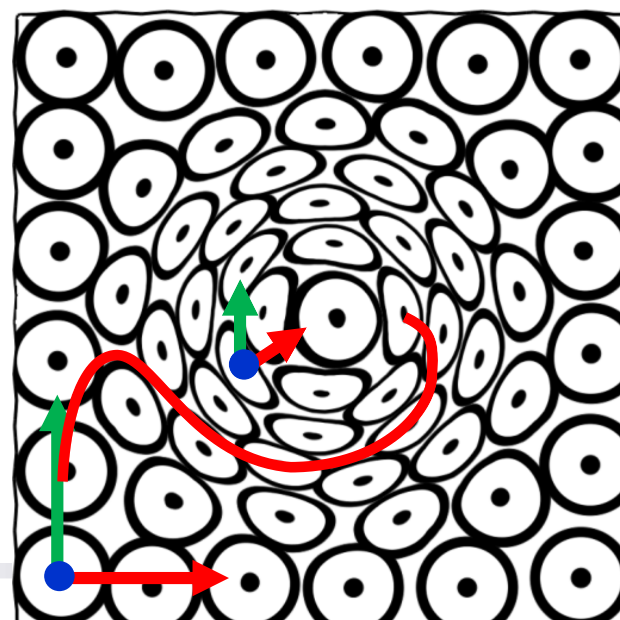
$$< v, w >_p = \cos(\alpha)\, \|v\|_p\, \|w\|_p$$

- Length of a curve

$$L(\gamma) = \int \| \dot{\gamma}(t) \|_{\gamma(t)}\, dt$$





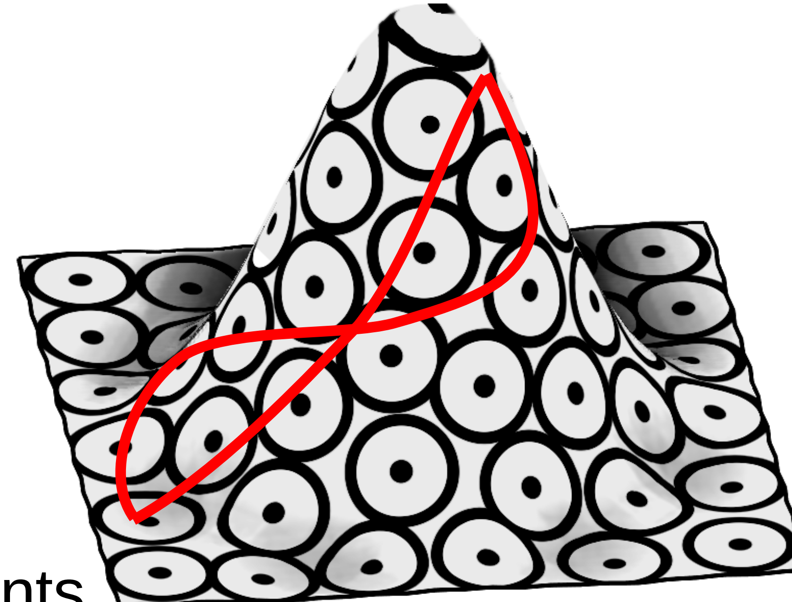


Riemannian manifolds

Basic tool: the scalar product

Bernhard Riemann
1826-1866

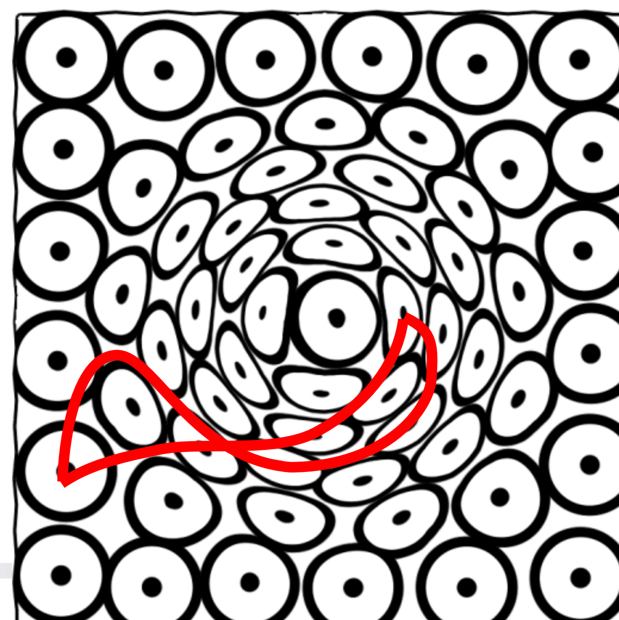
$$< v, w >_p = v^t G(p) w$$

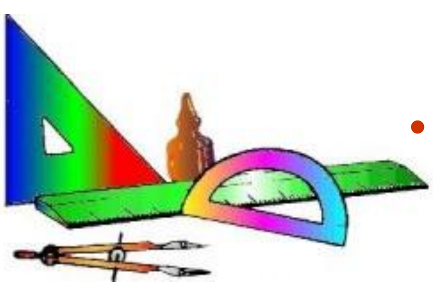


- Geodesic between 2 points
  - Shortest path
  - Calculus of variations (E.L.) : 2nd order differential equation (specifies acceleration)
  - Free parameters: initial speed and starting point

- Length of a curve

$$L(\gamma) = \int \| \dot{\gamma}(t) \|_{\gamma(t)} dt$$

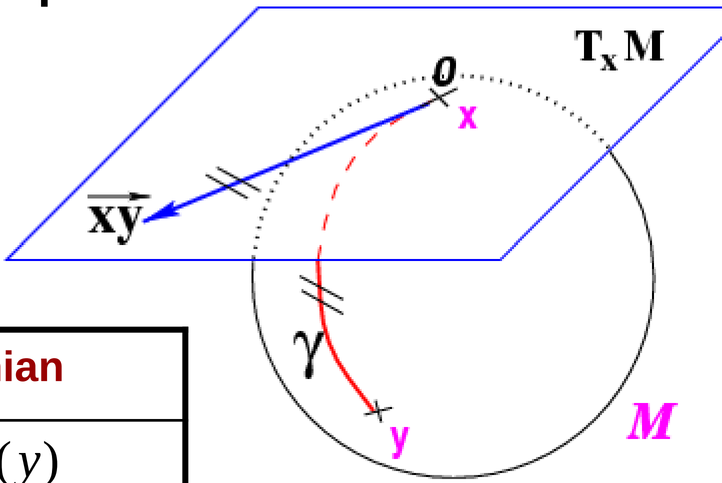




Bases of Algorithms in Riemannian Manifolds

Exponential map (Normal coordinate system):

- **$\text{Exp}_x(v)$** = geodesic shooting at x parameterized by the initial tangent vector v
- **$\text{Log}_x(y)$** = development of the manifold in the tangent space along geodesics
  - Geodesics = straight lines with Euclidean distance
  - Local → global domain: star-shaped, limited by the cut-locus
  - Covers all the manifold if **geodesically complete**



Reformulate algorithms with $\exp_x$ and $\log_x$

Vector -> Bi-point (no more equivalence classes)

Operation	Euclidean space	Riemannian
Subtraction	$\overrightarrow{xy} = y - x$	$\overrightarrow{xy} = \log_x(y)$
Addition	$y = x + \overrightarrow{xy}$	$y = \exp_x(\overrightarrow{xy})$
Distance	$\text{dist}(x, y) = \lVert y - x \rVert$	$\text{dist}(x, y) = \lVert \overrightarrow{xy} \rVert_x$
Gradient descent	$x_{t+\varepsilon} = x_t - \varepsilon \nabla C(x_t)$	$x_{t+\varepsilon} = \exp_{x_t}(-\varepsilon \nabla C(x_t))$

Outline

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- Dimension reduction

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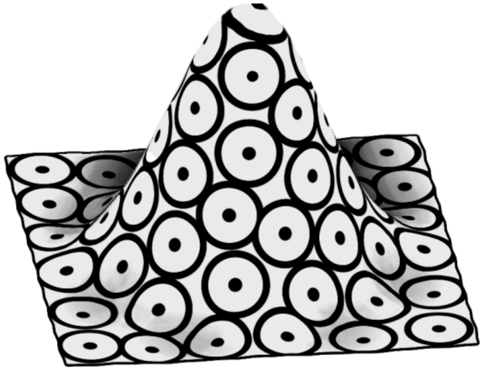
Conclusions

Random variable in a Riemannian Manifold



Intrinsic pdf of x

- For every set H

$$P(\mathbf{x} \in H) = \int_H p(y) dM(y)$$

- ~~Lebesgue's measure~~

→ Uniform Riemannian Mesure $dM(y) = \sqrt{\det(G(y))}\, dy$

Expectation of a real/vector function on M

- $\boldsymbol{E}_{\mathbf{x}}[\phi] = \int_M \phi(y) p(y) dM(y)$
- $\phi = dist^2$ (variance) : $\boldsymbol{E}_{\mathbf{x}}[dist(.,y)^2] = \int_M dist(y,z)^2 p(z) dM(z)$
- $\phi = \log(p)$ (information) : $\boldsymbol{E}_{\mathbf{x}}[\log(p)] = \int_M p(y) \log(p(y)) dM(y)$
- ~~$\phi = x$ (mean) : $\boldsymbol{E}_{\mathbf{x}}[\mathbf{x}] = \int_M y\, p(y) dM(y)$~~

First Statistical Tools: Moments

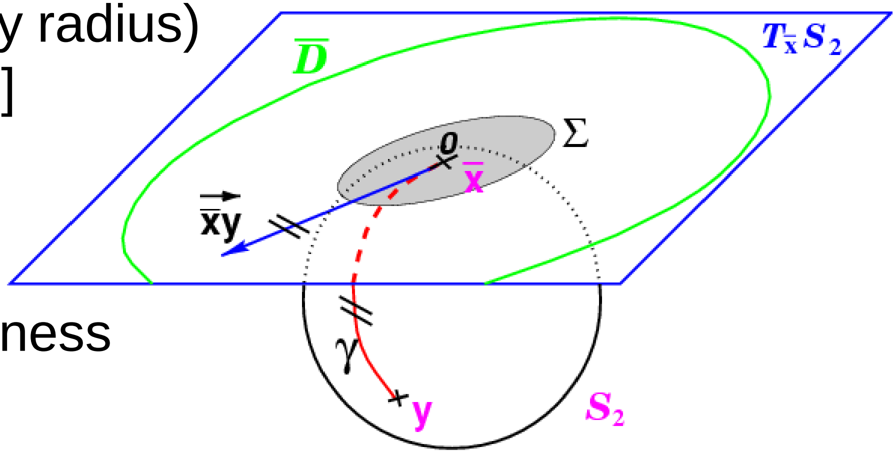
Frechet / Karcher mean minimize the variance

$$\mathrm{E}[\mathbf{x}] = \underset{y \in \mathrm{M}}{\operatorname{argmin}} \left(\mathrm{E}\left[\operatorname{dist}(y, \mathbf{x})^2 \right] \right) \quad \Rightarrow \quad \mathrm{E}\left[\overrightarrow{\bar{\mathrm{x}}\mathbf{x}} \right] = \int_{\mathrm{M}} \overrightarrow{\bar{\mathrm{x}}\mathbf{x}} . p_{\mathbf{x}}(z).d\mathrm{M}(z) = 0 \quad [P(C) = 0]$$

- Variational characterization: **Exponential barycenters**
- Existence and uniqueness (convexity radius) [Karcher 77 / Kendall 90 / Le / Afsari]

  Support in a regular geodesic ball with $r < r^* = \frac{1}{2}\min(inj(M), \pi/\sqrt{\kappa})$

- Empirical Fréchet mean: a.s. uniqueness [Arnaudon & Miclo 2013]



Gauss-Newton Geodesic marching

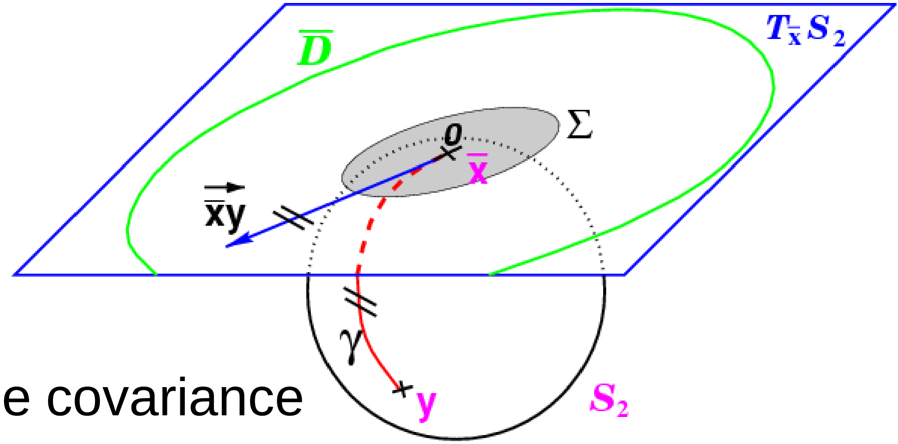
$$\bar{\mathrm{x}}_{t+1} = \exp_{\bar{\mathrm{x}}_t}(v) \quad \text{with} \quad v = \mathrm{E}\left[\overrightarrow{\mathrm{y}\mathbf{x}} \right] = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Log}_{\bar{\mathrm{x}}_t}(\mathrm{x}_i)$$

[Oller & Corcuera 95, Battacharya & Patrangenaru 2002, Pennec, NSIP'99 , JMIV06]

First Statistical Tools: Moments

Covariance (PCA) [higher moments]

$$\Sigma_{\mathbf{xx}} = \mathrm{E}\left[\left(\overrightarrow{\bar{\mathrm{x}}\mathbf{x}}\right).\left(\overrightarrow{\bar{\mathrm{x}}\mathbf{x}}\right)^{\mathrm{T}}\right] = \int_{\mathrm{M}} \left(\overrightarrow{\bar{\mathrm{x}}z}\right).\left(\overrightarrow{\bar{\mathrm{x}}z}\right)^{\mathrm{T}}.p_{\mathbf{x}}(z).d\mathrm{M}(z)$$



