Weighted automata: To be decidable, or not to be decidable... that is the question.

Laure Daviaud
University of East Anglia

And today we talk about...

And today we talk about...

First part: weighted automata

And today we talk about...

First part: weighted automata

Second part: weighted automata

And today we talk about...

First part: weighted automata
$\longrightarrow$ General stuff, a bit of a survey around decidability

Second part: weighted automata

And today we talk about...

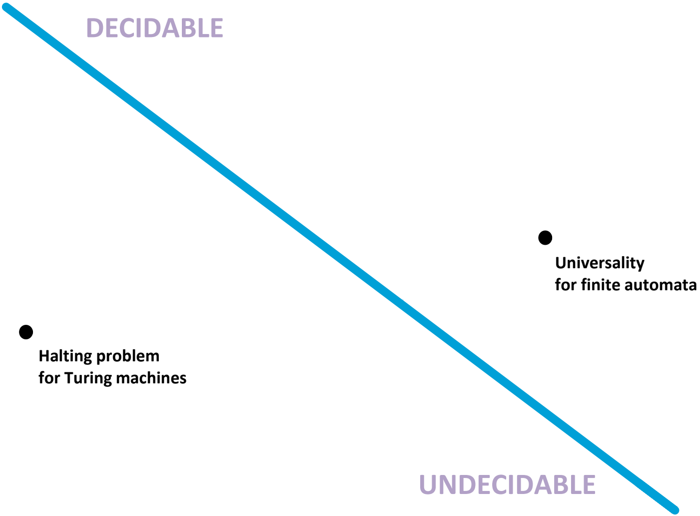
First part: weighted automata
$\longrightarrow$ General stuff, a bit of a survey around decidability

Second part: weighted automata
$\longrightarrow$ A specific result on max-plus automata

