Verification of programs manipulating ADTs using Shallow Horn Clauses

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What is the problem?

Objective

Proving relational properties on functional programs manipulating algebraic data types in a completely automatic way.

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Example: Program

```
type nat = Z | S of nat
type nlist = Nil | Cons of nat * nlist
```

```
let rec length l =
  match l with
  | Nil -> Z
  | Cons(_,t) ->
     S(length t)
```

```
let rec less n1 n2 =
  match (n1, n2) with
  | Z, S(_) -> true
  | _, Z -> false
  | S(n1'), S(n2') ->
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```

$P_1 \doteq \forall l, x,\ less\ Z\ (length\ (Cons(x,\ l)))$
$P_2 \doteq \forall l, x,\ less\ (length\ l)\ (length\ (Cons(x,\ l)))$

Algebraic non-relational properties: [1]

Property P_1

$\forall l, x,\ less\ Z\ (length\ (Cons(x,\ l)))$

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Type abstraction by regular languages (using tree automaton)

- Natural numbers: $\mathcal{L}_n \doteq \mathcal{L}_Z \cup \mathcal{L}_{S^+}$
- Lists: $\mathcal{L}_l \doteq \mathcal{L}_{Nil} \cup \mathcal{L}_{Cons^+}$

Abstract property: $less^\sharp\ \mathcal{L}_Z\ (length^\sharp\ (Cons^\sharp(\mathcal{L}_n, \mathcal{L}_l)))$

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true

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Algebraic non-relational properties: [1]

Property P_2

$\forall l, x,\ less\ (length\ l)\ (length\ (Cons(x,\ l)))$

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?

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? $\leftarrow$ Cannot be proved using non-relational abstraction

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Making relations explicit: from functions to clauses

From $len(l) = n$ to $\mathrm{Len}(l, n)$.

Program as clauses: Len and Less

$\mathrm{Len}(Nil,\ Z).$
$\mathrm{Len}(l,\ n) \Rightarrow \mathrm{Len}(Cons(x, l),\ S(n)).$
$\mathrm{Len}(l,\ n_1) \wedge \mathrm{Len}(l,\ n_2) \Rightarrow n_1 = n_2.$

$\mathrm{Less}(Z,\ S(m)).$
$\mathrm{Less}(n,\ Z) \Rightarrow \mathrm{False}.$
$\mathrm{Less}(n,\ m) \Leftrightarrow \mathrm{Less}(S(n),\ S(m)).$

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Unique Herbrand model $\mathcal{M}$ of the above-defined clauses

$\mathcal{M}(\mathrm{Len}) = \{(Nil, Z), (Cons(Z, Nil), S(Z)), \ldots\}$
$\mathcal{M}(\mathrm{Less}) = \{(Z, S(Z)), (Z, S(S(Z))), (S(Z), S(S(Z))), \ldots\}$

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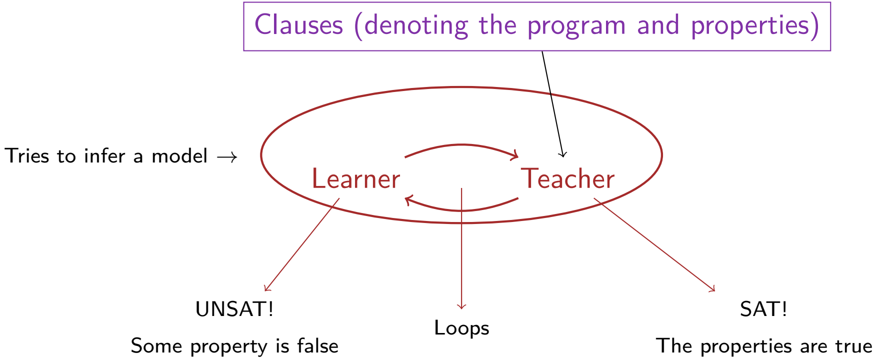
Property P_2 as clauses

$\mathrm{Len}(l,\ n) \wedge \mathrm{Len}(Cons(x, l),\ n') \Rightarrow \mathrm{Less}(n,\ n').$

Satisfiability procedure: our method at a glance



Model inference: Learner | Teacher

Guesses a model which generalizes ground clauses received from the teacher.

