On maximal order types of a lexicographic product of wpos

Mirna Džamonja, IRIF

Well orders and ordinals

A non-inductive order

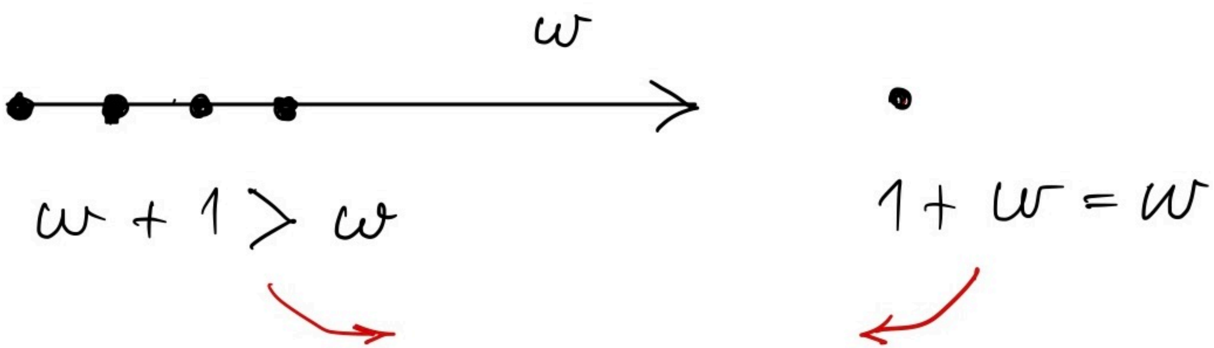
- A linear order is a **well order** if it has no infinite decreasing sequence.
- Examples: any finite order, ω (the order type of natural numbers)
- Non-examples: ω^*= the reverse of ω, the order types of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- The **Trichotomy Theorem** of Cantor (1878) shows that every two well orders are comparable: either they are isomorphic or one is isomorphic to an initial segment of the other. (This is not true of orders in general, consider for example $\mathbb{Q}$ and $\mathbb{R}$)
- Hence it is reasonable to define **ordinals** as isomorphism types of well orders (*the formal definition is somewhat more involved due to the axiomatisation of set theory).*

Well posets (wpos)

- A partial order is a **wpo** if it $(*)$ has no infinite decreasing sequence or an infinite antichain (=*set of incomparable elements*).
- (Non)-Example: the binary tree $2^{<\omega}$of finite sequences of 0 and 1 with the order of being an initial segment has no infinite decreasing sequence, but has infinite antichains such as $\{0^n 1 : n < \omega\}$
- Non-example: the full binary tree 2^{ω} of sequences of 0 and 1 of length ω, since there are $\mathfrak{c}$=continuum incomparable ones.
    - *A more general notion are **wqo**, which are quasi-orders satisfying $(*)$, but for all we have to say it is sufficient to work with their quotients, which are wpo.*

Linearisations

- A linear order $(L, <_L)$ is a **linearisation** of a poset $(P, <_P)$ if they have the same domain and $<_P \subseteq <_L$.
- Note that if P is a wpo, then every one of its linearisations is a well order (*since we are not going to add an infinite decreasing sequence by linearising*).
- One wpo can have linearisations of different order types :



$o(P)$

- If P is a wpo, let $O(P)$ denote the set of all the ordinals that we can obtain as order types of linearisations of P.
- de Jongh and Parikh (1977) proved that for every wpo P, the set $O(P)$ has a maximum. This is called **the maximal order type** and denoted by $o(P)$.