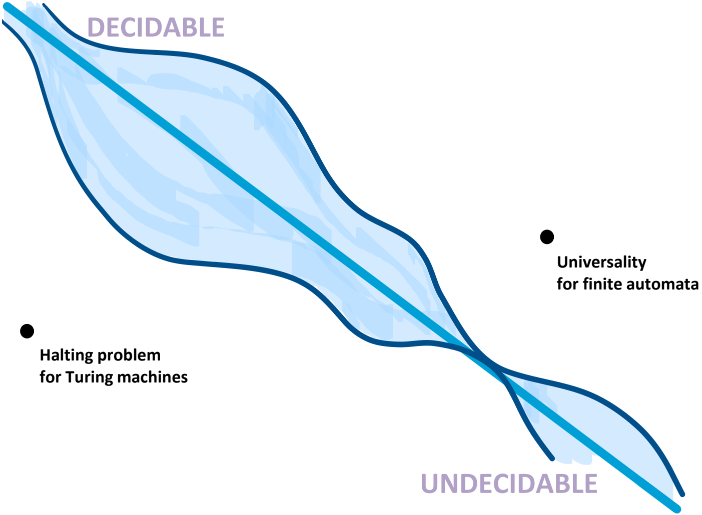
Part I

A split world

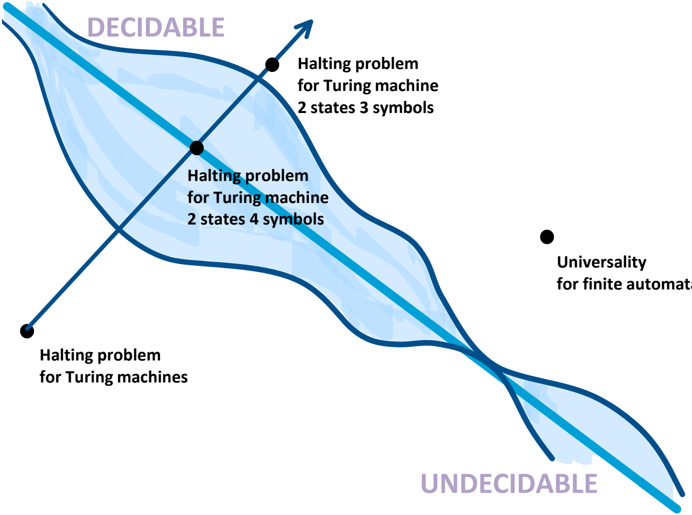
A split world



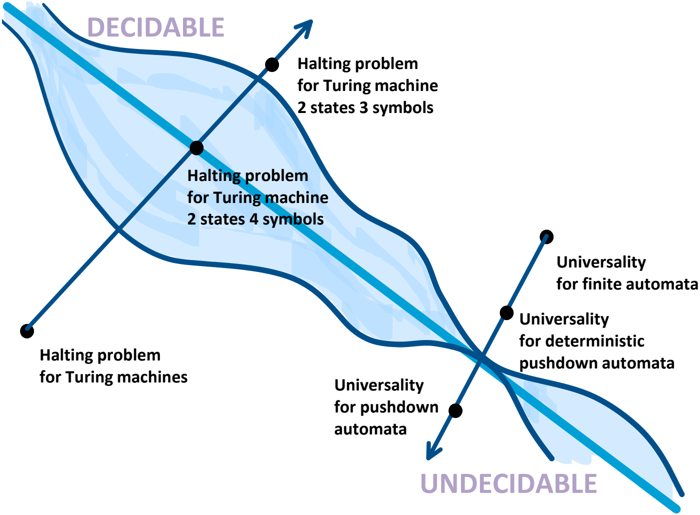
A split world



A split world



A split world



Weighted automata

Weighted automata:

- Finite non deterministic automata with weights (e.g. real values) on transitions
- Computes a function:

$$\Sigma^* \to \mathbb{R}$$

Weighted automata

Weighted automata:

- Finite non deterministic automata with weights (e.g. real values) on transitions
- Computes a function:

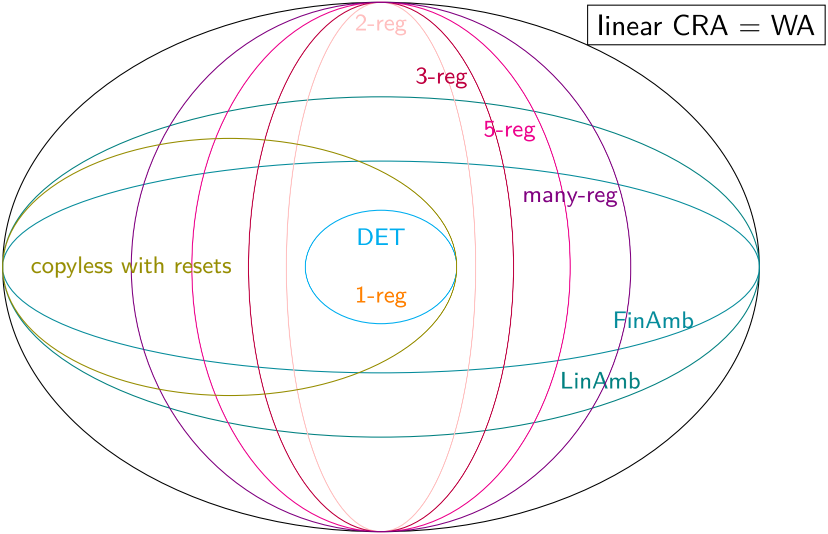
$$\Sigma^* \to \mathbb{R}$$

Cost register automata:

- Deterministic automaton with write-only registers storing weights (e.g. real values)
- Computes a function:

$$\Sigma^* \to \mathbb{R}$$

Classes


linear CRA = WA
2-reg
3-reg
5-reg
many-reg
DET
1-reg
copyless with resets
FinAmb
LinAmb

The decision problems

The decision problems

- Can I compute the same function with a simpler model?
  Determinisation, Ambiguity, Register complexity...

The decision problems

- Can I compute the same function with a simpler model?
  Determinisation, Ambiguity, Register complexity...
- Can I compare two models?
  Equivalence, Containement, Big-O...

The decision problems

- Can I compute the same function with a simpler model?
  Determinisation, Ambiguity, Register complexity...

- Can I compare two models?
  Equivalence, Containement, Big-O...

- Can I describe the function computed by a model?
  Boundedness, Universality, Approximation, Value...