- Proposes more and more refined models
- Stops with UNSAT if the ground clauses are inconsistent

Checks whether a given model meets the specification (clauses).

- Gives a ground counterexample to every clause that the model does not satisfy
- Stops with SAT if the proposed model satisfies every constraint

Learner	Teacher

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Learner | Teacher

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$\mathrm{Len}(Nil, Z)$
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Learner | Teacher

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Learner | Teacher

$\text{Len} \mapsto \{(Nil, Z)\}$
$\text{Less} \mapsto \{(Z, S(x)) \mid x \in Nat\}$

Learner:

$\text{Len}(Nil, Z)$

$\text{Less}(Z, S(Z))$

Teacher:

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$\text{Len}(Nil, Z) \Rightarrow \text{Len}(Cons(Z, Nil), S(Z))$
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Learner	Teacher
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$\mathrm{Len} \mapsto \{(Nil, Z)\} \cup \{(Cons(\cdot,\cdot), S(\cdot))\}$
$\mathrm{Less} \mapsto \{(\cdot, S(\cdot))\}$

Learner

$\mathrm{Len}(Nil, Z)$

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$\mathrm{Len}(Nil, Z) \Rightarrow \mathrm{Len}(Cons(Z, Nil), S(Z))$

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Teacher

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Learner | Teacher

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$\text{Less}(Z, S(Z)) \Rightarrow \text{Less}(S(Z), S(S(Z)))$

Teacher:

$\text{Len}(\mathit{Nil},\ Z)$

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$x\ \text{Len}(l,\ n_1) \wedge \text{Len}(l,\ n_2) \Rightarrow n_1 = n_2$

$\text{Less}(Z,\ S(m))$

$\text{Less}(n,\ Z) \Rightarrow \text{False}$

$x\ \text{Less}(n,\ m) \Leftrightarrow \text{Less}(S(n),\ S(m))$

$\text{Len}(l,\ n) \wedge \text{Len}(\mathit{Cons}(x, l),\ n') \Rightarrow \text{Less}(n,\ n')$

$$\begin{pmatrix} \text{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \ \wedge \\ \text{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \Rightarrow S(Z)) = S(S(Z))$$

$$\text{Less}(S(Z), S(Z)) \quad \Rightarrow \quad \text{Less}(Z, Z)$$

Learner

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$\mathrm{Less}(Z, S(Z))$
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$$\begin{pmatrix} \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \ \wedge \\ \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \implies \bot$$

$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

Teacher

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Learner | Teacher

Details coming soon

$\mathrm{Len} \mapsto \{([x_1, \ldots, x_n], S^n(Z)) \mid n \in \mathbb{N}\}$
$\mathrm{Less} \mapsto \{(n_1, n_2) \mid n_1 < n_2\}$

Learner

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$\mathrm{Len}(Nil, Z) \Rightarrow \mathrm{Len}(Cons(Z, Nil), S(Z))$