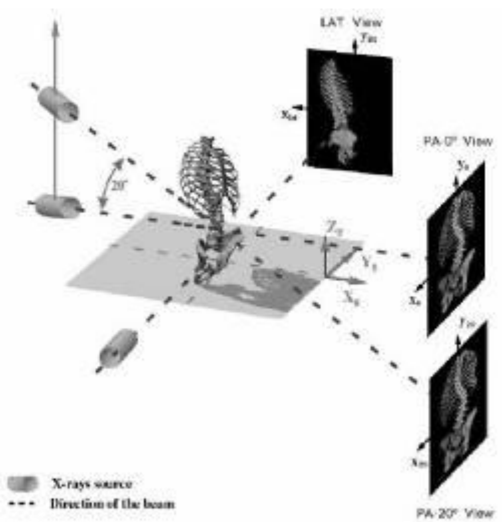
Principal component analysis

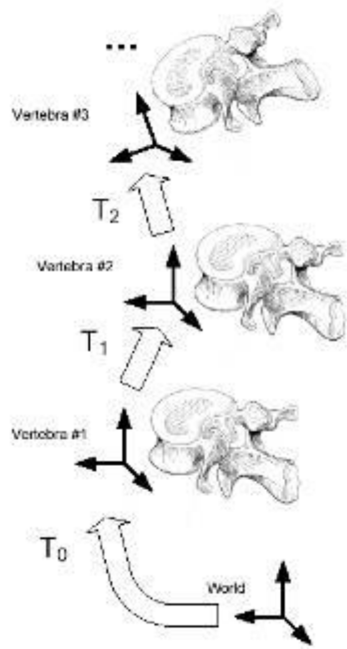
- Tangent-PCA: principal modes of the covariance
- Principal Geodesic Analysis (PGA) [Fletcher 2004]
- Barycentric subspace analysis (BSA) [Pennec 2015]

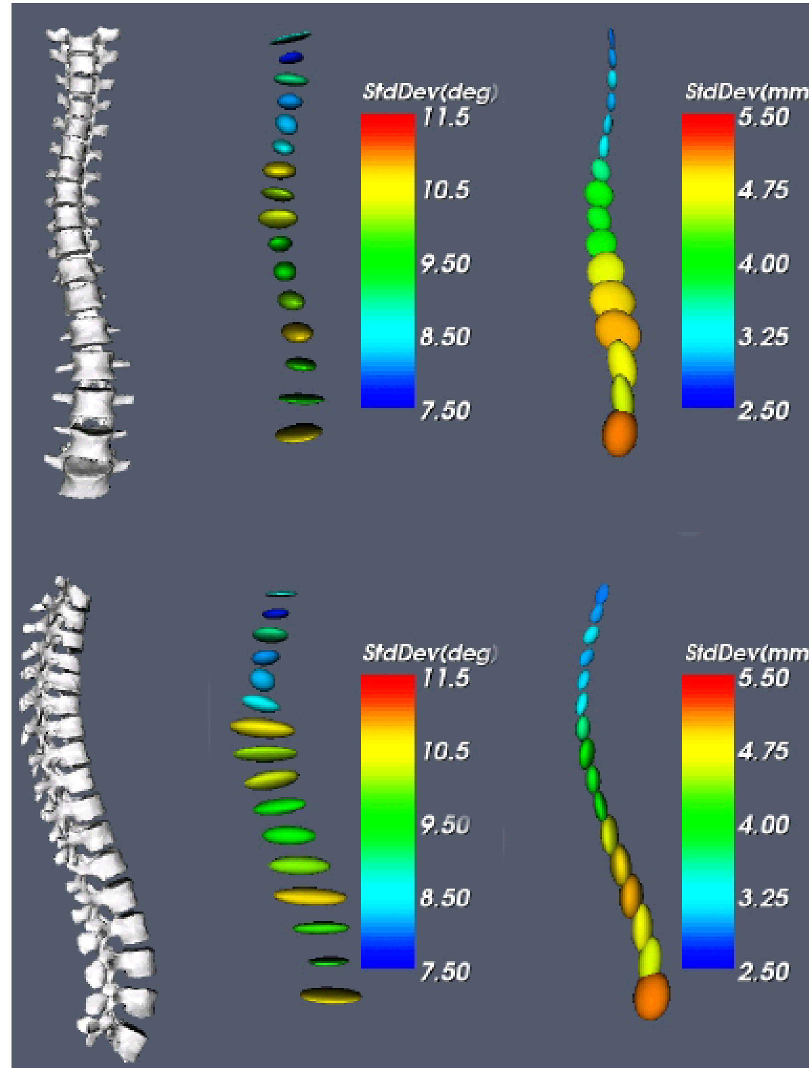
[Oller & Corcuera 95, Battacharya & Patrangenaru 2002, Pennec, NSIP'99 , JMIV06]

Statistical Analysis of the Scoliotic Spine

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]







Database

- 307 Scoliotic patients from the Montreal's Sainte-Justine Hospital.
- 3D Geometry from multi-planar X-rays

Mean

- Main translation variability is axial (growth?)
- Main rot. var. around anterior-posterior axis

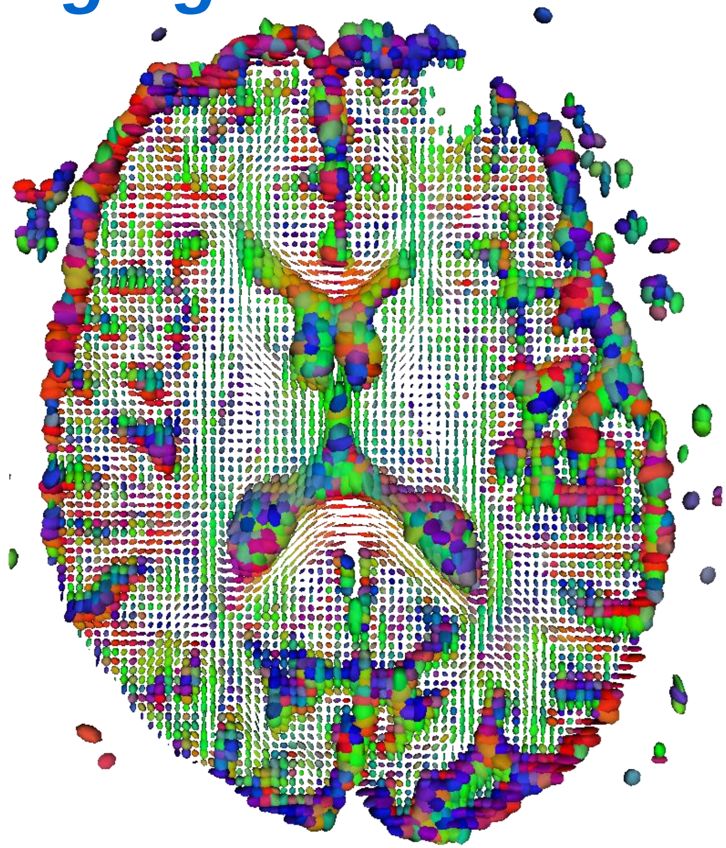
Statistical Analysis of the Scoliotic Spine

[J. Boisvert et al. ISBI'06, AMDO'06 and IEEE TMI 27(4), 2008]
AMDO'06 best paper award, Best French-Quebec joint PhD 2009

PCA of the Covariance:
4 first variation modes
have clinical meaning

- Mode 1: King's class I or III
- Mode 2: King's class I, II, III
- Mode 3: King's class IV + V
- Mode 4: King's class V (+II)

Diffusion Tensor Imaging



Covariance of the Brownian motion of water

- Filtering, regularization
- Interpolation / extrapolation
- Architecture of axonal fibers

Symmetric positive definite matrices

- Cone in Euclidean space (not complete)
- Convex operations are stable
  - mean, interpolation
- More complex operations are not
  - PDEs, gradient descent…

All invariant metrics under GL(n)

$$\langle W_1 \mid W_2 \rangle_{Id} = \mathrm{Tr}\left(W_1^T W_2\right) + \beta\, \mathrm{Tr}(W_1).\mathrm{Tr}(W_2) \quad (\beta > \text{-1/n})$$

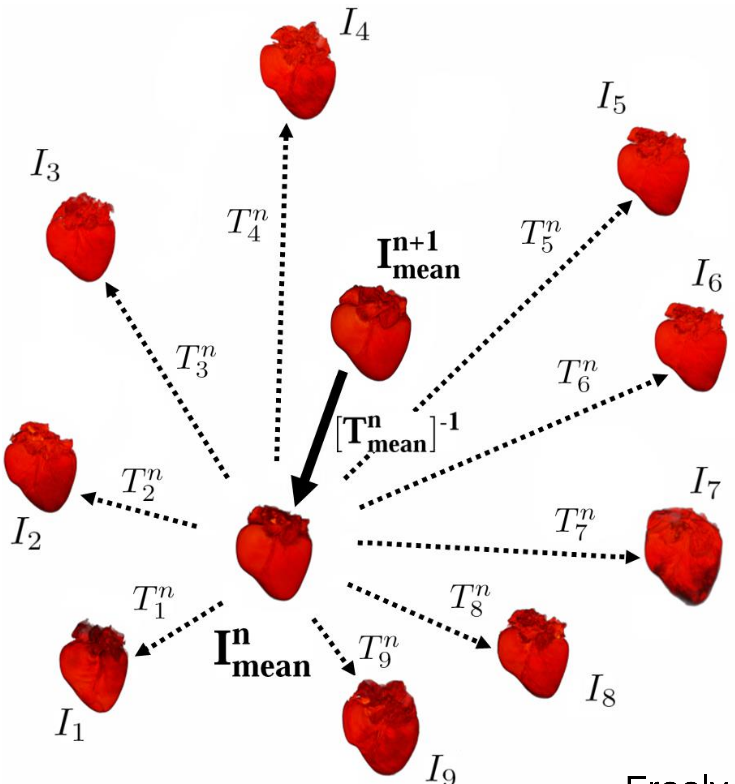
- Exponential map $\quad Exp_\Sigma(\overrightarrow{\Sigma\Psi}) = \Sigma^{1/2} \exp(\Sigma^{-1/2}.\overrightarrow{\Sigma\Psi}.\Sigma^{-1/2})\Sigma^{1/2}$
- Log map $\quad \overrightarrow{\Sigma\Psi} = Log_\Sigma(\Psi) = \Sigma^{1/2} \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2})\Sigma^{1/2}$
- Distance $\quad dist(\Sigma,\Psi)^2 = \left\langle \overrightarrow{\Sigma\Psi} \mid \overrightarrow{\Sigma\Psi} \right\rangle_\Sigma = \left\| \log(\Sigma^{-1/2}.\Psi.\Sigma^{-1/2}) \right\|_{Id}^2$

A Statistical Atlas of the Cardiac Fiber Structure

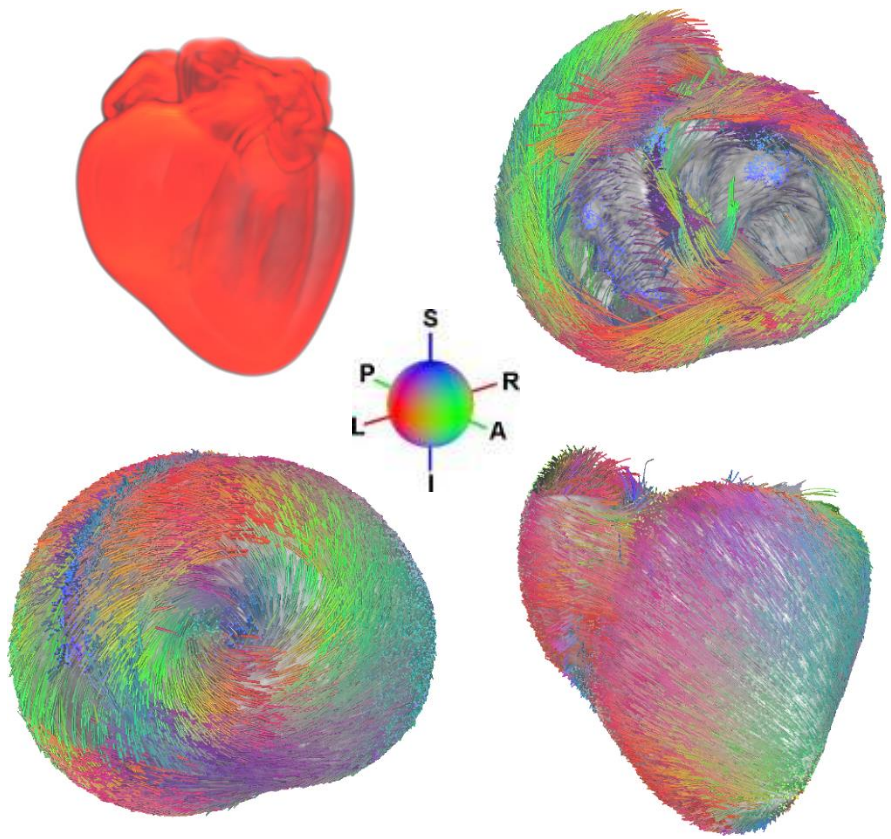
[J.M. Peyrat, et al., MICCAI'06, TMI 26(11), 2007]

Manifold data on a manifold

- Anatomical MRI and DTI
- Diffusion tensor on a 3D shape

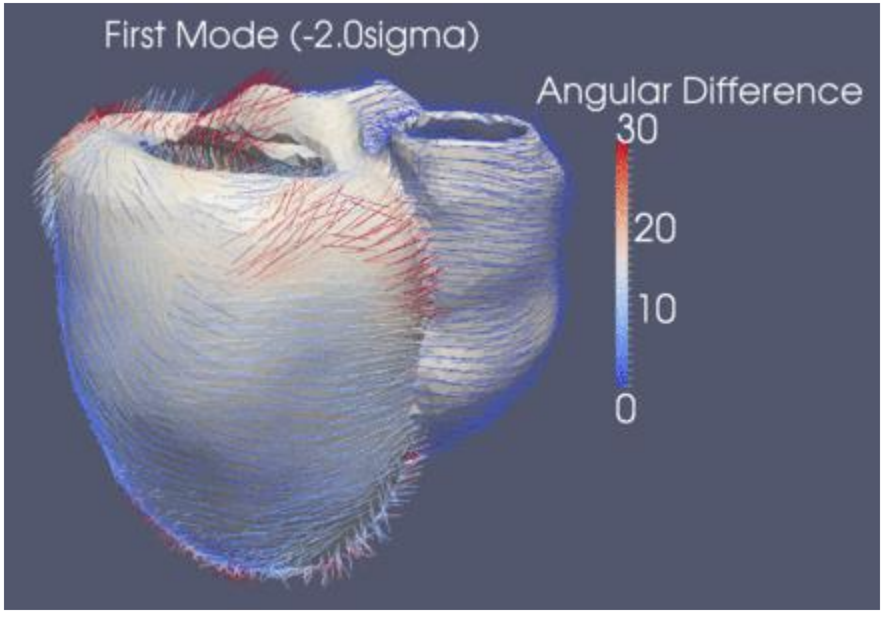


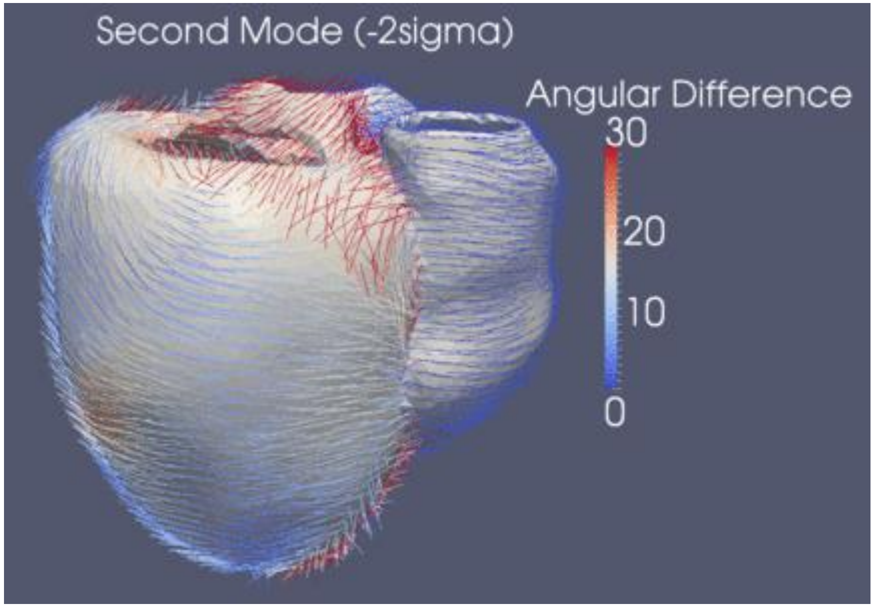
- *Average cardiac structure*
- Variability of fibers, sheets