The decision problems

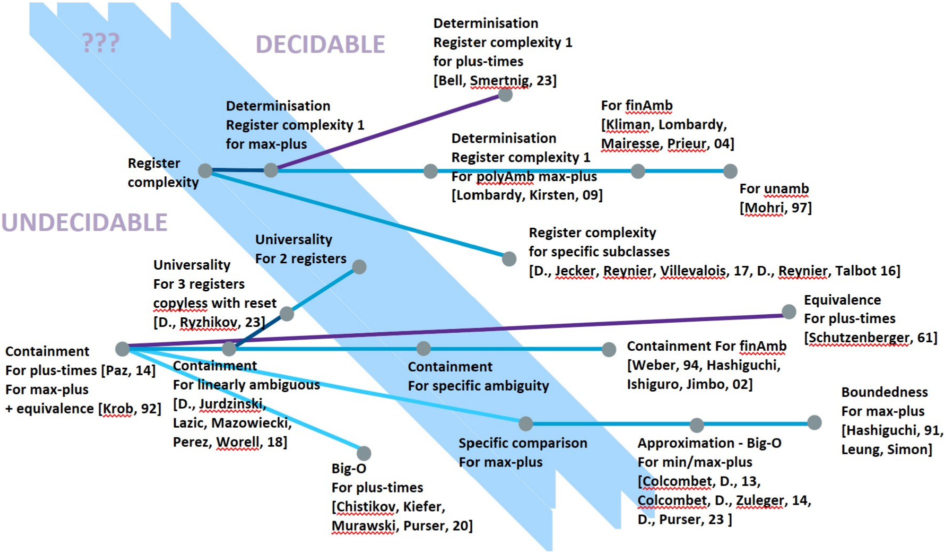
- Can I compute the same function with a simpler model?
  Determinisation, Ambiguity, Register complexity...

- Can I compare two models?
  Equivalence, Containement, Big-O...

- Can I describe the function computed by a model?
  Boundedness, Universality, Approximation, Value...

$\longrightarrow (\mathbb{R}, +, \times)$
$\longrightarrow (\mathbb{N}, \mathsf{max}, +)$

Some results - this is not exhaustive!



Part II

Big-O for max-plus

Big-O Problem

Given two max-plus automata computing functions f and g, is f big-O of g?

There exists C such that for all $w \in \Sigma^*$, $f(w) \leq Cg(w) + C$

Big-O for max-plus

Big-O Problem

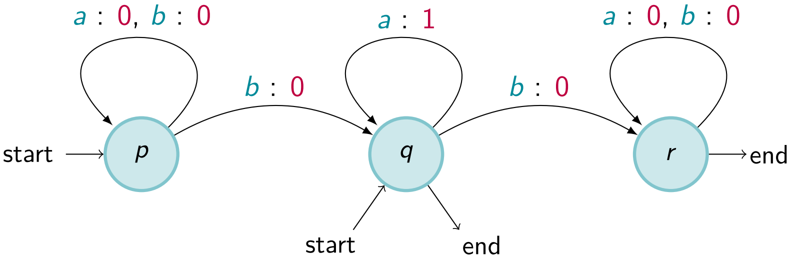
Given two max-plus automata computing functions f and g, is f big-O of g?

There exists C such that for all $w \in \Sigma^*$, $f(w) \leq Cg(w) + C$

Theorem [D., Purser]

The big-O problem is PSPACE-complete for max-plus automata.