$\mathrm{Less}(Z, S(Z)) \Rightarrow \mathrm{Less}(S(Z), S(S(Z)))$

$$\begin{pmatrix} \mathrm{Len}(Cons(Z, Nil), S(Z)) \ \wedge \\ \mathrm{Len}(Cons(Z, Nil), S(S(Z))) \end{pmatrix} \implies \bot$$

$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

Teacher

$\mathrm{Len}(Nil,\ Z)$
$\mathrm{Len}(l,\ n) \Rightarrow \mathrm{Len}(Cons(x, l),\ S(n))$
$\mathrm{Len}(l,\ n_1) \wedge \mathrm{Len}(l,\ n_2) \Rightarrow n_1 = n_2$

$\mathrm{Less}(Z,\ S(m))$
$\mathrm{Less}(n,\ Z) \Rightarrow \mathrm{False}$
$\mathrm{Less}(n,\ m) \Leftrightarrow \mathrm{Less}(S(n),\ S(m))$

$\mathrm{Len}(l,\ n) \wedge \mathrm{Len}(Cons(x, l),\ n') \Rightarrow \mathrm{Less}(n,\ n')$

Details coming soon

$\mathrm{Len} \mapsto \{([x_1, \ldots, x_n], S^n(Z)) \mid n \in \mathbb{N}\}$
$\mathrm{Less} \mapsto \{(n_1, n_2) \mid n_1 < n_2\}$

Learner:

$\mathrm{Len}(Nil, Z)$

$\mathrm{Less}(Z, S(Z))$

$\mathrm{Len}(Nil, Z) \Rightarrow \mathrm{Len}(Cons(Z, Nil), S(Z))$

$\mathrm{Less}(Z, S(Z)) \Rightarrow \mathrm{Less}(S(Z), S(S(Z)))$

$$\begin{pmatrix} \mathrm{Len}(Cons(Z, Nil), S(Z)) \ \wedge \\ \mathrm{Len}(Cons(Z, Nil), S(S(Z))) \end{pmatrix} \implies \bot$$

$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

Teacher:

$\mathrm{Len}(Nil,\ Z)$
$\mathrm{Len}(l,\ n) \Rightarrow \mathrm{Len}(Cons(x, l),\ S(n))$
$\mathrm{Len}(l,\ n_1) \wedge \mathrm{Len}(l,\ n_2) \Rightarrow n_1 = n_2$

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Details coming soon

$\mathrm{Len} \mapsto \{([x_1, \ldots, x_n], S^n(Z)) \mid n \in \mathbb{N}\}$
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$\mathrm{Len}(Nil, Z) \Rightarrow \mathrm{Len}(Cons(Z, Nil), S(Z))$
$\mathrm{Less}(Z, S(Z)) \Rightarrow \mathrm{Less}(S(Z), S(S(Z)))$
$$\begin{pmatrix} \mathrm{Len}(Cons(Z, Nil), S(Z)) \ \wedge \\ \mathrm{Len}(Cons(Z, Nil), S(S(Z))) \end{pmatrix} \implies \bot$$
$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

$\mathrm{Len}(Nil, \ Z)$
$\mathrm{Len}(l, \ n) \Rightarrow \mathrm{Len}(Cons(x, l), \ S(n))$
$\mathrm{Len}(l, \ n_1) \wedge \mathrm{Len}(l, \ n_2) \Rightarrow n_1 = n_2$

$\mathrm{Less}(Z, \ S(m))$
$\mathrm{Less}(n, \ Z) \Rightarrow \mathrm{False}$
$\mathrm{Less}(n, \ m) \Leftrightarrow \mathrm{Less}(S(n), \ S(m))$

$\mathrm{Len}(l, \ n) \wedge \mathrm{Len}(Cons(x, l), \ n') \Rightarrow \mathrm{Less}(n, \ n')$

SAT!

Models: what representation?

Len $\mapsto \emptyset$
Less $\mapsto \emptyset$

Len $\mapsto \{(Nil, Z)\}$
Less $\mapsto \{(Z, S(\cdot))\}$

Len $\mapsto \{(Nil, Z)\} \cup \{(Cons(\cdot,\cdot), S(\cdot))\}$
Less $\mapsto \{(\cdot, S(\cdot))\}$

Len $\mapsto \{([x_1, \ldots, x_n], S^n(Z)) \mid n \in \mathbb{N}\}$
Less $\mapsto \{(n_1, n_2) \mid n_1 < n_2\}$

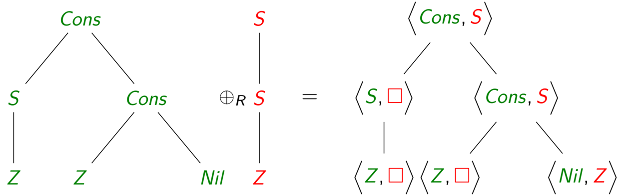
We need a finite representation of these models that

- Allows to *express* various relation between recursive algebraic datatypes;
- Can be *synthesized* from examples (Learner);
- Admits a procedure for *model-checking* constrained clauses (Teacher).