Operations on posets

we care only about wpos, to be later connected with operations on ordinals

- If we are given two wpos P and Q, we can perform various operations with them and still have a wpo:
- the disjoint union $P \sqcup Q$
- the lexicographic sum $P +_l Q$: all points of P before all points of Q
- the Cartesian product $P \times Q$, with $(p_0, q_0) \leq (p_1, q_1)$ iff $[p_0 \leq_P p_1 \wedge q_0 \leq_Q q_1]$
- the lexicographic product $P \cdot Q$ with $(p_0, q_0) \leq (p_1, q_1)$ iff $q_0 <_Q q_1$ or $[q_0 = q_1 \wedge p_0 <_P p_1]$

Operations on ordinals

- $\alpha + \beta$: a copy of α followed by a copy of β . Note that this is note commutative because of the phenomenon of *absorption*, for example $1 + \omega = \omega$, since 1 gets absorbed into $\omega < \omega + 1$
- $\alpha \cdot \beta$: take α and replace each of its elements by a copy of β. For example, $2 \cdot \omega = \omega + \omega$, while $\omega \cdot 2 = \omega$
- Exponentiation: defined recursively, knowing that every ordinal is either a successor such as $\beta + 1$ or an unattained limit such as 0 or ω. Then we have: $\alpha^0 = 1, \quad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha, \quad \alpha^\delta = \bigcup_{\beta<\delta} \alpha^\beta$ for δ limit > 0

Cantor Normal Form and Hessenberg Operations

- Cantor proved that every ordinal has a unique representation of the form $\omega^{\alpha_0}m_0 + \omega^{\alpha_1}m_1 + \cdots \omega^{\alpha_n}m_n$ where $\alpha_0 > \alpha_1 > \ldots \alpha_n$ and $1 \leq m_0, m_1, \ldots m_n < \omega$
- Hessenberg (1911) defined *the natural operations* on ordinals, as follows: $\alpha \oplus \beta$: represent α and β using their normal forms and then **add** the resulting polynomials (treating ω as a variable). Reinterpret the ordinal.
- $\alpha \otimes \beta$: represent α and β using their normal forms and then **multiply** the resulting polynomials (treating ω as a variable). Reinterpret the ordinal.

Operations on ordinals versus those on wpos

As far as the maximal order type is concerned

- de Jongh and Parikh (1977) proved the following formulas
- $o(P \sqcup Q) = o(P) \oplus o(Q)$
- $o(P +_l Q) = o(P) + o(Q)$
- $o(P \times Q) = o(P) \otimes o(Q)$
- They did not deal with the lexicographic product $P \cdot Q$

A wrong attribution

- Dž, Schnoebelen and Schmitz (2020) write an article on an ordinal invariant, the width w and decide to extend it to a survey of three invariants o, h, w
- They state « the obvious » $o(P \cdot Q) = o(P) \cdot o(Q)$ and, moreover, they wrongly attribute it to Abraham and Bonnet who never dealt with the invariant o (and did write a very important paper on w). 🤣
- In March 2024, Harry Altman informed us that by looking at another preprint on the arxiv which cited us and repeated our mistake, he found that there was a problem.
- Luckily both of the papers that cited the « equation » never made any use of it, just « cited it ».

A counterexample

- Let $P = \omega + 1$.
- If $Q = \omega + 1$, then $P \cdot Q = (\omega + 1)(\omega + 1)$ is a linear order and its only « linearisation » is the ordinal $\omega^2 + 1$, which conforms with the formula $o(P) \cdot o(Q)$
- However, if we take Q to be the poset $\omega \cup \{\mathbf{1}\}$, then a possible linearisation of $P \cdot Q$ is to first take $(\omega + 1) \cdot \omega = \omega^2 + 1$ and then put the $(\omega + 1) \cdot \{1\}$, all together giving $(\omega^2 + 1) + (\omega + 1) = \omega^2 + \omega + 1$

Isa Vialard's formula

- A day after Altman's message, Philippe Schnoebelen's student Isa Vialard found the right formula for $o(P \cdot Q)$
- If Q is a poset with k maximal elements, then $o(Q) = \delta + m$ for some limit ordinal δ and $m \geq k$.
- $o(P \cdot Q) = o(P) \cdot [\delta + (m - k)] + o(P) \otimes k$.
- In particular, if $k = 0$ (equivalently, $o(Q)$ is a limit) then $o(P \cdot Q) = o(P) \cdot o(Q)$.

Isa’s proof

- The difficult case is when Q is finite
- Isa’s methods are a mixture of game theory, estimates and formulas developed in his upcoming thesis (July 2024)
- We are working on a self-contained inductive proof to be posted in the corrigendum of our paper on the arxiv