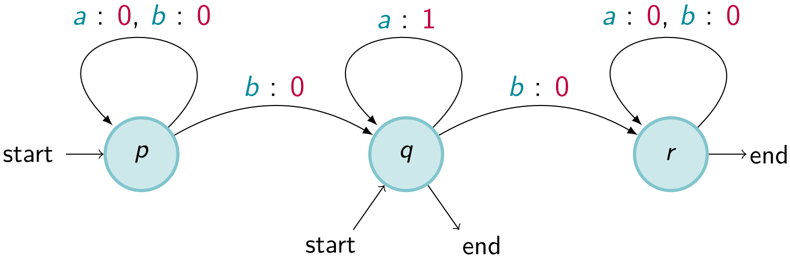
An example of max-plus automata



$$\Sigma^* \to \mathbb{N} \cup \{-\infty\}$$

- weights in $\mathbb{N}$

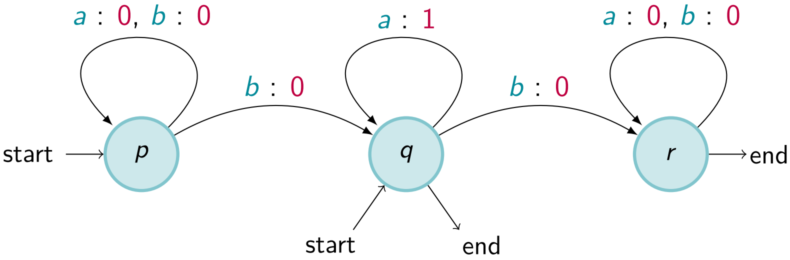
An example of max-plus automata



$$\Sigma^* \to \mathbb{N} \cup \{-\infty\}$$

- weights in $\mathbb{N}$
- run $\to$ sum the weights

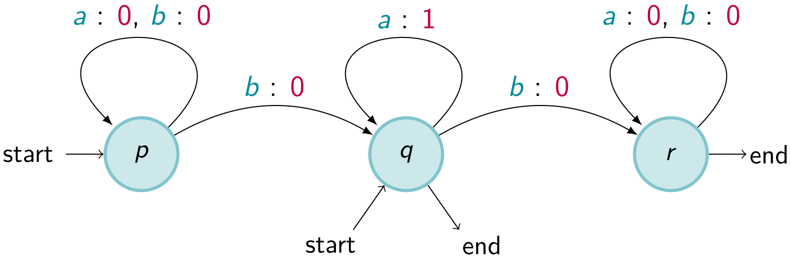
An example of max-plus automata



$$\Sigma^* \to \mathbb{N} \cup \{-\infty\}$$

- weights in $\mathbb{N}$
- run $\to$ sum the weights
- word $\to$ max of the accepting runs labelled by the word (or $-\infty$)

An example of max-plus automata

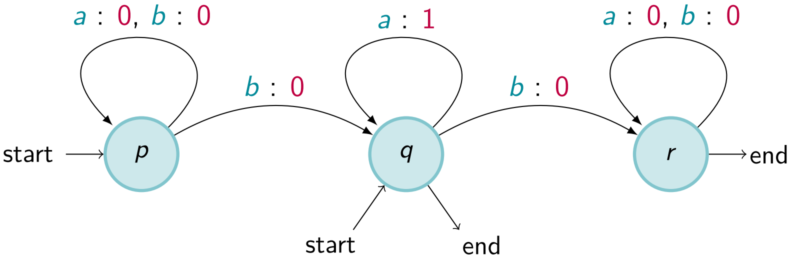


$$\Sigma^* \to \mathbb{N} \cup \{-\infty\}$$

- weights in $\mathbb{N}$
- run $\to$ sum the weights
- word $\to$ max of the accepting runs labelled by the word (or $-\infty$)

  What is computed on *aaabbaabbbaaaaabaa*?

An example of max-plus automata



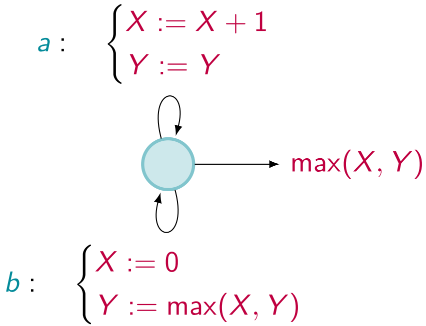
$$\Sigma^* \to \mathbb{N} \cup \{-\infty\}$$

- weights in $\mathbb{N}$
- run $\to$ sum the weights
- word $\to$ max of the accepting runs labelled by the word (or $-\infty$)

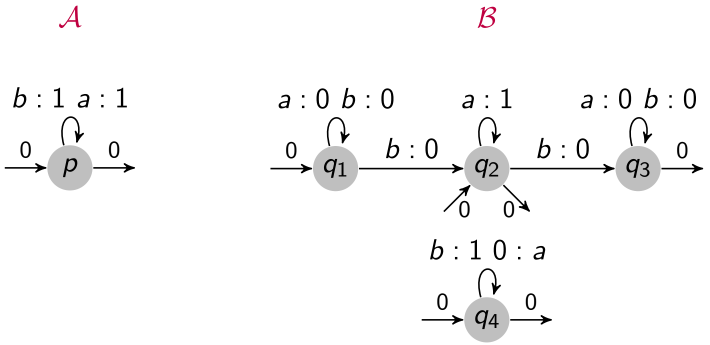
  What is computed on *aaabbaabbbaaaaabaa*?

  $w \mapsto$ maximal number of consecutive *a*'s in w

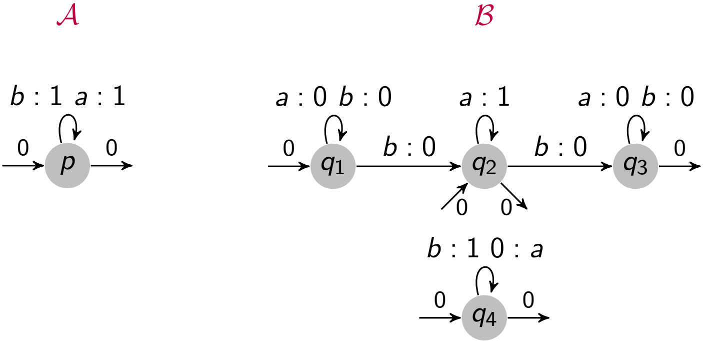
Same example with registers



Our running example

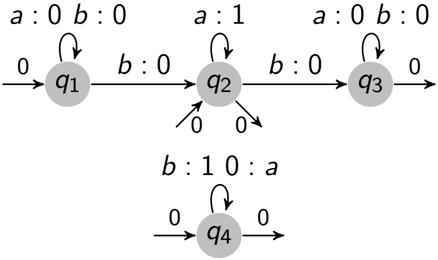


Our running example



$\mathcal{B}$: $w \mapsto$ max of number of b's and number of consecutive a's in w

And matrices...