Convolution

Core idea

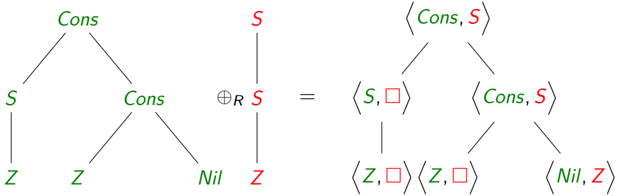
Overlay terms to read them at the same time using a classical tree automaton on a product alphabet



Convolution

Core idea

Overlay terms to read them at the same time using a classical tree automaton on a product alphabet



Convoluted tree automaton denoting the relation Len

Automaton $\mathcal{A}_{\mathrm{Len}}$ has final states $\{q_l\}$ and transitions

$$\left\{ \begin{array}{ll} \langle Nil, Z\rangle() \to q_l & \langle Z, \square\rangle() \to q_n \\ \langle Cons, S\rangle(q_n, q_l) \to q_l & \langle S, \square\rangle(q_n) \to q_n \end{array} \right\}$$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $\mathit{Vars}(B) \subseteq \mathit{Vars}(H)$

A set of SHoCs is a model that inductively defines relations.

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A set of SHoCs is a model that inductively defines relations.

Length

$\mathrm{Len}(\mathit{Nil}, Z)$
$\mathrm{Len}(\mathit{Cons}(x, l),\ S(n)) \Leftarrow \mathrm{Len}(l,\ n)$

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A set of SHoCs is a model that inductively defines relations.

Height using Max

$$\mathrm{Height}(\mathit{Leaf}, Z)$$

$$\mathrm{Height}(\mathit{Node}(l, x, r),\ S(n)) \Leftarrow \mathrm{Height}(l, h_l) \land \mathrm{Height}(r, h_r) \land \mathrm{Max}(h_l, h_r, n)$$

$$\mathrm{Max}(\ldots) \Leftarrow \ldots$$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $\mathit{Vars}(B) \subseteq \mathit{Vars}(H)$

A set of SHoCs is a model that inductively defines relations.

Height using Max: forbidden

$$\mathrm{Height}(\mathit{Leaf}, Z)$$

$$\mathrm{Height}(\mathit{Node}(l, x, r),\ S(n)) \Leftarrow \mathrm{Height}(l, h_l) \wedge \mathrm{Height}(r, h_r) \wedge \mathrm{Max}(h_l, h_r, n)$$

$$\mathrm{Max}(\ldots) \Leftarrow \ldots$$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $\mathit{Vars}(B) \subseteq \mathit{Vars}(H)$

A set of SHoCs is a model that inductively defines relations.

Height using Shallower

$\mathrm{Height}(\mathit{Leaf}, Z)$
$\mathrm{Height}(\mathit{Node}(l, x, r),\ S(n)) \Leftarrow \mathrm{Height}(l,\ n) \land \mathrm{Shal}(r, n)$
$\mathrm{Height}(\mathit{Node}(l, x, r),\ S(n)) \Leftarrow \mathrm{Height}(r,\ n) \land \mathrm{Shal}(l, n)$
$\mathrm{Shal}(\mathit{Leaf}, Z)$
$\mathrm{Shal}(\mathit{Leaf}, S(n))$
$\mathrm{Shal}(\mathit{Node}(l, x, r),\ S(n)) \Leftarrow \mathrm{Shal}(l, n) \land \mathrm{Shal}(l, n)$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $\mathit{Vars}(B) \subseteq \mathit{Vars}(H)$

A set of SHoCs is a model that inductively defines relations.

Double

$\mathit{Double}(Z, Z)$
$\mathit{Double}(S(S(x_1)), S(x_2)) \Leftarrow \mathit{Double}(x_1, x_2)$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $Vars(B) \subseteq Vars(H)$

A set of SHoCs is a model that inductively defines relations.