Freely available at **http://www-sop.inria.fr/asclepios/data/heart**

A Statistical Atlas of the Cardiac Fiber Structure

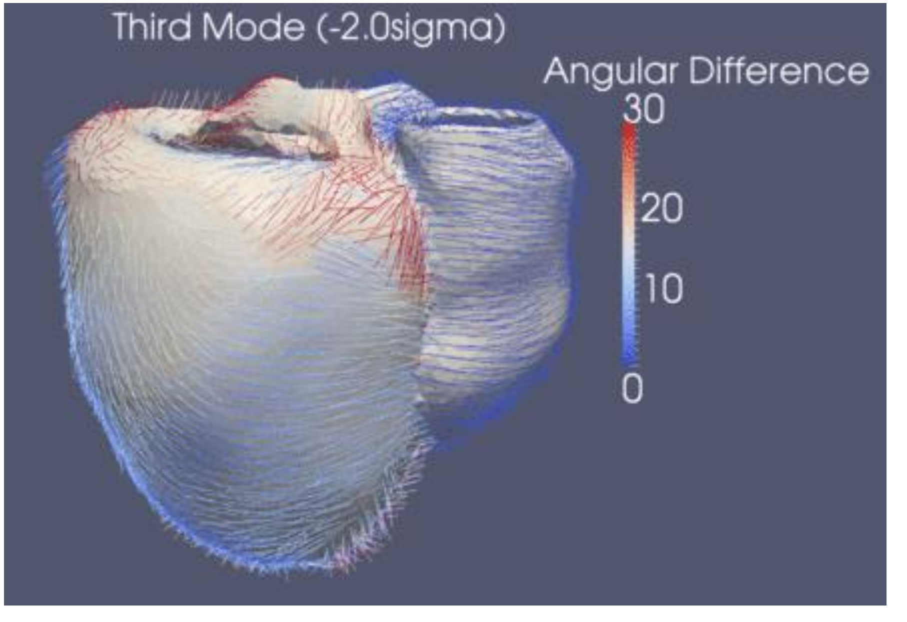




10 human ex vivo hearts (CREATIS-LRMN, France)

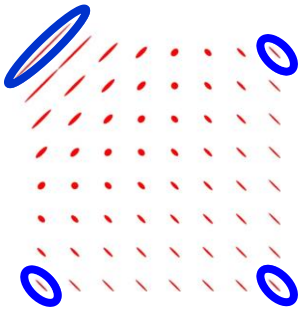
- Classified as healthy (controlling weight, septal thickness, pathology examination)
- Acquired on 1.5T MR Avento Siemens
  - bipolar echo planar imaging, 4 repetitions, 12 gradients
- Volume size: 128×128×52, 2 mm resolution

[R. Mollero, M.M Rohé, et al, FIMH 2015]



Manifold-valued image algorithms

Integral or sum in M: weighted Fréchet mean



- Interpolation
  - Linear between 2 elements: interpolation geodesic
  - Bi- or tri-linear or spline in images: weighted means
- Gaussian filtering: convolution = weighted mean

$$\Sigma(x) = \min \sum_i G_\sigma(x - x_i)\ \mathrm{dist}^2(\Sigma, \Sigma_i)$$

PDEs for regularization and extrapolation: the exponential map (partially) accounts for curvature

- Gradient of Harmonic energy = Laplace-Beltrami

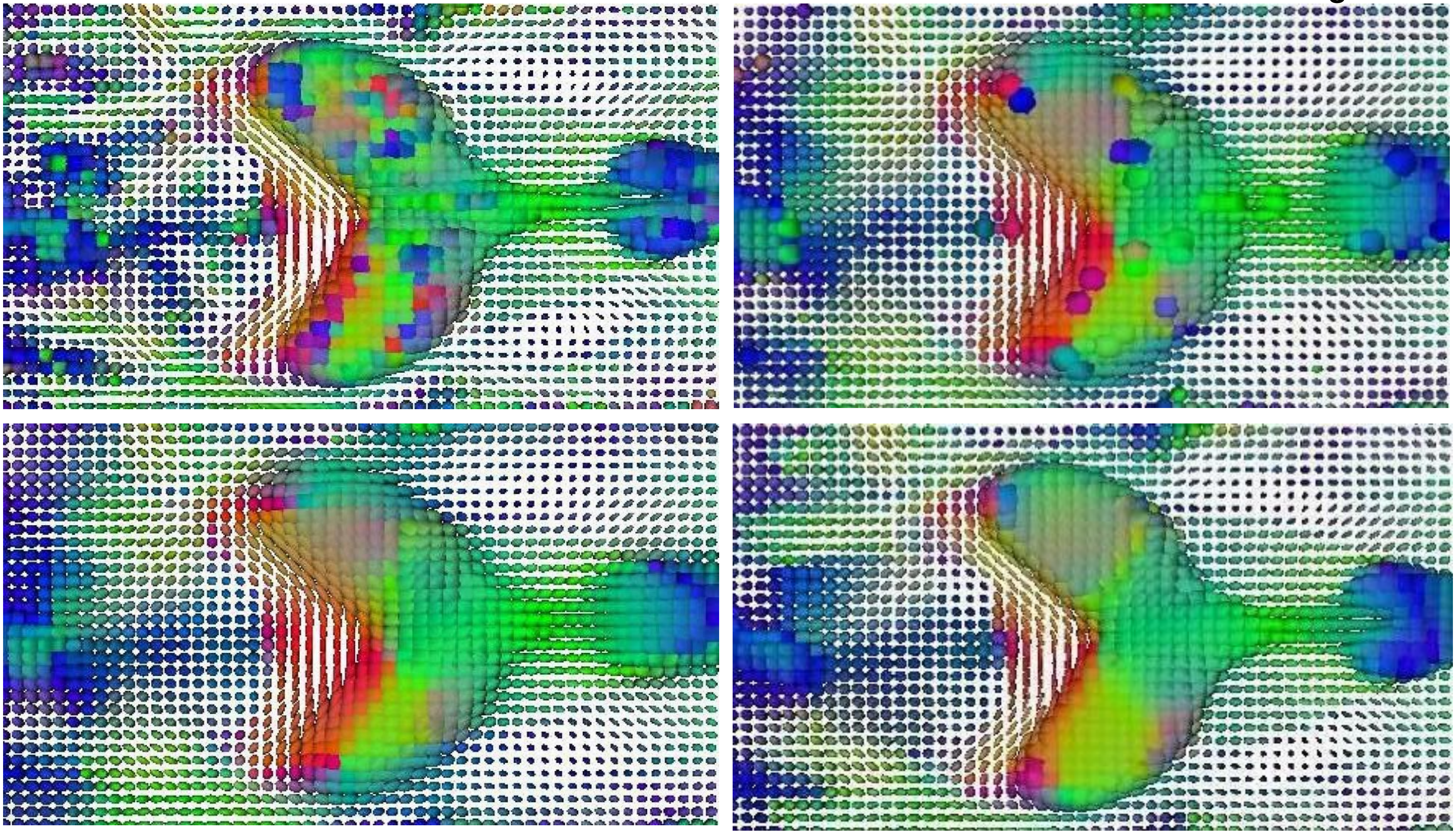
$$\Delta\Sigma(x) = \frac{1}{\varepsilon} \sum_{u \in S} \overrightarrow{\Sigma(x)\Sigma(x + \varepsilon u)} + O\left(\varepsilon^2\right)$$

- Anisotropic regularization using robust functions $\mathrm{Reg}(\Sigma) = \int \Phi\left(\left\|\nabla\Sigma(x)\right\|^2_{\Sigma(x)}\right) dx$
- Simple intrinsic numerical schemes thanks the exponential maps!

[Pennec, Fillard, Arsigny, IJCV 66(1), 2005, ISBI 2006]

Filtering and anisotropic regularization of DTI

Raw **Euclidean Gaussian smoothing**



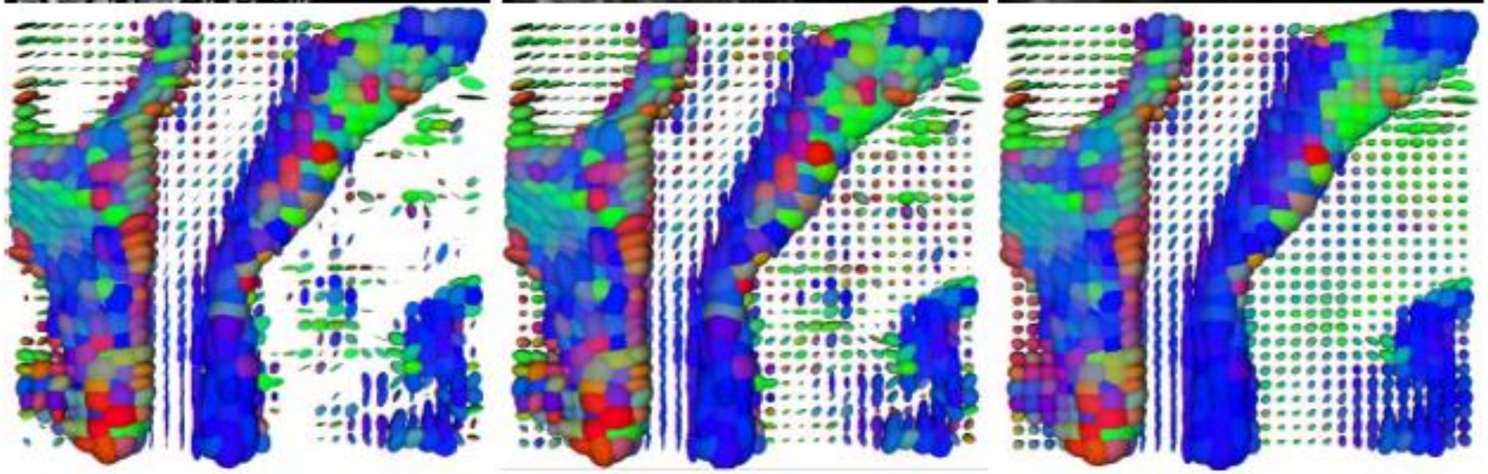
Riemann Gaussian smoothing **Riemann anisotropic smoothing**

Rician MAP estimation with Riemannian spatial prior

$$MAP(\Sigma) = -\sum_{i=1}^{N} \int \log\left(\frac{\hat{S}_i}{\sigma^2} \exp\left(-\frac{\hat{S}_i^{\,2} + S_i(\Sigma)^2}{2\sigma^2} \right) I_0\left(\frac{S_i(\Sigma)\hat{S}_i}{\sigma^2} \right) \right).dx + \int \Phi\left(\left\| \nabla\Sigma(x) \right\|^2_{\Sigma(x)} \right) dx$$

FA

Estimated tensors



Standard ML Rician MAP Rician

[Fillard, Arsigny, Pennec, Ayache ISBI’06, TMI 26(11) 2007]

Outline

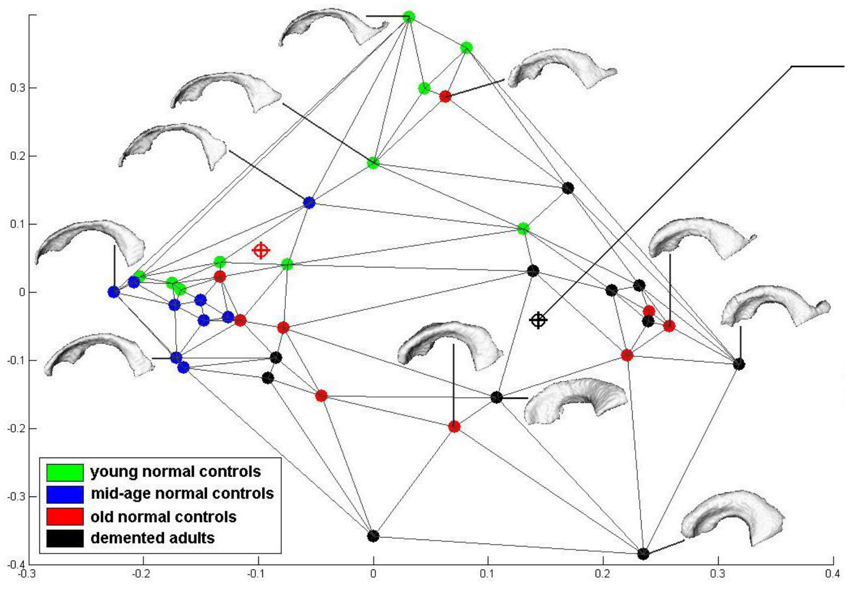
Statistical computing on Riemannian manifolds

- Computing on Riemannian manifolds
- Simple statistics on manifolds
- Dimension reduction

An affine setting for Lie groups

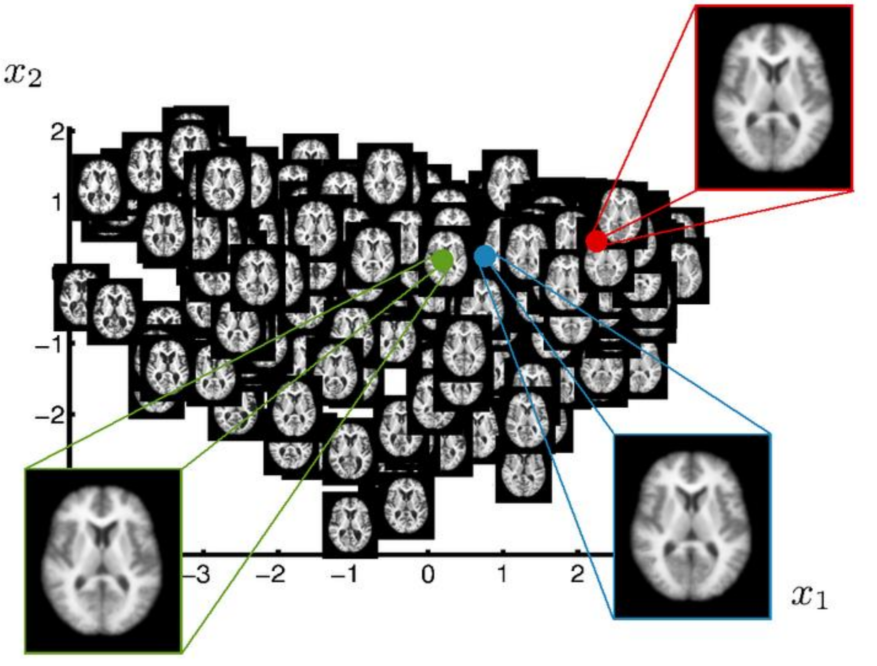
Conclusions

Low dimensional subspace approximation?