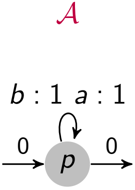


$$M(a) = \begin{pmatrix} 0 & - & - & - \\ - & 1 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{pmatrix}$$

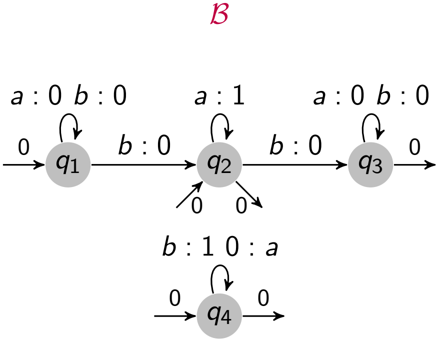
$$M(b) = \begin{pmatrix} 0 & 0 & - & - \\ - & - & 0 & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix}$$

with $(I)_{q_1} = (I)_{q_2} = (I)_{q_4} = (F)_{q_2} = (F)_{q_3} = (F)_{q_4} = 0$

Witnesses

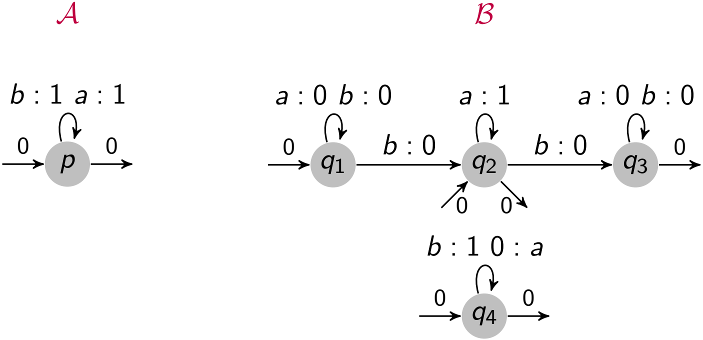


$\mathcal{A}$
$b:1$ $a:1$
0
p
0



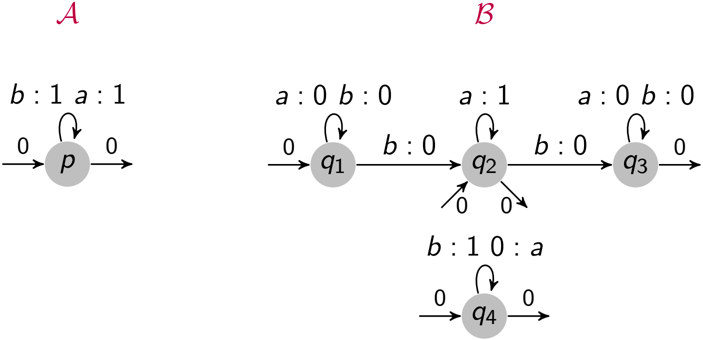
$\mathcal{B}$
$a:0$ $b:0$
$a:1$
$a:0$ $b:0$
0
q_1
$b:0$
q_2
$b:0$
q_3
0
0
0
$b:1$ $0:a$
0
q_4
0

Witnesses



Key sequence: $(a^n b)^n a^n$

Witnesses



Key sequence: $(a^n b)^n a^n$

$$\left(n^2, \begin{pmatrix} 0 & n & n & - \\ - & - & n & - \\ - & - & 0 & - \\ - & - & - & n \end{pmatrix} \right)$$

Witnesses

$$\left(n^2, \begin{pmatrix} 0 & n & n & - \\ - & - & n & - \\ - & - & 0 & - \\ - & - & - & n \end{pmatrix} \right)$$

Becomes:

$$\left(\infty, \begin{pmatrix} 0 & 1 & 1 & - \\ - & - & 1 & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix} \right)$$

- $-$: no run at all
- 0: all runs have weight 0
- 1: some runs with positive weights but not the largest growth rate
- ∞: runs with largest growth rate

Game plan

Game plan

Aim: Decide some property P on an abstract model M

Game plan

Aim: Decide some property P on an abstract model M

$\longrightarrow$ Find the right finite algebraic structure S to represent M

Game plan

Aim: Decide some property P on an abstract model M

$\longrightarrow$ Find the right finite algebraic structure S to represent M

$\longrightarrow$ Find the right operations on S to capture P (no more, no less)

