The Freyd–Schützenberger Completion

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The Freyd–Schützenberger completion of a monoid (or category)

Appeared twice:

- *Stable homotopy*, P. Freyd, Proceedings of Conference on Categorical Algebra, La Jolla (1966).
- *The Schützenberger category of a semigroup*, A. Costa and B. Steinberg, Semigroup Forum (2015).

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How does the Green-Rees local theory of finite monoids emerges from it?

A bit of motivation

A finite state machine: states X and actions $X \to X$.

Actions compose $\rightsquigarrow$ a monoid $M \circlearrowright X$.

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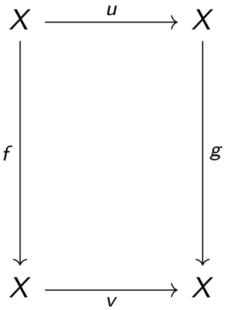
How to representing algebraically the *images* of actions?

Examples:

- $\mathrm{im}(f \circ g) \subseteq \mathrm{im}(f)$
- $\mathrm{im}(g) \twoheadrightarrow \mathrm{im}(f \circ g)$

Examples

$$fv = ug$$



Examples

$$fv = ug$$

$$\begin{array}{ccc} X & \xrightarrow{u} & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{im}(f) & \dashrightarrow & \mathrm{im}(g) \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[v]{} & X \end{array}$$

Concrete description (1/3) – The category $\mathbb{D}(M)$

Objects of $\mathbb{D}(M) = \mathrm{im}(f)$ where $f \in M$.

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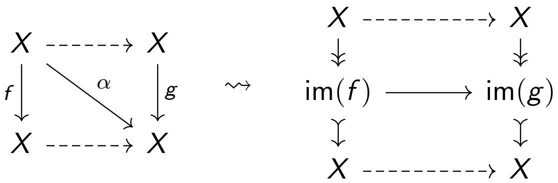

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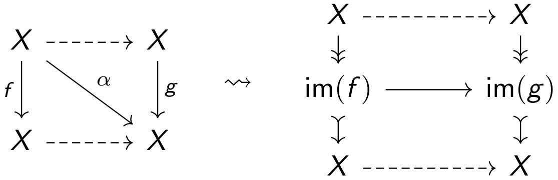
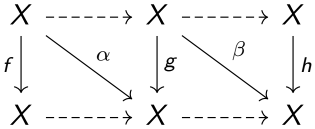


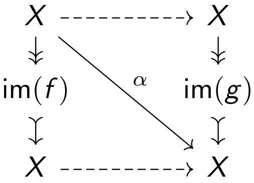
Composition is given by gluing the squares:



We recover M as the endomorphisms $\text{im}(\text{id}) \to \text{im}(\text{id})$.

Concrete description (2/3) – The factorization system

What is the image of $\alpha : \mathrm{im}(f) \to \mathrm{im}(g)$?



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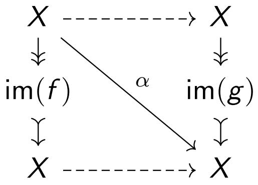
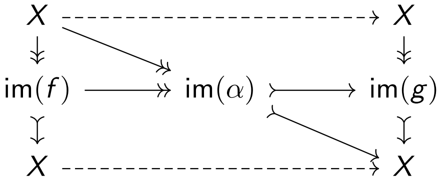
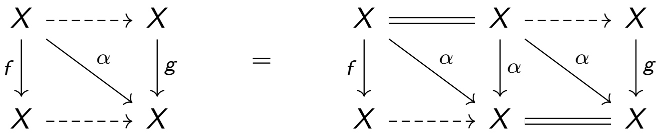


It is $\mathrm{im}(\alpha)$:

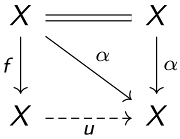


Concrete description (2/3) – The factorization system

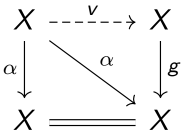
More formally:



Left class:



Right class:



Concrete description (3/3) – Green's relations

The two classes of the factorization system of $\mathbb{D}(\mathbf{C})$ are preorders:

- $f \twoheadrightarrow \alpha$ if $fu = \alpha$ for some u: Green's $\mathscr{R}$ preorder, $f \geqslant_{\mathscr{R}} \alpha$.
- $\alpha \rightarrowtail g$ if $\alpha = vg$ for some v: Green's $\mathscr{L}$ preorder, $\alpha \leqslant_{\mathscr{L}} g$.

The arrows $f \to g$ are the α such that $f \longrightarrow\!\!\!\!\!\rightarrow \alpha \rightarrowtail\!\!\longrightarrow g$

Example: Green's Lemma

Let $\mathrm{Sub}(f) = \{\alpha \mid \alpha \rightarrowtail f\}$.

Lemma.
If $f \cong g$, then $\mathrm{Sub}(f) \cong \mathrm{Sub}(g)$.