Manifold of cerebral ventricles

Etyngier, Keriven, Segonne 2007.



Manifold of brain images

S. Gerber et al, Medical Image analysis, 2009.

- Manifold dimension reduction
- When embedding structure is already manifold (e.g. Riemannian):
  Not manifold learning (LLE, Isomap,…) but **submanifold learning**

Dimension reduction: PCA in manifolds

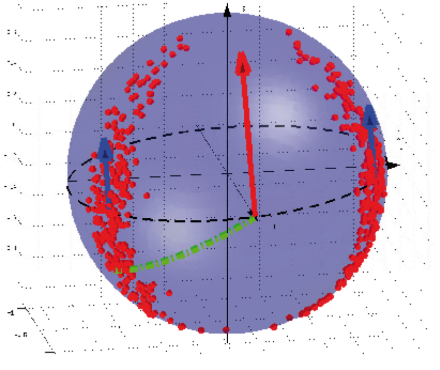
Tangent PCA:

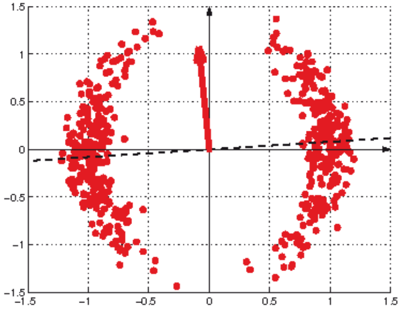
- **Maximize the explained variance in tangent space**

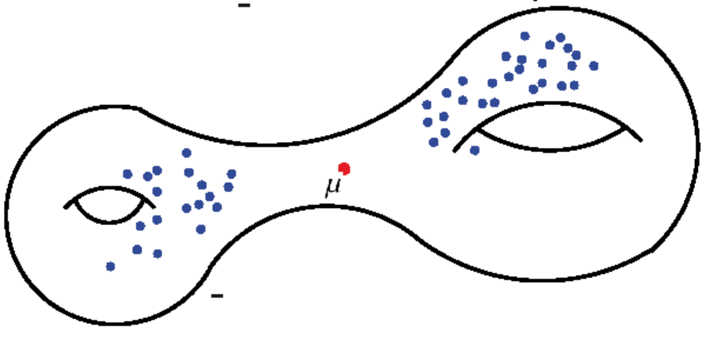
PGA [Fletcher 2004, Sommer 2014], GPCA [Huckeman 2010]

- Find the geodesic subspace generated by rays from a point that **minimizes the unexplained variance**

- Analysis still done around a « central point »:
  Problem for multimodal distributions







Affine span in Euclidean spaces

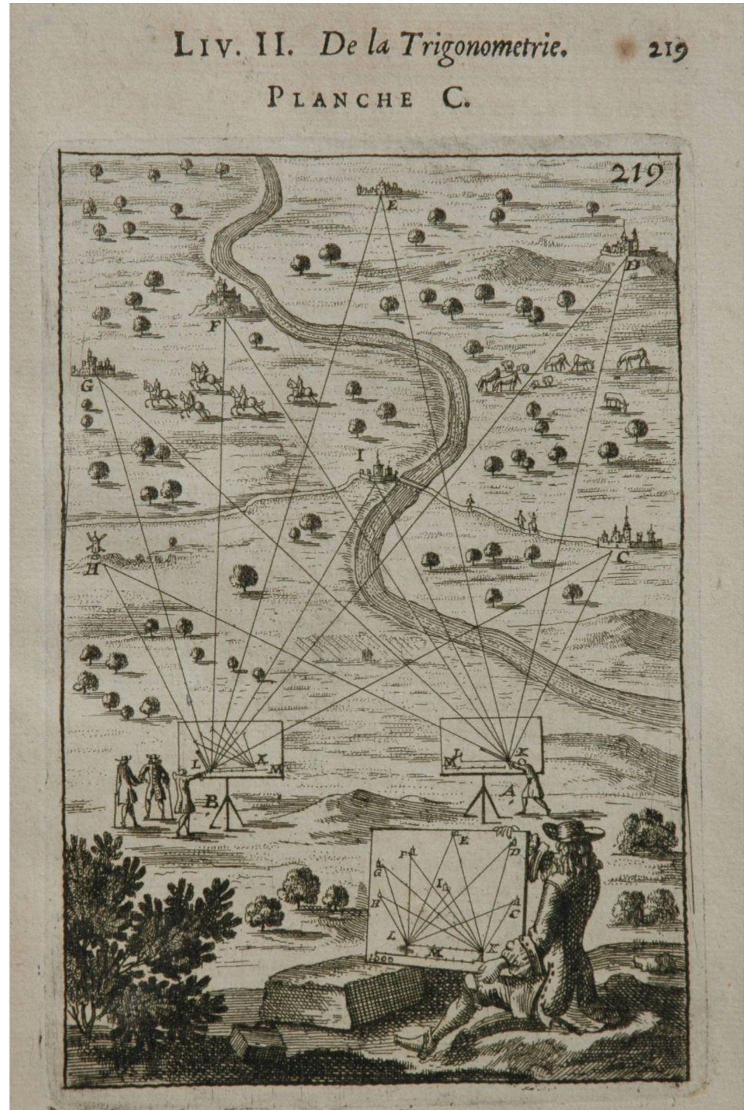
Affine span of (k+1) points: weighted barycentric equation

$$\mathrm{Aff}(x_0, x_1, \ldots x_k) = \{x = \textstyle\sum_i \lambda_i x_i \;\; with \; \sum_i \lambda_i = 1\}$$
$$= \{x \in R^n \;\; s.t \; \textstyle\sum_i \lambda_i (x_i - x) = 0, \lambda \in P_k^*\}$$

Key ideas:

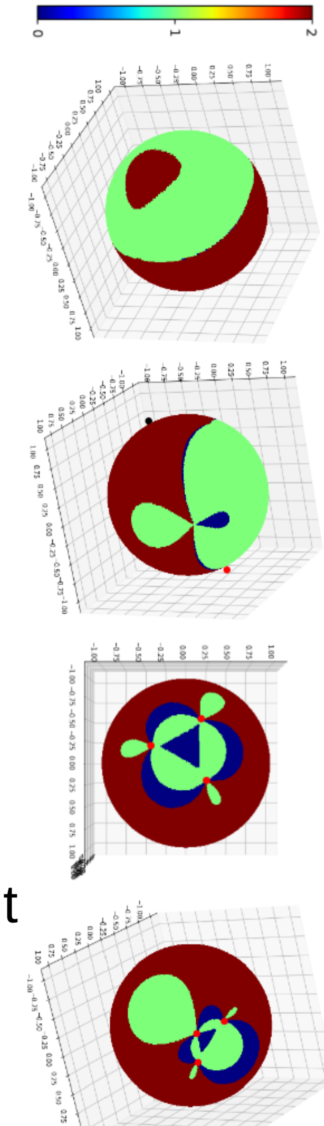
- ~~tPCA, PGA: Look at data points from the mean~~

- Triangulate position in submanifold from **several references**: locus of weighted mean



A. Manesson-Mallet. La géométrie Pratique, 1702

Barycentric subspaces and Affine spans



Non-linear subspaces in manifolds

- Fréchet / Karcher Barycentric subspaces
  Locus of weighted Fréchet / Karcher means
- Exponential barycentric subspace (EBS)
  critical points of weighted variance
- Affine span: completion of EBS

Properties

- K-dim submanifold around reference points
- Generalize geodesic subspaces [Fletcher et al.]
- EBS partitioned in cell complex by index of critical point
  brown = -2 (min) = KBS / green = -1 (saddle) / blue = 0 (max)

[X.P. Barycentric Subspace Analysis on Manifolds, Annals of Statistics 2017]

The natural object for PCA: Flags of subspaces in manifolds

Subspace approximations with variable dimension

- Optimal unexplained variance → non nested subspaces
- Nested forward / backward procedures → not optimal
- Optimize first, decide dimension later → Nestedness required
  [Principal nested relations: Damon, Marron, JMIV 2014]

Barycentric subspace analysis (BSA):

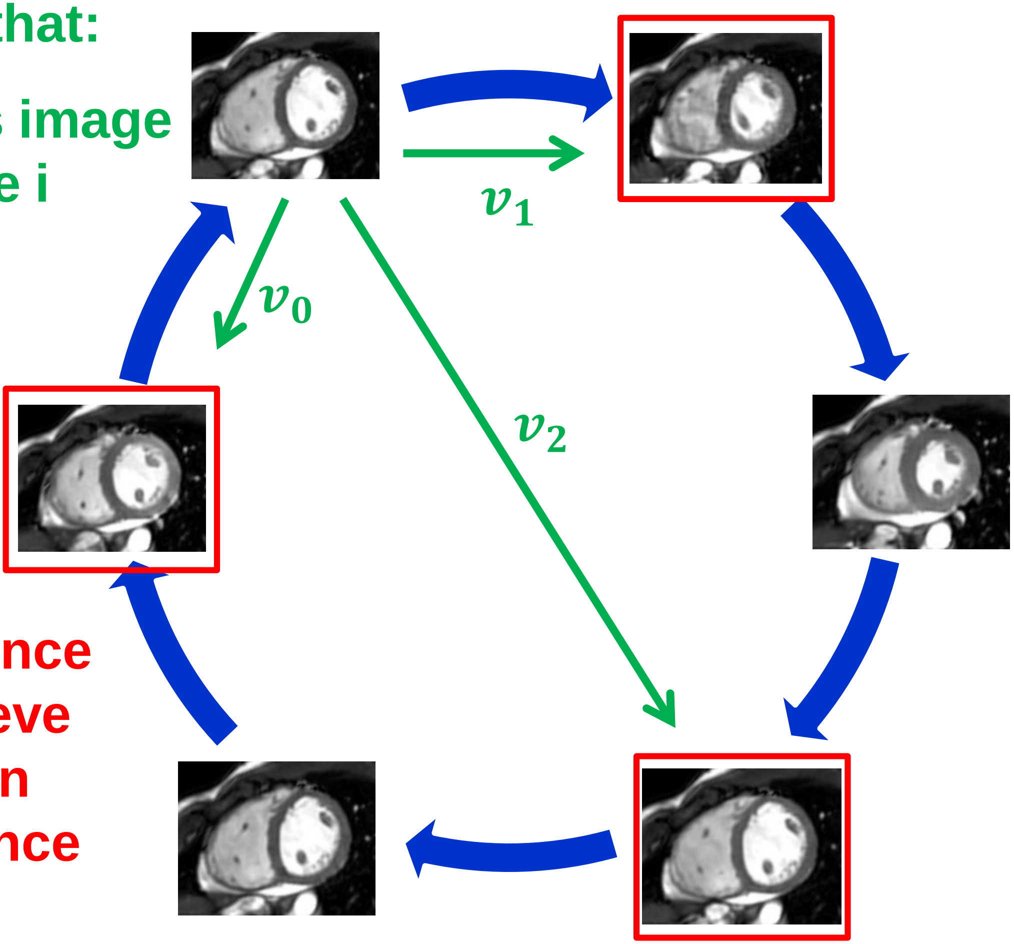
- Flags of affine spans in manifolds: sequence of (nested) $Aff(x_0, \dots x_i)$
- Energy on flags: Accumulated Unexplained Variance
  → produce the right ordered flags of subspaces in Euclidean spaces

[X.P. Barycentric Subspace Analysis on Manifolds, Annals of Statistics 2017]

Application in Cardiac motion analysis

Find weights λ_i and SVFs v_i such that:

- v_i **registers image to reference i**
- $\sum_i \lambda_i v_i = 0$



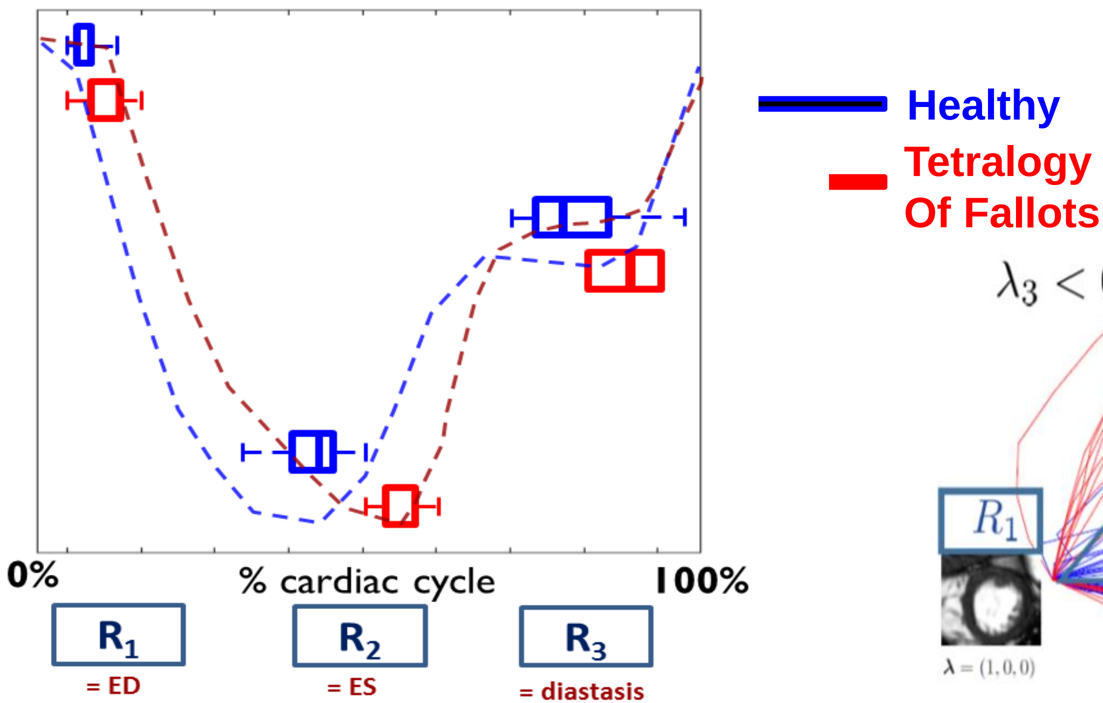
Optimize reference images to achieve best registration over the sequence