Double: forbidden

$Double(Z, Z)$
$Double(S(S(x_1)), S(x_2)) \Leftarrow Double(x_1, x_2)$

Shallow Horn Clauses (SHoCs)

Syntactic restriction of Horn clauses

A SHoC is a definite Horn clause $H \Leftarrow B$ such that

- The head H is linear and shallow
- The body B is flat
- There is no existential variable, i.e., $\mathit{Vars}(B) \subseteq \mathit{Vars}(H)$

A set of SHoCs is a model that inductively defines relations.

SHoCs are closed by boolean operations

$\text{Len} \mapsto \{([x_1, \ldots, x_n], S^n(Z)) \mid n \in \mathbb{N}\}$
$\text{Less} \mapsto \{(n_1, n_2) \mid n_1 < n_2\}$

Learner

$\text{Len}(Nil, Z)$

$\text{Less}(Z, S(Z))$

$\text{Len}(Nil, Z) \Rightarrow \text{Len}(Cons(Z, Nil), S(Z))$

$\text{Less}(Z, S(S(Z)))$

$$\begin{pmatrix}\text{Len}(Cons(Z, Nil), S(Z)) \;\; \wedge \\ \text{Len}(Cons(Z, Nil), S(S(Z)))\end{pmatrix} \implies \bot$$

$\text{Less}(S(Z), S(Z)) \Rightarrow \text{Less}(Z, Z)$

Teacher

$\text{Len}(Nil, \; Z)$
$\text{Len}(l, \; n) \Rightarrow \text{Len}(Cons(x, l), \; S(n))$
$\text{Len}(l, \; n_1) \wedge \text{Len}(l, \; n_2) \Rightarrow n_1 = n_2$

$\text{Less}(Z, \; S(m))$
$\text{Less}(n, \; Z) \Rightarrow \text{False}$
$\text{Less}(n, \; m) \Leftrightarrow \text{Less}(S(n), \; S(m))$

$\text{Len}(l, \; n) \wedge \text{Len}(Cons(x, l), \; n') \Rightarrow \text{Less}(n, \; n')$

Len $\mapsto$
Less $\mapsto$
SHoCs that defines Len and Less

Learner

$\mathrm{Len}(\mathit{Nil}, Z)$

$\mathrm{Less}(Z, S(Z))$

$\mathrm{Len}(\mathit{Nil}, Z) \Rightarrow \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$

$\mathrm{Less}(Z, S(S(Z)))$

$$\begin{pmatrix} \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \ \wedge \\ \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \Longrightarrow \bot$$

$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

Teacher

$\mathrm{Len}(\mathit{Nil}, Z)$
$\mathrm{Len}(l, n) \Rightarrow \mathrm{Len}(\mathit{Cons}(x, l), S(n))$
$\mathrm{Len}(l, n_1) \wedge \mathrm{Len}(l, n_2) \Rightarrow n_1 = n_2$

$\mathrm{Less}(Z, S(m))$
$\mathrm{Less}(n, Z) \Rightarrow \mathrm{False}$
$\mathrm{Less}(n, m) \Leftrightarrow \mathrm{Less}(S(n), S(m))$

$\mathrm{Len}(l, n) \wedge \mathrm{Len}(\mathit{Cons}(x, l), n') \Rightarrow \mathrm{Less}(n, n')$

Details coming now:
Model generalisation

$\mathrm{Len} \mapsto$
$\mathrm{Less} \mapsto$
SHoCs that defines Len and Less

Learner:

$\mathrm{Len}(Nil, Z)$

$\mathrm{Less}(Z, S(Z))$

$\mathrm{Len}(Nil, Z) \Rightarrow \mathrm{Len}(Cons(Z, Nil), S(Z))$

$\mathrm{Less}(Z, S(S(Z)))$

$$\begin{pmatrix} \mathrm{Len}(Cons(Z, Nil), S(Z)) \ \wedge \\ \mathrm{Len}(Cons(Z, Nil), S(S(Z))) \end{pmatrix} \implies \bot$$

$\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)$

Teacher:

$\mathrm{Len}(Nil,\ Z)$
$\mathrm{Len}(l,\ n) \Rightarrow \mathrm{Len}(Cons(x, l),\ S(n))$
$\mathrm{Len}(l,\ n_1) \wedge \mathrm{Len}(l,\ n_2) \Rightarrow n_1 = n_2$

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$\mathrm{Less}(n,\ m) \Leftrightarrow \mathrm{Less}(S(n),\ S(m))$

$\mathrm{Len}(l,\ n) \wedge \mathrm{Len}(Cons(x, l),\ n') \Rightarrow \mathrm{Less}(n,\ n')$

Learner: Model synthesis

$$\begin{array}{l}
\mathrm{Len}(\mathit{Nil}, Z) \\
\mathrm{Less}(Z, S(Z)) \\
\mathrm{Len}(\mathit{Nil}, Z) \Rightarrow \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \\
\mathrm{Less}(Z, S(S(Z))) \\
\begin{pmatrix} \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \ \wedge \\ \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \implies \bot \\
\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)