Game plan

Aim: Decide some property P on an abstract model M

⟶ Find the right finite algebraic structure S to represent M

⟶ Find the right operations on S to capture P (no more, no less)

⟶ Identify witnesses, present in S if and only if M satisfies P

Running example

$$a = (p, 1, p, \begin{pmatrix} 0 & \text{-} & \text{-} & \text{-} \\ \text{-} & 1 & \text{-} & \text{-} \\ \text{-} & \text{-} & 0 & \text{-} \\ \text{-} & \text{-} & \text{-} & 0 \end{pmatrix}) \text{ and } b = (p, 1, p, \begin{pmatrix} 0 & 0 & \text{-} & \text{-} \\ \text{-} & \text{-} & 0 & \text{-} \\ \text{-} & \text{-} & 0 & \text{-} \\ \text{-} & \text{-} & \text{-} & 1 \end{pmatrix})$$

Running example

$$a = (p, 1, p, \begin{pmatrix} 0 & - & - & - \\ - & 1 & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{pmatrix}) \text{ and } b = (p, 1, p, \begin{pmatrix} 0 & 0 & - & - \\ - & - & 0 & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix})$$

$$ab = (p, 1, p, \begin{pmatrix} 0 & 0 & - & - \\ - & - & 1 & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix}) \quad bb = (p, 1, p, \begin{pmatrix} 0 & 0 & 0 & - \\ - & - & 0 & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix})$$

Running example

⟶ Sharp

$$a^{\#} = (p, \infty, p, \begin{pmatrix} 0 & - & - & - \\ - & \infty & - & - \\ - & - & 0 & - \\ - & - & - & 0 \end{pmatrix}) \quad a^{\#}b = (p, \infty, p, \begin{pmatrix} 0 & 0 & - & - \\ - & - & \infty & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix})$$

$$a^{\#}ba^{\#}b = (p, \infty, p, \begin{pmatrix} 0 & 0 & \infty & - \\ - & - & \infty & - \\ - & - & 0 & - \\ - & - & - & 1 \end{pmatrix})$$

$$(a^{\#}ba^{\#}b)^{\#} = (p, \infty, p, \begin{pmatrix} 0 & 0 & \infty & - \\ - & - & \infty & - \\ - & - & 0 & - \\ - & - & - & \infty \end{pmatrix})$$

Running example

$\longrightarrow$ Flat

$$(a^{\#}ba^{\#}b)^{\flat} = (p, \infty, p, \begin{pmatrix} 0 & 0 & 1 & \text{-} \\ \text{-} & \text{-} & 1 & \text{-} \\ \text{-} & \text{-} & 0 & \text{-} \\ \text{-} & \text{-} & \text{-} & 1 \end{pmatrix})$$

Game plan

Aim: Decide whether $\mathcal{A}$ big-O of $\mathcal{B}$

$\longrightarrow$ Find the right finite algebraic structure S to represent M

$\longrightarrow$ Find the right operations on S to capture P (no more, no less)

$(p, \overline{x_a}, q, \overline{M(a)})$ closed under product, sharp and flat

$\longrightarrow$ Identify witnesses, present in S if and only if M satisfies P

$$(p, x, q, M)$$

with:

- p initial, q final
- $x = \infty$
- $IMF \neq \infty$

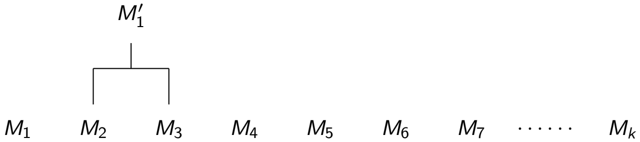
The factorisation forest theorem of Simon

In any finite semigroup...

$M_1 \quad M_2 \quad M_3 \quad M_4 \quad M_5 \quad M_6 \quad M_7 \quad \cdots\cdots \quad M_k$

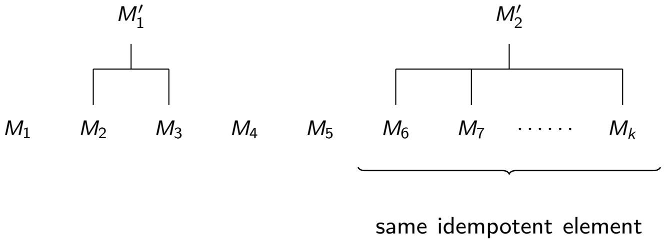
The factorisation forest theorem of Simon

In any finite semigroup...