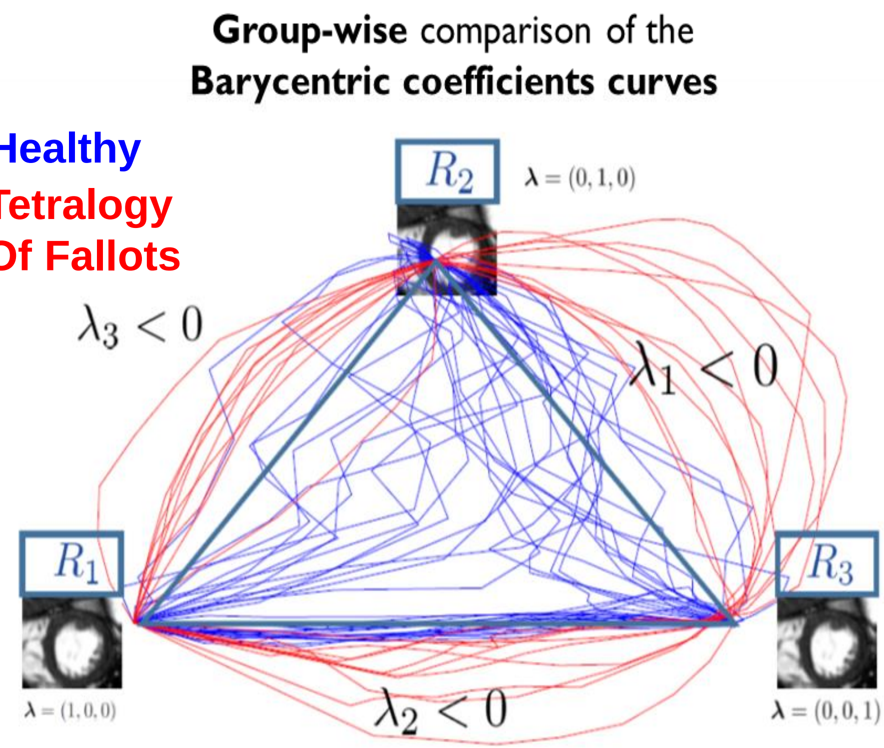
[Marc-Michel Rohé et al., MICCAI 2016]

Cardiac Motion Signature

Efficient low-dimensional representation of cardiac motion

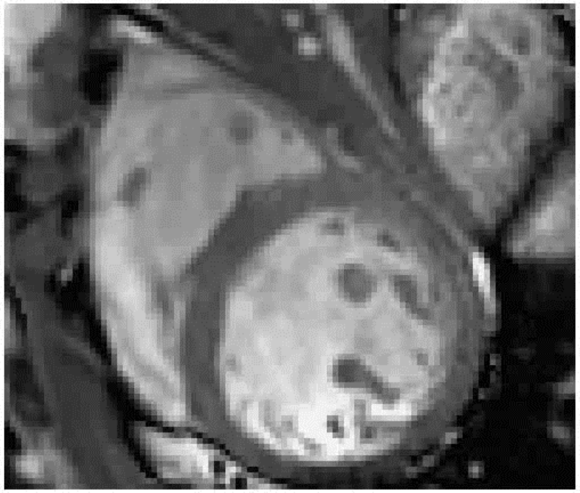
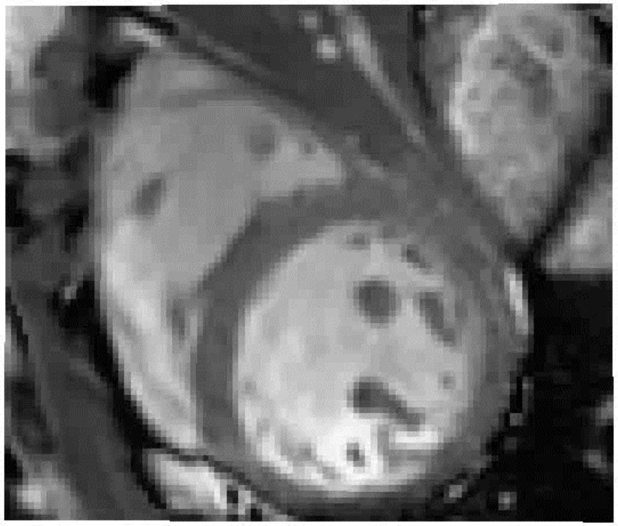
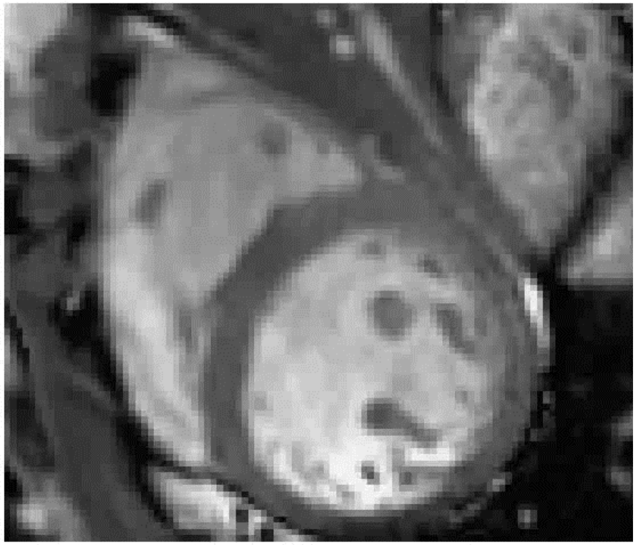
- **3 references** frame + **2 barycentric coeff * 30 frames**





[Marc-Michel Rohé et al., MICCAI 2016]

Application in Cardiac motion analysis

Original sequence	Barycentric Reconstruction (3 images)	PCA Reconstruction (2 modes)
		
30 images	**3 images + 2 coeff.**	**1 image + 2 SVF + 2 coeff.**
	Reconstr. error: 18.75 **Compression ratio: 1/10**	**Reconstr. error: 26.32 (+40%)** **Compression ratio: 1/4**

[Marc-Michel Rohé et al., MICCAI 2016]

Take home messages

Natural subspaces in manifolds

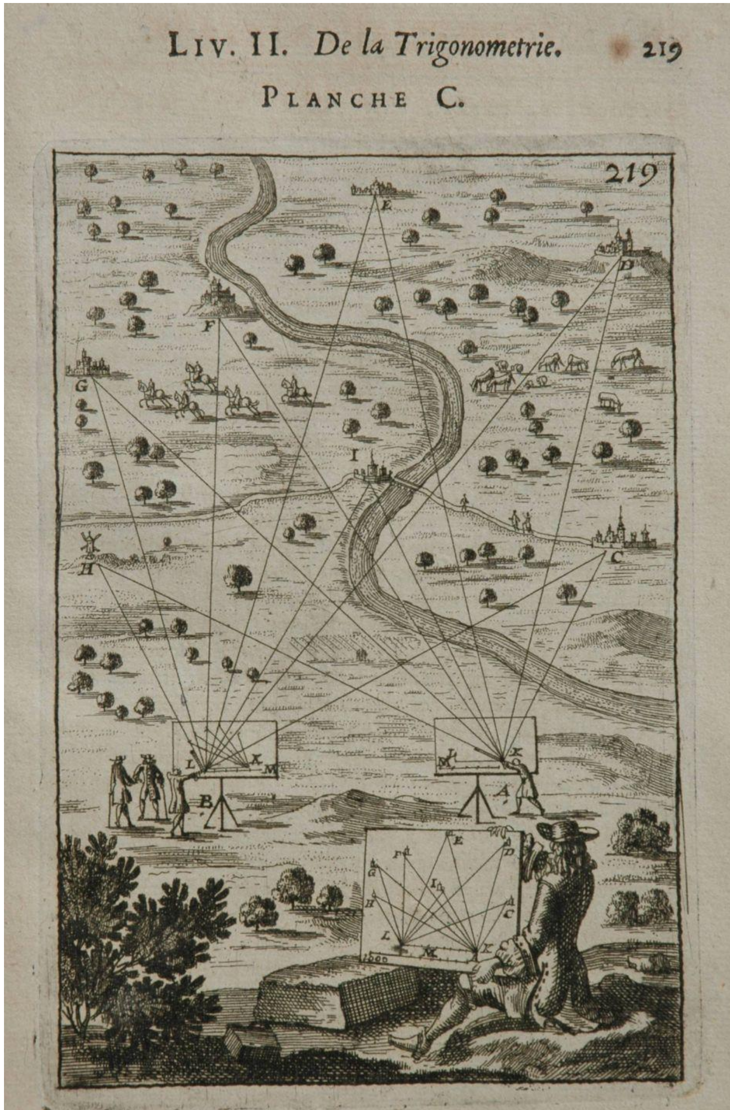
- PGA & Godesic subspaces:
  look at data points from the (unique) mean
- Barycentric subspaces:
  « triangulate » several reference points
  - Justification of multi-atlases?

Natural flag structure for PCA

- Hierarchically embedded approximation subspaces to summarize / describe data

Critical points (affine span) rather than minima (FBS/KBS)

- Barycentric coordinates need not be positive (convexity is a problem)
- Affine notion (more general than metric)



A. Manesson-Mallet. La géométrie Pratique, 1702

Outline

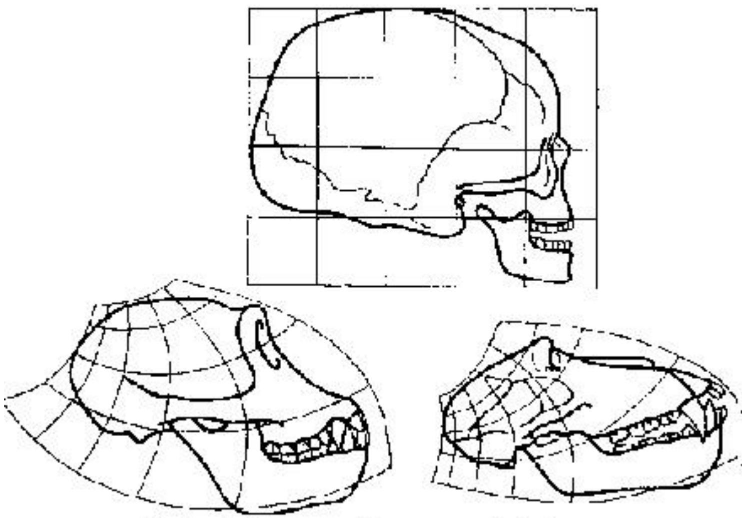
Statistical computing on Riemannian manifolds

An affine setting for Lie groups

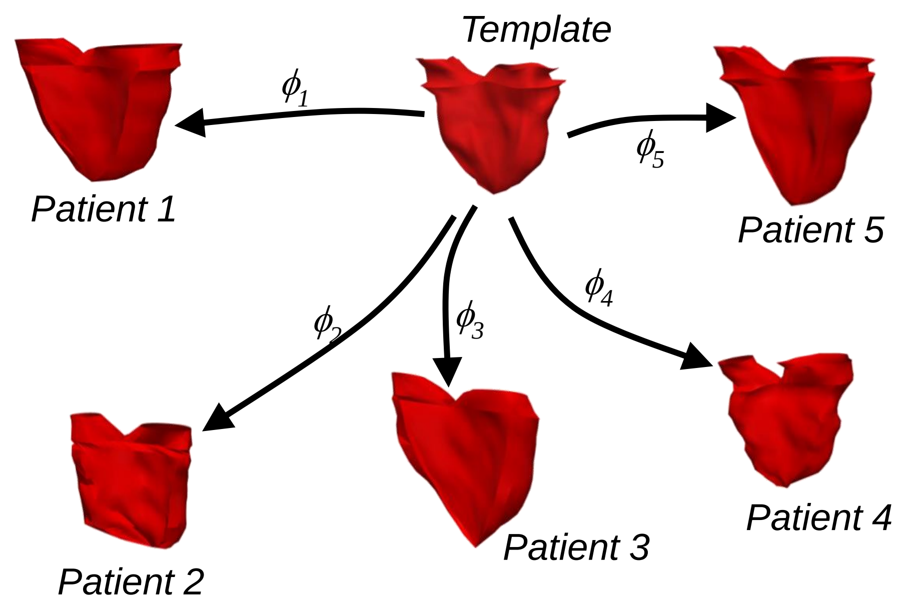
- The bi-invariant Cartan connection structure
- Extending statistics without a metric
- The SVF framework for diffeomorphisms

Conclusions

Cross-sectional Deformation-based Morphometry



Skulls of a human, a chimpanzee and a baboon
and transformations between them



Measure of deformation [D'Arcy Thompson 1917, Grenander & Miller]

- Observation = **random deformation** of a reference template
- Deterministic template = anatomical invariants [Atlas ~ mean]
- Random deformations = geometrical variability [Covariance matrix]

Limits of the Riemannian Framework

Lie group: Smooth manifold with group structure

- Composition $g \circ h$ and inversion g^{-1} are smooth
- Left and Right translation $L_g(f) = g \circ f$ $R_g(f) = f \circ g$
- Natural Riemannian metric choices
  - Chose a metric at Id: $<x,y>_{Id}$
  - Propagate at each point g using left (or right) translation $<x,y>_g = < DL_{g^{(-1)}} .x , DL_{g^{(-1)}} .y >_{Id}$

No bi-invariant metric in general

- **Incompatibility of the Fréchet mean with the group structure**
  - Left of right metric: different Fréchet means
  - The inverse of the mean is not the mean of the inverse
- Examples with simple 2D rigid transformations

- **Can we design a mean compatible with the group operations?**
- **Is there a more convenient structure for statistics on Lie groups?**

Properties of Lie groups

Flow of a left invariant vector field $\tilde{X} = DL.x$ from identity

- $\gamma_x(t)$ exists for all time
- One parameter subgroup: $\gamma_x(s+t) = \gamma_x(s).\,\gamma_x(t)$

Lie group exponential

- $Exp(x \in \mathfrak{g}) = \gamma_x(1) \in G$
- Diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G (not true in general for inf. dim)

3 curves parameterized by the same tangent vector

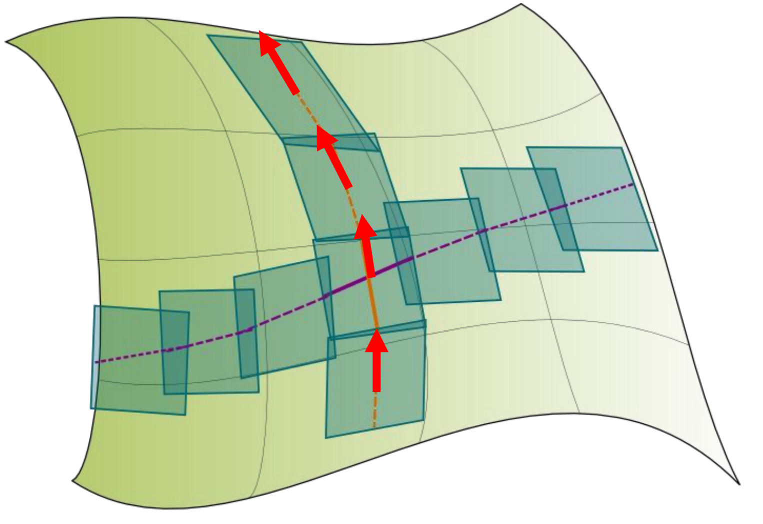
- Left / Right-invariant geodesics, one-parameter subgroups

Question: Can one-parameter subgroups be geodesics?