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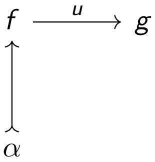

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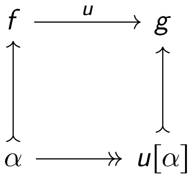

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If $f \cong g$, then $\mathrm{Sub}(f) \cong \mathrm{Sub}(g)$.

$$\begin{array}{ccccc} f & \xrightarrow{u} & g & \xrightarrow{u^{-1}} & f \\ \uparrow & & \uparrow & & \uparrow \\ \alpha & \twoheadrightarrow & u[\alpha] & \twoheadrightarrow & \alpha \end{array}$$

Other definitions of $\mathbb{D}(M)$

(1) $\mathbb{D}(M)$ is the free category with an epi–mono factorization system over M.
(2) The evaluation $\mathbf{C} \to \mathbf{Set}^{[\mathbf{C},\mathbf{Set}]}$ extends to a full embedding $\mathbb{D}(\mathbf{C}) \subseteq \mathbf{Set}^{[\mathbf{C},\mathbf{Set}]}$.

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Lemma.

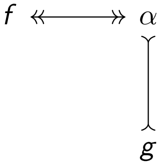
(1) $f \rightarrowtail g$ iff $\mathrm{im}(\sigma_f) \subseteq \mathrm{im}(\sigma_g)$ for all $\sigma : M \to \mathrm{End}(X)$.
(2) $f \twoheadrightarrow g$ iff $\ker(\sigma_f) \subseteq \ker(\sigma_g)$ for all $\sigma : M \to \mathrm{End}(X)$.

$\downarrow$

If $\sigma_f(x) = \sigma_f(y)$ then $\sigma_g(x) = \sigma_g(y)$.

Isomorphism classes

$f \twoheadleftarrow\!\!\!\twoheadrightarrow \alpha$ means $f \twoheadrightarrow \alpha$ and $\alpha \twoheadrightarrow f$; $\alpha \rightarrowtail\!\!\!\leftarrowtail g$ means $\alpha \rightarrowtail g$ and $g \rightarrowtail \alpha$
Isomorphisms $f \to g$ = the α such that $f \twoheadleftarrow\!\!\!\twoheadrightarrow \alpha \rightarrowtail\!\!\!\leftarrowtail g$.



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$$\begin{array}{ccc} f & \twoheadleftarrow\!\!\!-\!\!\!-\!\!\!\twoheadrightarrow & \alpha \\ \curlyvee\!\!\!\!\!\mid & & \curlyvee\!\!\!\!\!\mid \\ \curlywedge\!\!\!\!\!\mid & & \curlywedge\!\!\!\!\!\mid \\ \alpha^{-1} & \twoheadleftarrow\!\!\!-\!\!\!-\!\!\!\twoheadrightarrow & g \end{array}$$

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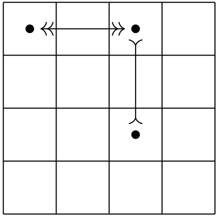
$$\begin{array}{ccc} f & \twoheadleftarrow\!\!\!\longrightarrow\!\!\!\twoheadrightarrow & \alpha \\ \Updownarrow & & \Updownarrow \\ \alpha^{-1} & \twoheadleftarrow\!\!\!\longrightarrow\!\!\!\twoheadrightarrow & g \end{array}$$

$f \cong g$ is written $f \mathscr{D} g$ in finite monoid theory.

Structure of the isomorphism classes



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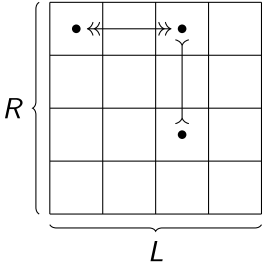

All the information is contained in:

- the numbers $|L|$ and $|R|$, and
- the group G of automorphisms.

Topological visualization of compositions

A composition $f_1 f_2 f_3$ in M becomes in $\mathbb{D}(M)$:

$$\begin{array}{ccccccc} 1 & & & & & & \\ \downarrow & & & & & & \\ f_1 & \rightarrowtail & 1 & & & & \\ & & \downarrow & & & & \\ & & f_2 & \rightarrowtail & 1 & & \\ & & & & \downarrow & & \\ & & & & f_3 & \rightarrowtail & 1 \end{array}$$

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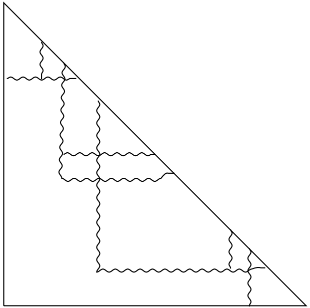
1 & & & & & & \\
\downarrow\!\!\!\!\downarrow & & & & & & \\
f_1 & \rightarrowtail & 1 & & & & \\
\downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & & & \\
f_1 f_2 & \rightarrowtail & f_2 & \rightarrowtail & 1 & & \\
\downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \\
f_1 f_2 f_3 & \rightarrowtail & f_2 f_3 & \rightarrowtail & f_3 & \rightarrowtail & 1
\end{array}
$$

Topological visualization of compositions

Forget the arrows and draw the boundaries of isomorphism classes.

For each non-invertible arrow:

We get pictures of the form:



Thank you!