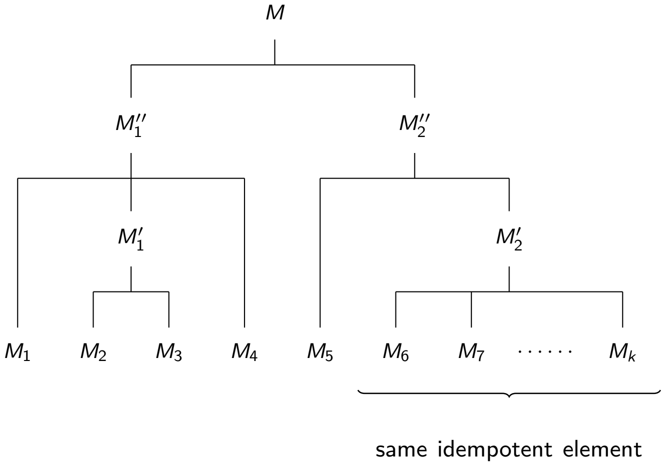
The factorisation forest theorem of Simon

In any finite semigroup...



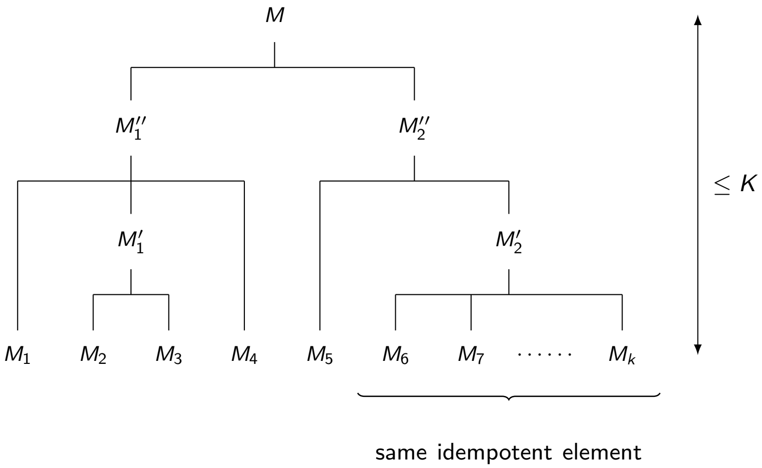
The factorisation forest theorem of Simon

In any finite semigroup...



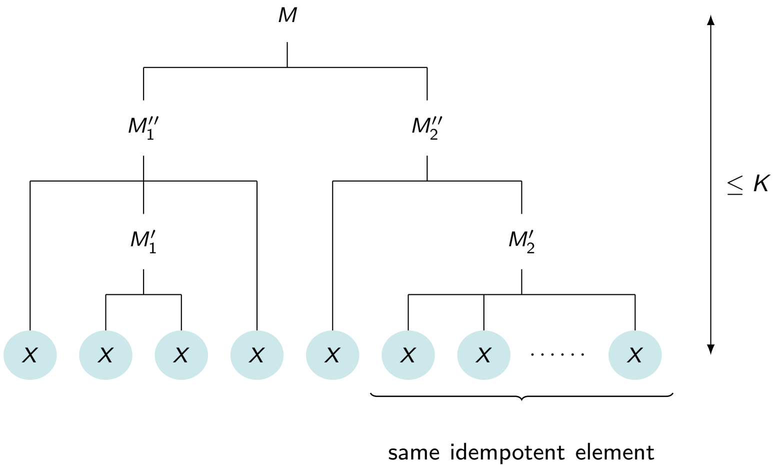
The factorisation forest theorem of Simon

In any finite semigroup...



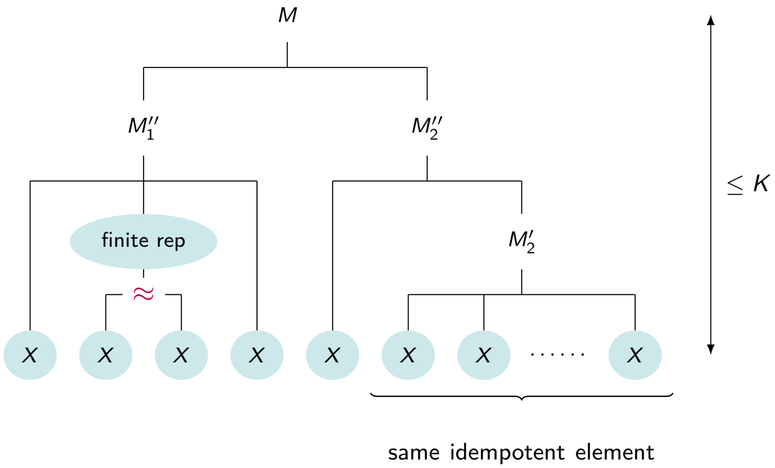
The factorisation forest theorem of Simon

In any finite semigroup...