Affine connection spaces

Affine Connection (infinitesimal parallel transport)

- Acceleration = derivative of the tangent vector along a curve
- Projection of a tangent space on a neighboring tangent space



Adapted from Lê Nguyên Hoang, science4all.org

Geodesics = straight lines

- Null acceleration: $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
- 2nd order differential equation: Normal coordinate system
- Local exp and log maps

Canonical Connections on Lie Groups

A unique Cartan-Schouten connection

- Symmetric (no torsion) and bi-invariant
- For which geodesics through Id are one-parameter subgroups (group exponential)
  - Matrices : M(t) = A.exp(t.V)
  - Diffeos : **translations of Stationary Velocity Fields (SVFs)**

Levi-Civita connection of a bi-invariant metric (if it exists)

- Continues to exists in the absence of such a metric (e.g. for rigid or affine transformations)

Two flat connections (left and right)

- **Absolute parallelism**: no curvature but torsion (Cartan / Einstein)

Outline

Statistical computing on Riemannian manifolds

Computing on Lie groups

- The bi-invariant affine Cartan connection structure
- Extending statistics without a metric
- The SVF framework for diffeomorphisms

Towards more complex geometries

Statistics on an affine connection space

~~Fréchet mean~~: exponential barycenters

- $\sum_i Log_x(y_i) = 0$ **[Emery, Mokobodzki 91, Corcuera, Kendall 99]**
- Existence & **local uniqueness** if local convexity **[Arnaudon & Li, 2005]**

For Cartan-Schouten connections **[Pennec & Arsigny, 2012]**

- Locus of points *x* such that $\sum Log\left(x^{-1}.y_i\right) = 0$
- Algorithm: fixed point iteration (**local convergence**)

$$x_{t+1} = x_t \circ Exp\left(\frac{1}{n}\sum Log(x_t^{-1}.y_i)\right)$$

- **Mean stable by left / right composition and inversion**
  - If m is a mean of $\{g_i\}$ and h is any group element, then

    $h \circ m$ is a mean of $\{h \circ g_i\}$, $m \circ h$ is a mean of the points $\{g_i \circ h\}$

    and $m^{(-1)}$ is a mean of $\left\{g_i^{(-1)}\right\}$

Special matrix groups

Heisenberg Group (resp. Scaled Upper Unitriangular Matrix Group)

- No bi-invariant metric
- Group geodesics defined globally, all points are reachable
- **Existence and uniqueness of bi-invariant mean** (closed form resp. solvable)

Rigid-body transformations

- Logarithm well defined iff log of rotation part is well defined, i.e. if the Givens rotation have angles $|\theta_i| < \pi$
- **Existence and uniqueness with same criterion as for rotation parts (same as Riemannian)**

SU(n) and GL(n)

- Logarithm does not always exists (need 2 exp to cover the group)
  - If it exists, it is unique if no complex eigenvalue on the negative real line
- **Generalization of geometric mean**

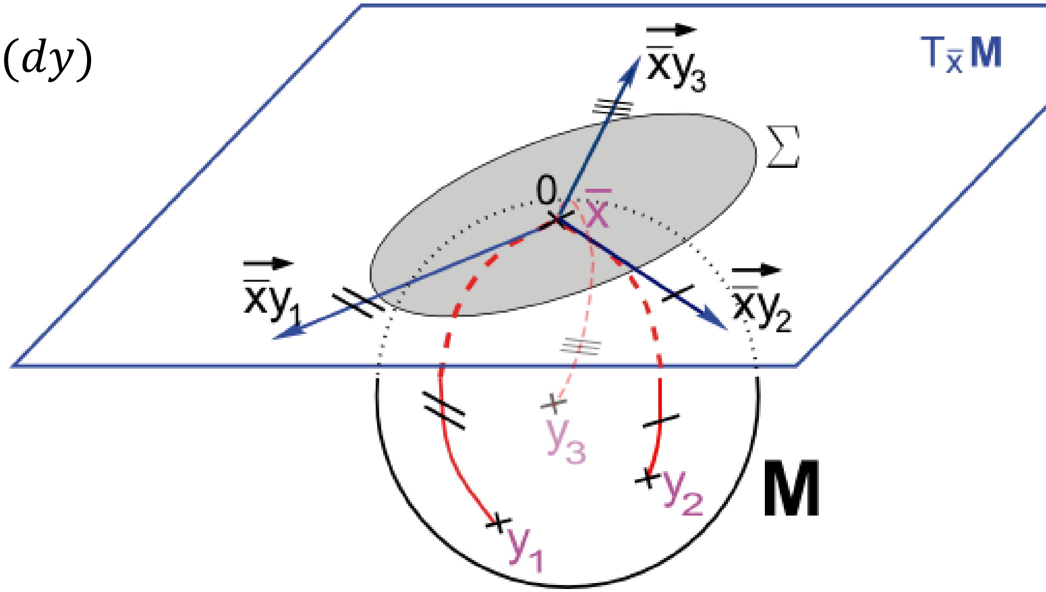
Generalization of the Statistical Framework

Covariance matrix & higher order moments

- Defined as tensors in tangent space

$$\Sigma = \int Log_x(y) \otimes Log_x(y)\, \mu(dy)$$

- Matrix expression changes according to the basis



Other statistical tools

- Mahalanobis distance well defined and bi-invariant
- ~~Tangent Principal Component Analysis (t-PCA)~~
- PGA & BSA, provided a data likelihood
- Independent Component Analysis (ICA)?

Cartan Connections vs Riemannian

What is similar

- Standard differentiable geometric structure [curved space without torsion]
- Normal coordinate system with Exp_x et Log_x [finite dimension]

Limitations of the affine framework

- No metric (but no choice of metric to justify)
- The exponential does always not cover the full group
  - Pathological examples close to identity in finite dimension
  - In practice, similar limitations for the discrete Riemannian framework

What we gain

- A globally invariant structure invariant by composition & inversion
- Simple geodesics, efficient computations (stationarity, group exponential)
- The simplest linearization of transformations for statistics?

Outline

Statistical computing on Riemannian manifolds

Computing on Lie groups

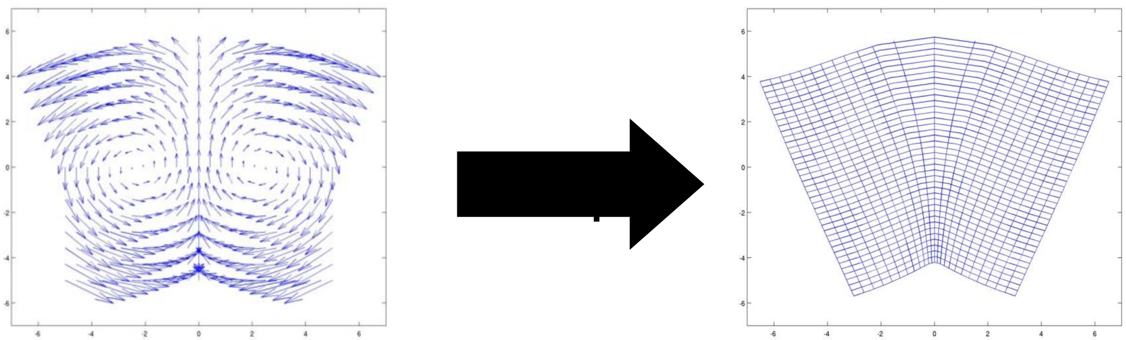
- The bi-invariant affine Cartan connection structure
- Extending statistics without a metric
- The SVF framework for diffeomorphisms

Conclusion

The SVF framework for Diffeomorphisms

Idea: **[Arsigny MICCAI 2006, Bossa MICCAI 2007, Ashburner Neuroimage 2007]**

- Exponential of a smooth vector field is a diffeomorphism
- **Parameterize deformation by ~~time-varying~~ Stationary Velocity Fields**



Stationary velocity field   Diffeomorphism

Direct generalization of numerical matrix algorithms

- Computing the deformation: **Scaling and squaring** **[Arsigny MICCAI 2006]**
  recursive use of exp($\boldsymbol{v}$)=exp($\boldsymbol{v}$/2) o exp($\boldsymbol{v}$/2)

- Updating the deformation parameters: **BCH formula** **[Bossa MICCAI 2007]**

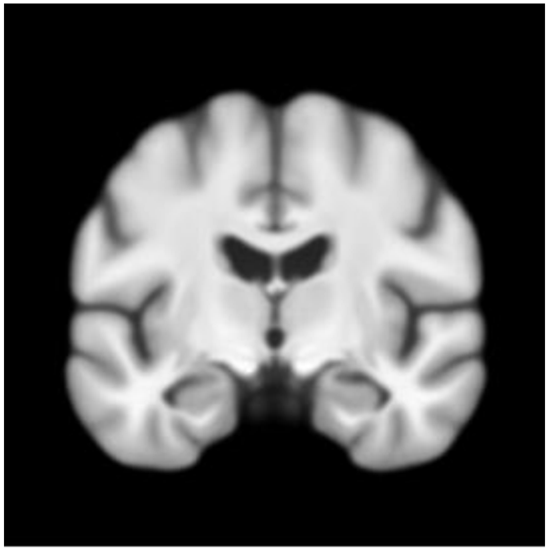
  $\exp(\boldsymbol{v}) \circ \exp(\varepsilon \boldsymbol{u}) = \exp(\, \boldsymbol{v} + \varepsilon \boldsymbol{u} + [\boldsymbol{v},\varepsilon \boldsymbol{u}]/2 + [\boldsymbol{v},[\boldsymbol{v},\varepsilon \boldsymbol{u}]]/12 + \ldots\,)$

  - Lie bracket $[\boldsymbol{v},\boldsymbol{u}](p) = \mathrm{Jac}(\boldsymbol{v})(p).\boldsymbol{u}(p) - \mathrm{Jac}(\boldsymbol{u})(p).\boldsymbol{v}(p)$

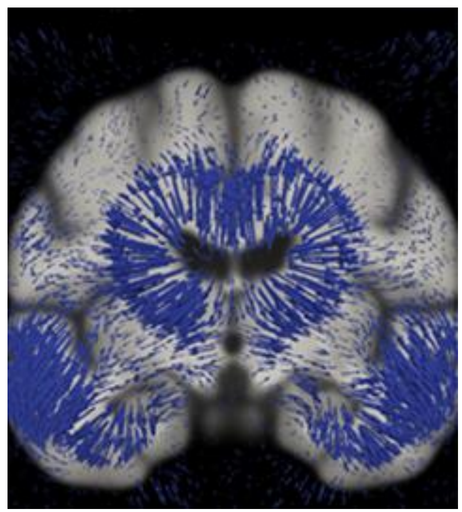
Temporal Evolution with Deformation-based Morphometry

Alzheimer's atrophy trajectory

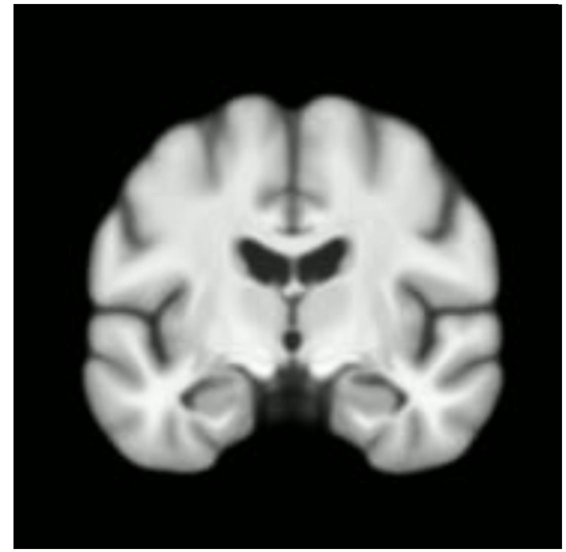
Baseline MRI



$\varphi = \exp(v)$



Follow-up MRI



Atrophy flow encoded by the dense stationary velocity field

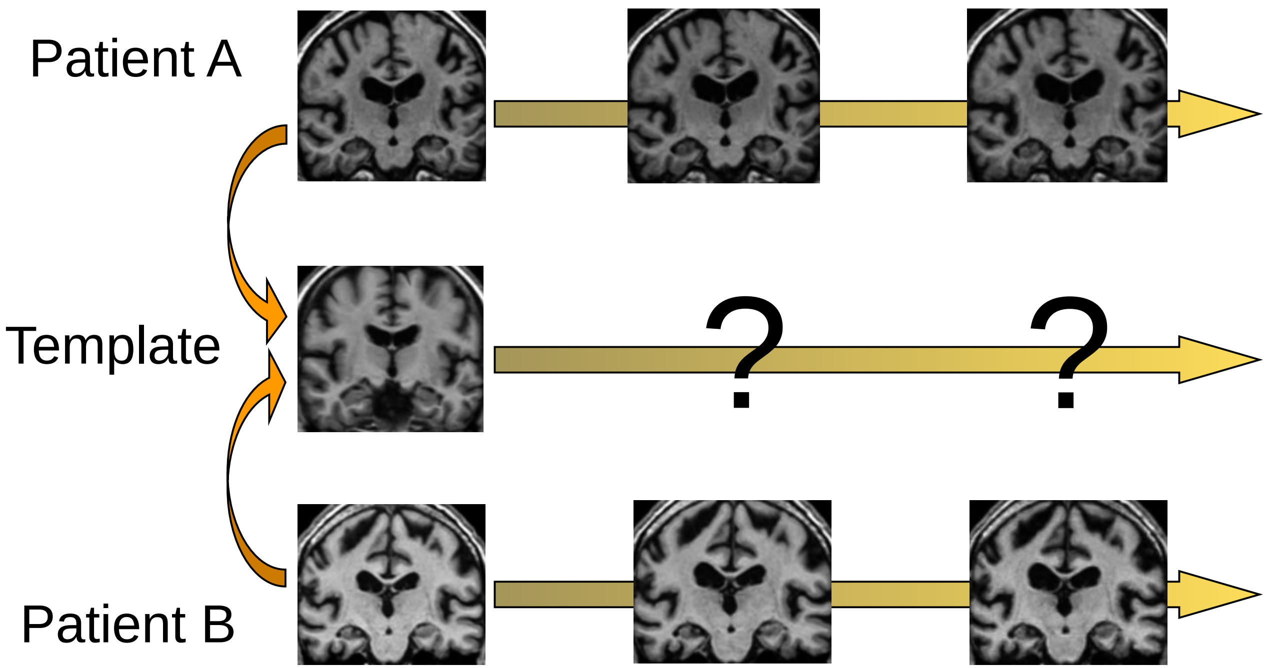
[Lorenzi, Ayache, Frisoni, Pennec, Neuroimage 81, 1 (2013) 470-483]

https://team.inria.fr/asclepios/software/lcclogdemons/

Longitudinal deformation analysis in AD

From patient specific evolution to population trend
(parallel transport of SVS parameterizing deformation trajectories)

- Inter-subject and longitudinal deformations are of different nature and might require different deformation spaces/metrics
- Consistency of the numerical scheme with geodesics?