\end{array}$$

Learner: Model synthesis

$$\mathrm{Len}(\mathit{Nil}, Z)$$

$$\sout{\mathrm{Less}(Z, S(Z))}$$

$$\mathrm{Len}(\mathit{Nil}, Z) \Rightarrow \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$$

$$\sout{\mathrm{Less}(Z, S(S(Z)))}$$

$$\begin{pmatrix} \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \;\; \wedge \\ \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \implies \bot$$

$$\sout{\mathrm{Less}(S(Z), S(Z)) \Rightarrow \mathrm{Less}(Z, Z)}$$

Learner: Model synthesis

$$\mathrm{Len}(\mathit{Nil}, Z)$$
$$\mathrm{Len}(\mathit{Nil}, Z) \Rightarrow \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$$
$$\begin{pmatrix} \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \ \wedge \\ \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z))) \end{pmatrix} \implies \bot$$

Learner: Model synthesis

$$\mathrm{Len}(\mathit{Nil}, Z)$$
$$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$$
$$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))$$

Learner: Model synthesis

Input

$\mathrm{Len}(\mathit{Nil}, Z)$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))$

Learner: Model synthesis

Input

$\mathrm{Len}(\mathit{Nil}, Z)$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))$

Output

Learner: Model synthesis

Input

$\text{Len}(\mathit{Nil}, Z)$

$\neg\text{Len}(\mathit{Nil}, Z) \quad \vee \quad \text{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))$

$\neg\text{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \neg\text{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))$

- **Choose** one literal per clause that must be satisfied

Output

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$

$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$

$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
-
-
-

Output

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$

$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$

$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
-
-

Output

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
-
-

Output

$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow$
$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(Nil, Z)}$
$\neg\mathrm{Len}(Nil, Z) \quad \vee \quad \underline{\mathrm{Len}(Cons(Z, Nil), S(Z))}$
$\neg\mathrm{Len}(Cons(Z, Nil), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(Cons(Z, Nil), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
-

Output

$\mathrm{Len}(Nil, Z) \Leftarrow$
$\mathrm{Len}(Cons(x, l), S(n)) \Leftarrow$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
-

Output

$$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$$
$$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow \top$$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$
$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow \top$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$
$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow \top$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$$
$$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow \top$$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(Nil, Z)}$
$\neg\mathrm{Len}(Nil, Z) \quad \vee \quad \underline{\mathrm{Len}(Cons(Z, Nil), S(Z))}$
$\neg\mathrm{Len}(Cons(Z, Nil), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(Cons(Z, Nil), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC ←
- Verify that every chosen negative literal is not satisfied

Output

$\mathrm{Len}(Nil, Z) \Leftarrow \top$