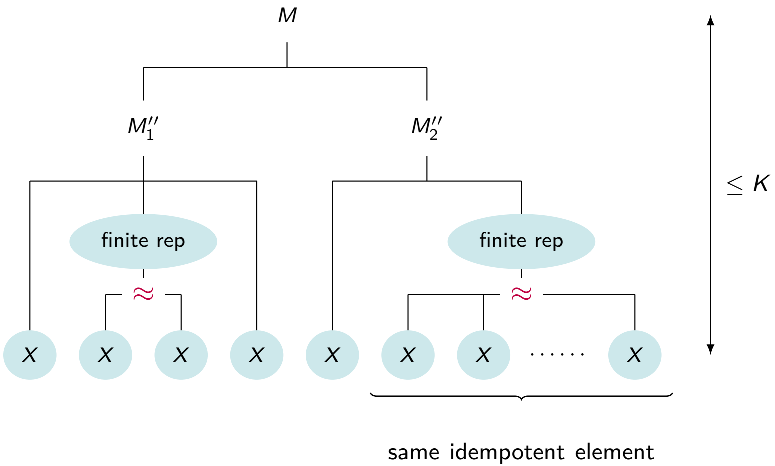
The factorisation forest theorem of Simon

In any finite semigroup...



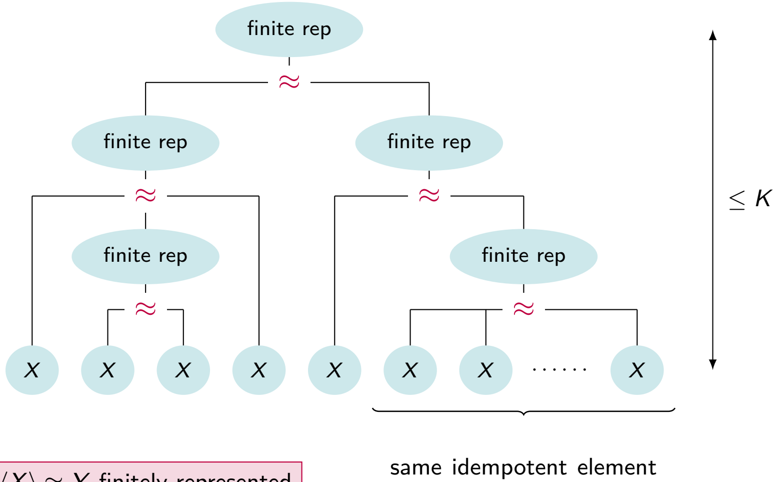
The factorisation forest theorem of Simon

In any finite semigroup...



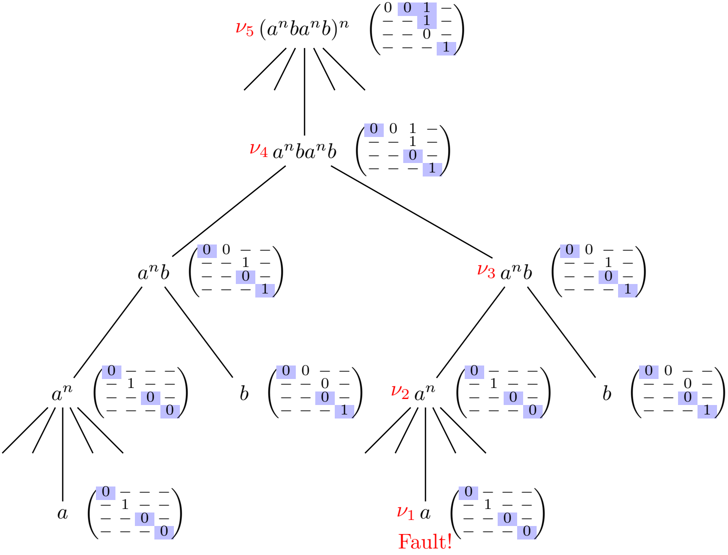
The factorisation forest theorem of Simon

In any finite semigroup...



$\langle X \rangle \approx Y$ finitely represented

Just a fancy picture



The End (or just the beginning)

Theorem [D., Purser]

The big-O problem is PSPACE-complete for max-plus automata.

Proposition [D., Purser, Tcheng]

Construction of witnesses with increasing complexity.