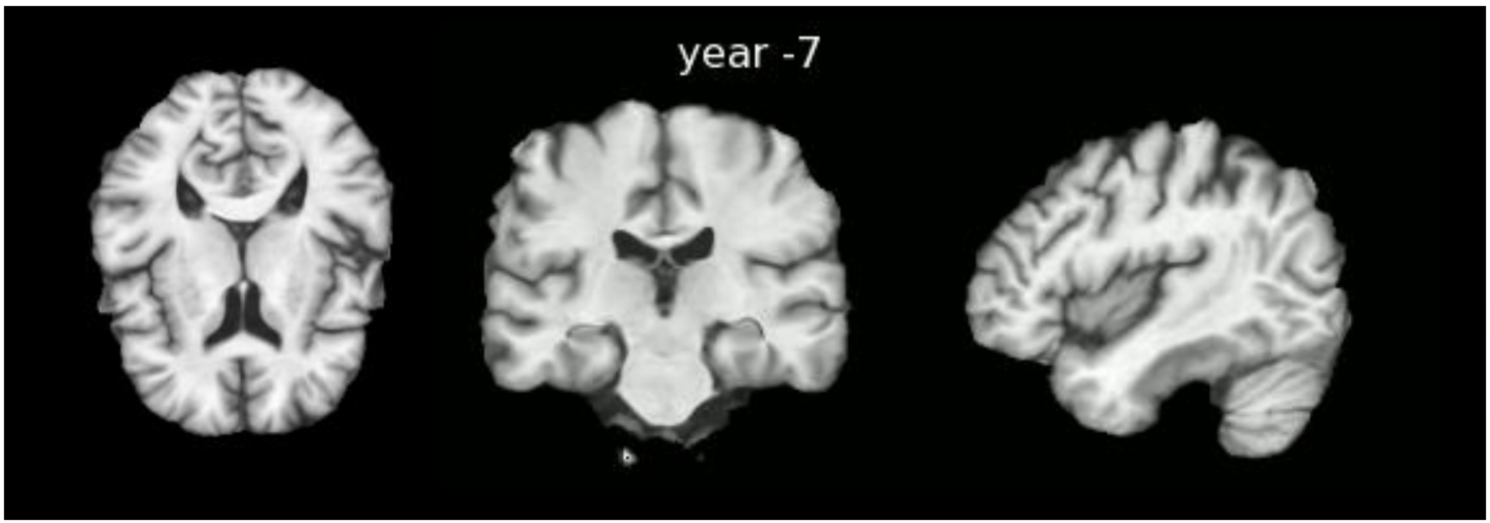


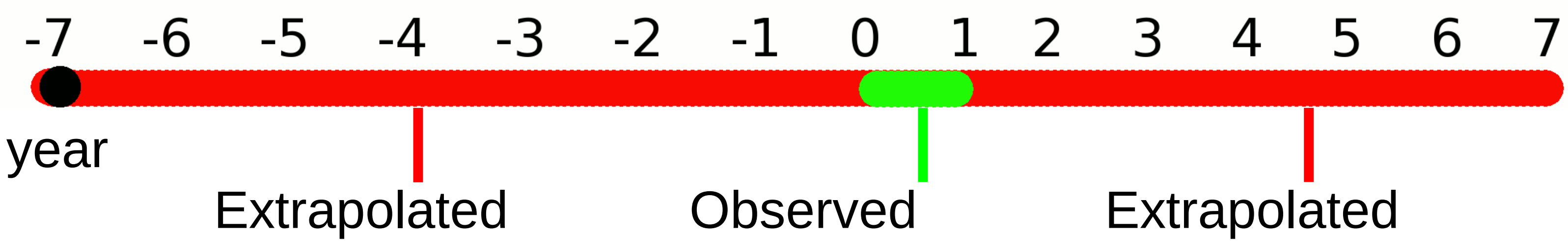
[Lorenzi, Pennec: Efficient Parallel Transport of Deformations in Time Series of Images: from Schild's to pole Ladder, JMIV 50(1-2):5-17, 2013]

Longitudinal model for AD

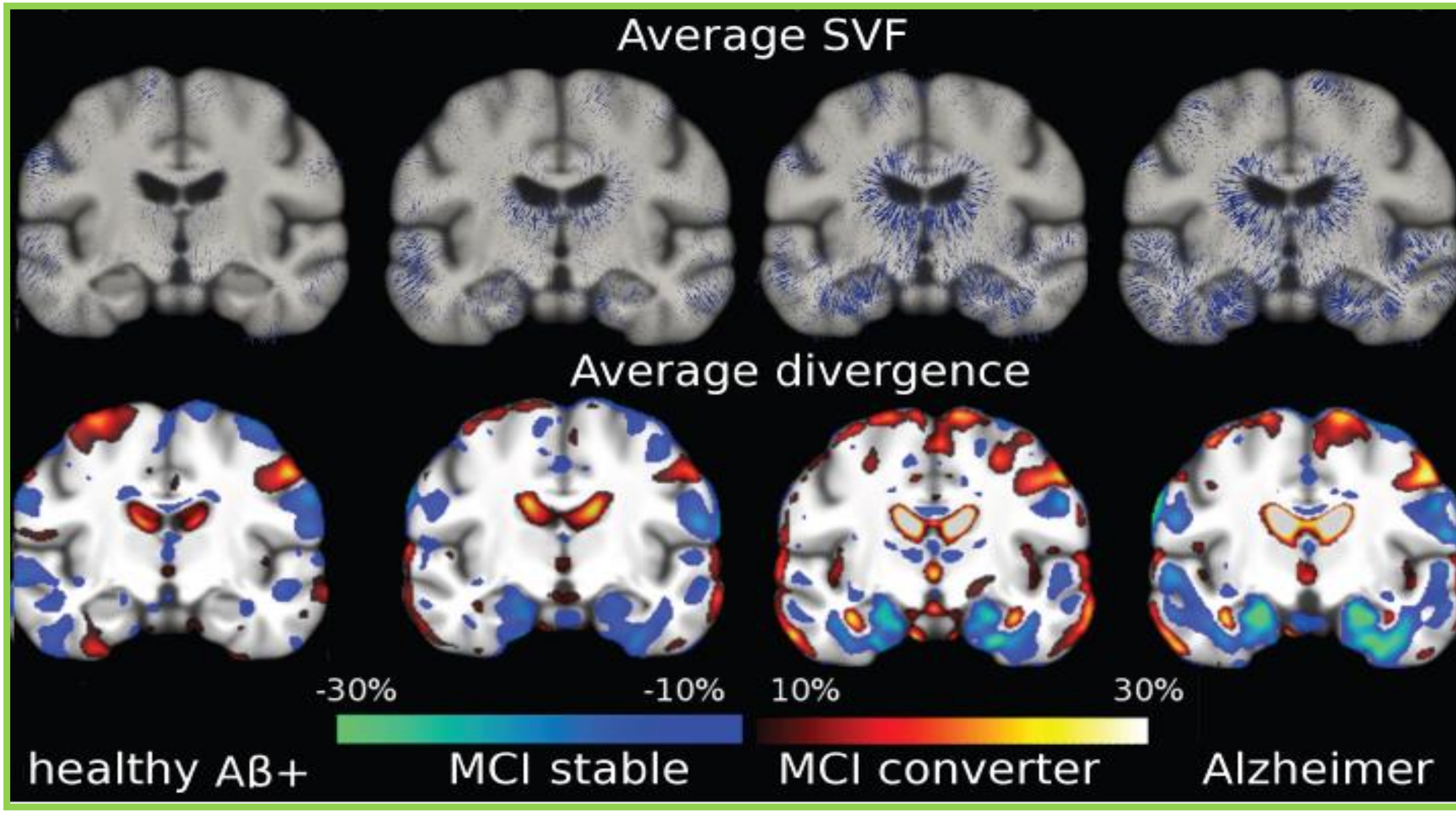
Estimated from 1 year changes – Extrapolation to 15 years

70 AD subjects (ADNI data)





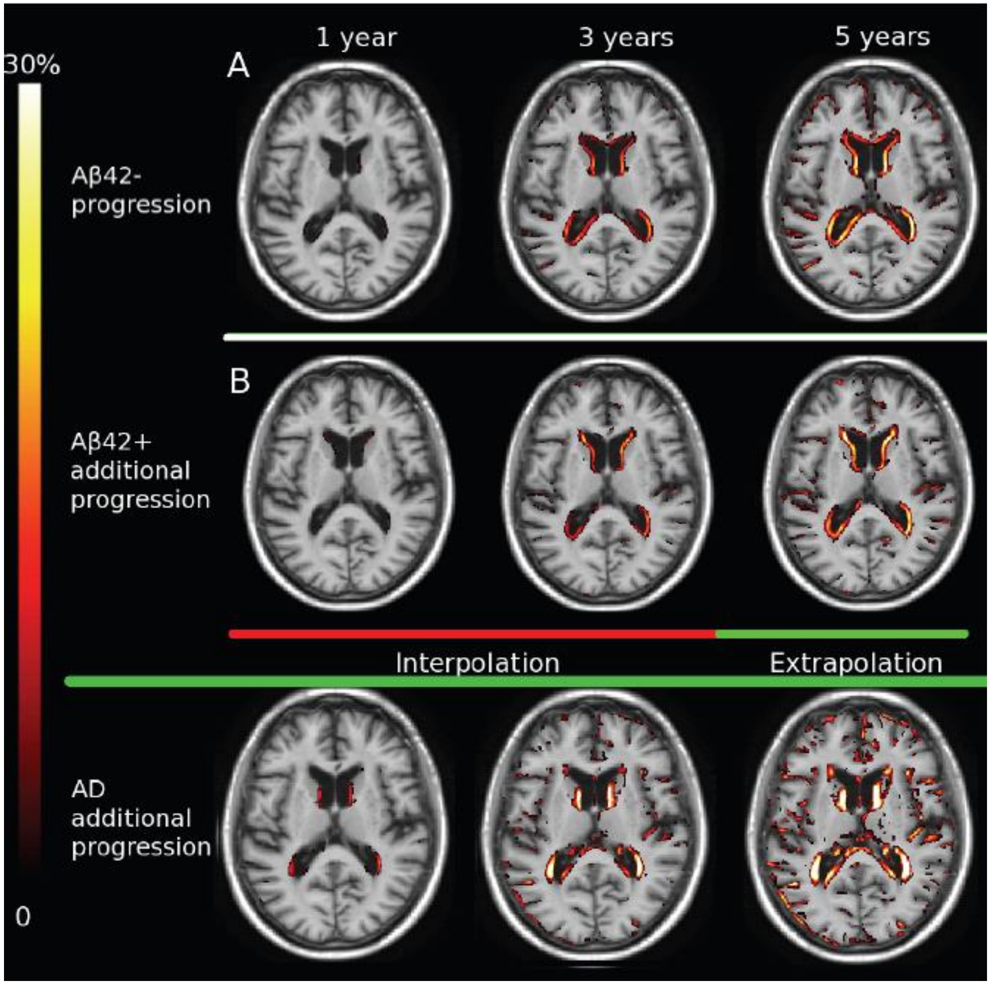
Mean deformation / atrophy per group



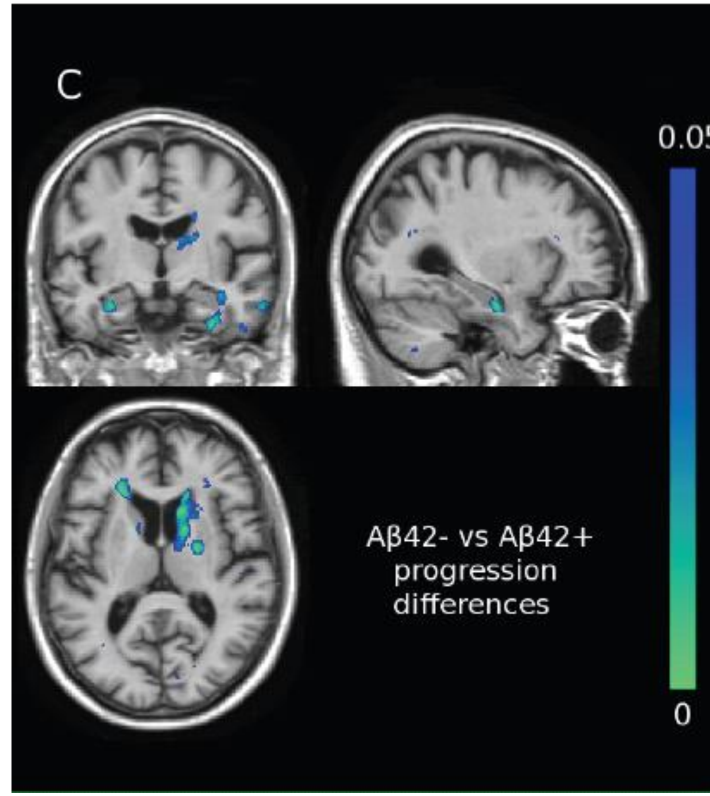
M Lorenzi, N Ayache, X Pennec G B. Frisoni, for ADNI. Disentangling the normal aging from the pathological Alzheimer's disease progression on structural MR images. 5th Clinical Trials in Alzheimer's Disease (CTAD'12), Monte Carlo, October 2012. (see also MICCAI 2012)

Study of prodromal Alzheimer's disease

Linear regression of the SVF over time: interpolation + prediction



$$T(t) = Exp(\tilde{v}(t)) * T_0$$



Multivariate group-wise comparison of the transported SVFs shows statistically significant differences (nothing significant on log(det))

[Lorenzi, Ayache, Frisoni, Pennec, in Proc. of MICCAI 2011]

Outline

Statistical computing on Riemannian manifolds

An affine setting for Lie groups

Beyond Riemannian and affine geometries?