$\mathrm{Len}(Cons(x, l), S(n)) \Leftarrow \top$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$
$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow P(n)$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(Nil, Z)}$
$\neg\mathrm{Len}(Nil, Z) \quad \vee \quad \underline{\mathrm{Len}(Cons(Z, Nil), S(Z))}$
$\neg\mathrm{Len}(Cons(Z, Nil), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(Cons(Z, Nil), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$$\mathrm{Len}(Nil, Z) \Leftarrow \top$$
$$\mathrm{Len}(Cons(x, l), S(n)) \Leftarrow P(n) \wedge Q(l)$$

Learner: Model synthesis

Input

$$\underline{\mathrm{Len}(Nil, Z)}$$
$$\neg\mathrm{Len}(Nil, Z) \quad \lor \quad \underline{\mathrm{Len}(Cons(Z, Nil), S(Z))}$$
$$\neg\mathrm{Len}(Cons(Z, Nil), S(Z)) \quad \lor \quad \underline{\neg\mathrm{Len}(Cons(Z, Nil), S(S(Z)))}$$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$$\mathrm{Len}(Nil, Z) \Leftarrow \top$$
$$\mathrm{Len}(Cons(x, l), S(n)) \Leftarrow P(n) \land Q(l) \land R(x, l)$$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(\mathit{Nil}, Z)}$
$\neg\mathrm{Len}(\mathit{Nil}, Z) \quad \vee \quad \underline{\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z))}$
$\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(\mathit{Cons}(Z, \mathit{Nil}), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$$\mathrm{Len}(\mathit{Nil}, Z) \Leftarrow \top$$
$$\mathrm{Len}(\mathit{Cons}(x, l), S(n)) \Leftarrow P(n) \wedge Q(l) \wedge R(x, l) \wedge \mathrm{Len}(l, n)$$

Learner: Model synthesis

Input

$\underline{\mathrm{Len}(Nil, Z)}$
$\neg\mathrm{Len}(Nil, Z) \quad \vee \quad \underline{\mathrm{Len}(Cons(Z, Nil), S(Z))}$
$\neg\mathrm{Len}(Cons(Z, Nil), S(Z)) \quad \vee \quad \underline{\neg\mathrm{Len}(Cons(Z, Nil), S(S(Z)))}$

- **Choose** one literal per clause that must be satisfied
- Create a SHoC to recognize each chosen positive atom
- **Choose** atoms for the body of every created SHoC
- Verify that every chosen negative literal is not satisfied

Output

$$\mathrm{Len}(Nil, Z) \Leftarrow \top$$
$$\mathrm{Len}(Cons(x, l), S(n)) \Leftarrow \mathrm{Len}(l, x)$$

Learner: Model synthesis

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$\neg\text{Len}(Nil, Z) \quad \vee \quad \underline{\text{Len}(Cons(Z, Nil), S(Z))}$
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Implementation

`https://gitlab.inria.fr/tlosekoo/auto-forestation`.

- Library for terms, clauses, SHoCs, etc...
  $\sim$ 5k lines of Ocaml code.
- **Learner** implemented with
  - $\sim$ 100 tricky lines of Clingo[1] for SHoCs inference
  - $\sim$ 500 lines of Ocaml for instance encoding/decoding
- **Teacher** implemented with heuristics to make it practical
  $\sim$ 2k lines of Ocaml code

[1] Clingo in an Answer Set Programming solver

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Some relations cannot be expressed by SHoCs

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✓	$\forall l, x,$	$less\ (length\ l)\ (length\ (Cons(x,\ l)))$
✓	$\forall p, l,$	$(prefix\ p\ (concat\ p\ l))$
✓	$\forall l,$	$length\ l\ =\ length\ (insert\_sort\ l)$
✓	$\forall t,$	$leq\ (height\ t)\ (size\ t)$
×	$\forall t,$	$height\ t\ =\ height\ (mirror\ t)$
×	$\forall x, l,$	$(count\ x\ (Cons(x, l)))\ =\ S(count\ x\ l)$

Conclusion

What is the problem, again?

Automatically prove relational properties on first-order monomorphic programs transforming algebraic data structures

In this talk

- Learner/Teacher overview
- Shallow Horn Clauses to represent models

Shallow Horn Clauses = Relational Alternating Tree Automata