Geometric Statistics: Riemannian, affine structures and beyond

Riemannian / affine manifolds

- Impact of curvature on non-asymptotic Fréchet mean estimations?
- Sharp theorems for existance and uniqueness? For Karcher mean?
- A CLT for multiple Karcher p-means / exponential barycenters?

Flag manifolds for hierarchical subspace approximations

- Metrics on flag-manifolds and limit towards multi-jets?
- Generalization of ICA or iterative least-squares methods (PLS)?
- Algorithms for manifold dimension reduction?

Quotient spaces

- Kendall shape spaces; curves, surfaces, images / parameterization
- Inconsistency of Fréchet mean in q-space (extrinsic curvature of orbit)
- Orbifolds and stratified spaces: Continuous and discrete geometry?

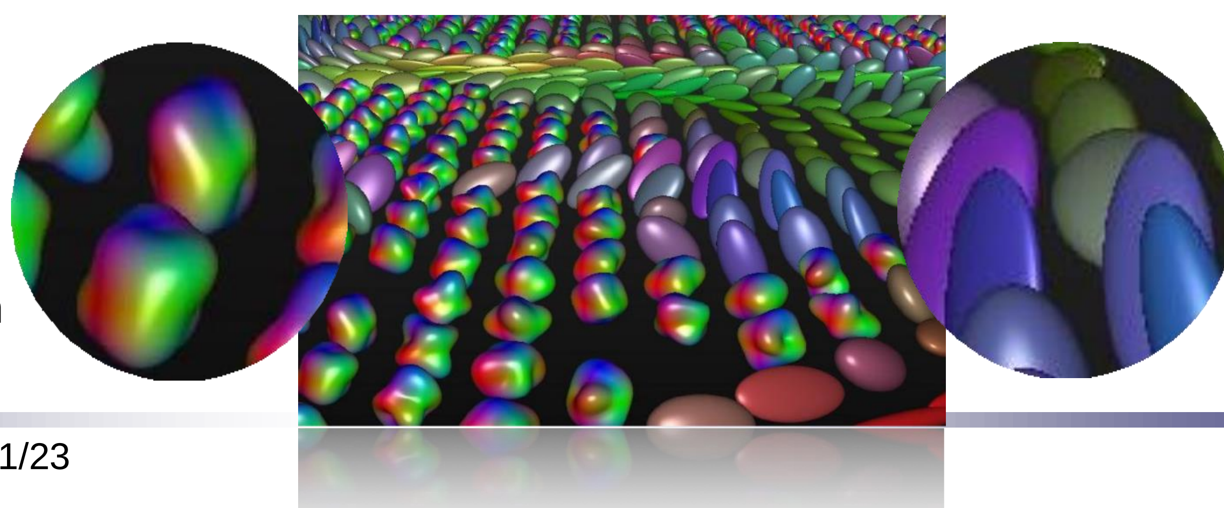
Towards more complex geometries?

Non quadratic metric: Statistics on Finsler spaces?

[Image from Sepasian, Thije Boonkkamp, Florack, Ter Haar Romeny, Vilanova
Riemann-Finsler Multi-valued Geodesic Tractography for HARDI]

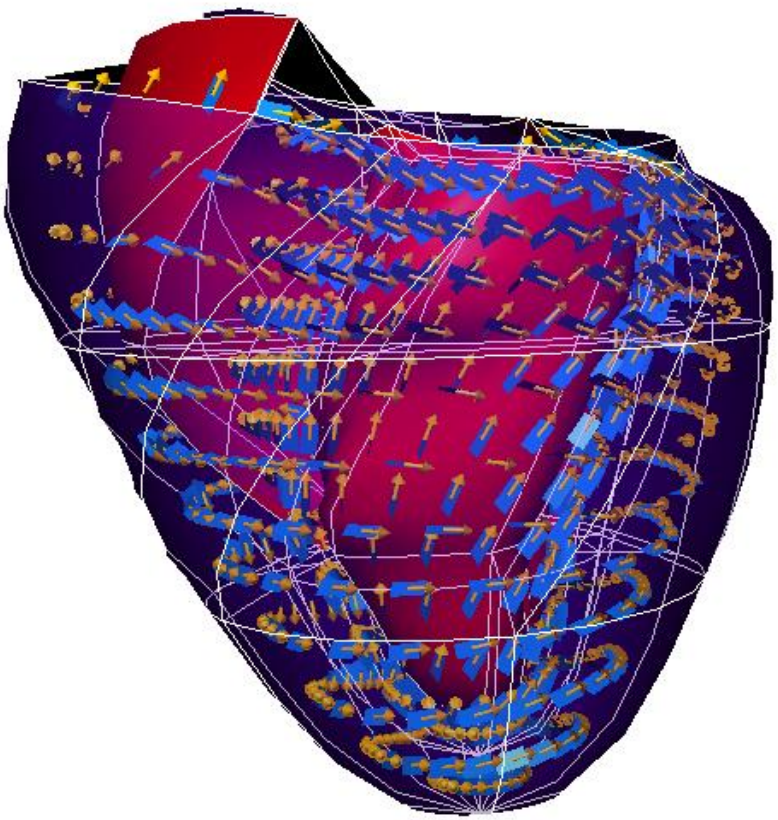
Finsler manifold-valued image processing?

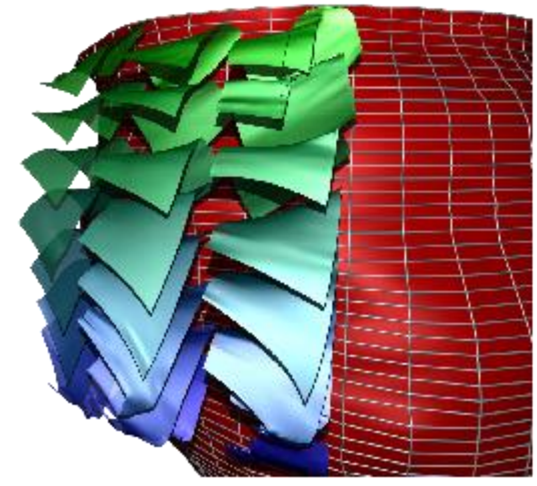
[Image shamelessly stolen
from Luc Florac's talk]

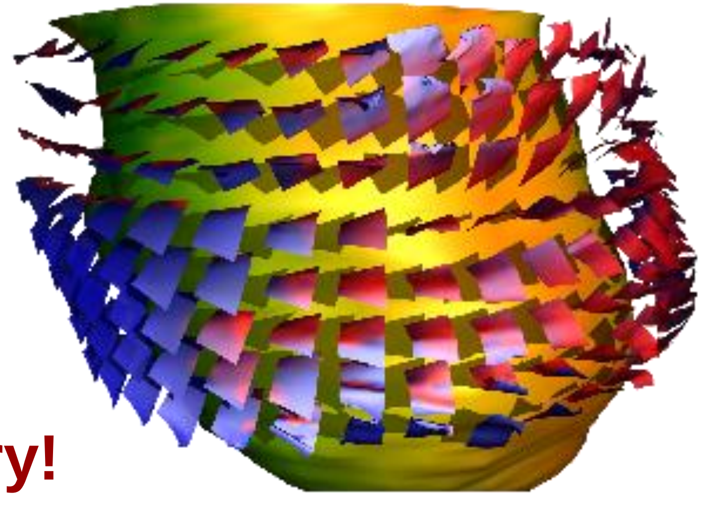


Towards more complex geometries?

Laminar sheets in the myocardium:







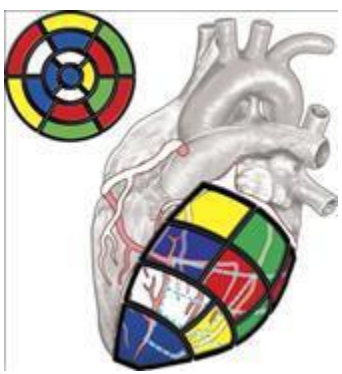
→ **Torsion: Non-integrable geometry!**

Towards more complex geometries?

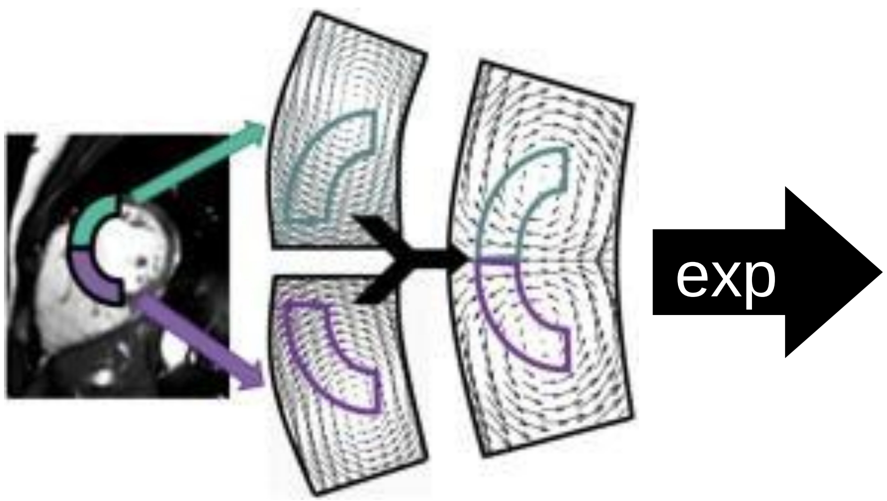
Fibre bundles

- Multiscale LDDMM [Sommer et al, JMIV 2013]
- Locally affine atoms of transformation:
  - Jetlets diffeomorphisms [Sommer SIIMS 2013, Jacobs / Cotter 2014]
  - Parametric Polyaffine deformations [Arsigny et al., MICCAI 06, JMIV 09]

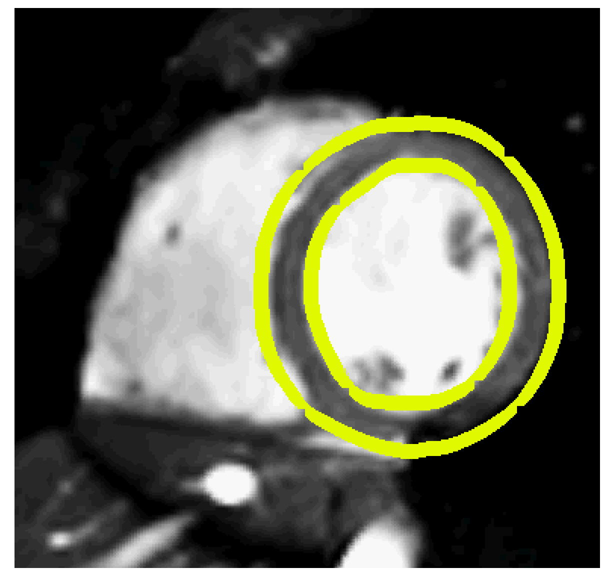
  Log demons projected but with 204 parameters instead of a few millions



AHA regions



Stationary velocity fields



Diffeomorphism with 204 parameters [McLeod, Miccai 2013]

Geometric Statistics for anatomical shapes

Study geometric structures

- Riemannian, Finsler, affine, bundles, Lie groups

Generalize statistics

- Real data have noise
- Approximate invariance, factor analysis…

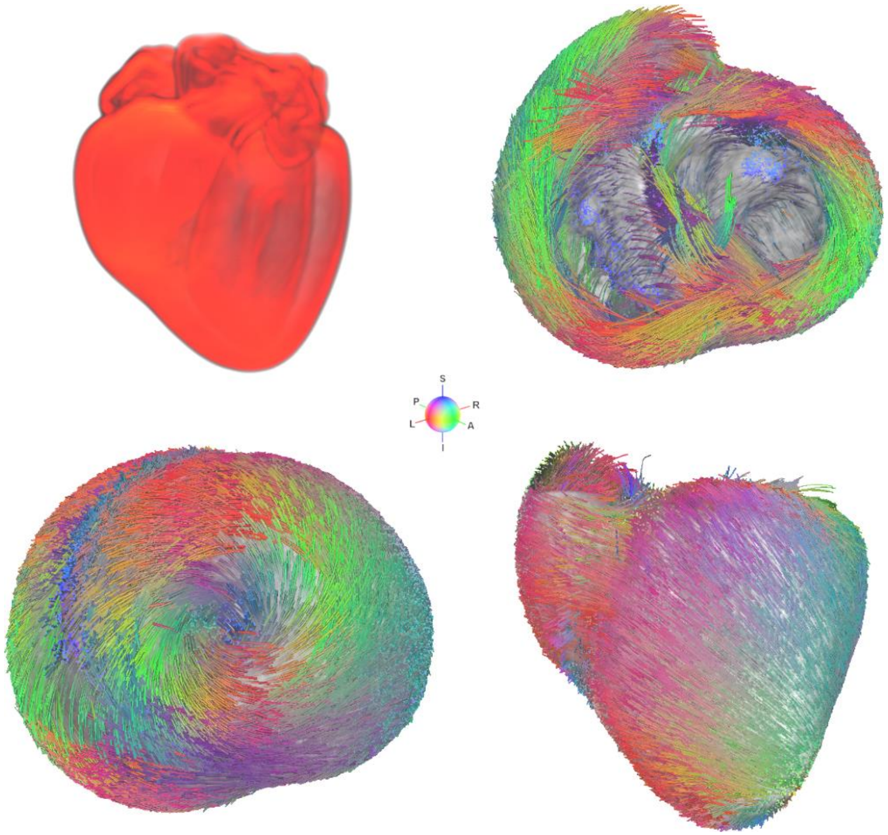
Design algorithm

- Dimension reduction, Image processing…

With important medical applications

- Heart, brain diseases

Thank You!



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Software: https://team.inria.fr/asclepios/software/

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