Stackelberg Games of Water Extraction

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Ideas of the paper

We analyze a simple problem of water extraction.

- Myopic agents harvesting a common water resource
- A regulator has some control on the cost function
- The regulator takes into account rainfall and strategic interaction between agents

Ideas of the paper (ctd.)

- Groundwater extraction: marginal cost depends on the level of the aquifer
- In general, resources with accessibility problems: cost depends on scarcity
- The main ingredient: make the cost depend on the projected evolution of the resource: before or after the extraction or rainfall
- The goal: deduce its economical and environmental consequences
- The method: a Stackelberg Game
  linear-quadratic static game between agents,
  different kinds of optimization for the regulator.

The model of Provencher and Burt

We consider the extraction of groundwater by K players.
Dynamics of groundwater:

$$G_{t+1} = G_t + R - \sum_{k=1}^{K} u_t^k, \quad G_0, \quad \text{given.}$$

We suppose R is a constant.

Instantaneous profit:

$$\pi^i(u_t^i, G_t) = F_i(u_t^i) - C_i(G_t) \; \times \; u_t^i.$$

Marginal extraction cost ($C_i(.)$): depends on the current level of the groundwater.

The extended model

Introduce the more general instantaneous profit function:

$$\pi^i(u_t) = F_i(u_t^i) - C_i(G_t + mR - n\sum_j u_t^j)\, u_t^i$$

where $n, m \in [0, 1]$.

The extreme cases:

- $n = 0, m = 0$ (the standard case): cost based on current resource
- $n = 1, m = 1$: cost based on the state of the resource in the following period.

When $n \neq 0$ the profit function of player i depends on the action of the other player: strategic interaction not just through the dynamics.

Angles of analysis

In a previous work, we developped the non-cooperative, Nash-feedback solution for agents.

Here, we develop the supervised setting

- myopic followers, static Stackelberg leader
- myopic followers, dynamic Stackelberg leader

$\rightarrow$ an exercise in sensitivity analysis of LQ dynamic games, and non-LQ, non-concave optimal control problems.
Mostly work in progress...

Contents

The case of myopic agents

Assume agents play Nash with their instantaneous profit:

$$\max_{u^i} \left\{ F_i(u_t^i) - C_i(G_t + mR - n \sum_{k=1}^{K} u_t^k) \; u_t^i \right\}.$$

For convenience, we continue with the particular linear-quadratic functional form:

$$F_i(u) = a_i u - \frac{b_i}{2} u^2, \quad C_i(x) = z_i - c_i x > 0.$$

Myopic agent reaction

In the symmetric case we find:

$$u(G) = \underbrace{\frac{c}{b+(K+1)cn}}_{\alpha} G + \underbrace{\frac{a-z+cmR}{b+(K+1)cn}}_{\gamma}.$$

The value function of each player is:

$$\pi(G_0) = \frac{(cG_0+a-z+cmR)^2}{(b+(K+1)cn)^2} \frac{b+2cn}{2}.$$

Myopic stock dynamics

The stock dynamics is:

$$\begin{aligned} G_{t+1} &= G_t \;+\; R \;-\; K(\alpha G_t + \gamma) \\ &= (1 - K\alpha)^t G_0 + (R - K\gamma)\frac{1-(1-K\alpha)^t}{K\alpha} \;. \end{aligned}$$

The asymptotic stock is:

$$G_\infty \;=\; \frac{R - K\gamma}{K\alpha} \;=\; \frac{Rb}{Kc} + \frac{K+1}{K}\, Rn - Rm - \frac{a-z}{c} \;.$$

Contents

Supervised setting

A regulator is in charge of selecting the cost model, by choosing the cost parameters n and m.
The choice is announced to players, who play Nash.
Followers are myopic:

$$u_t^i = \frac{c}{b + (K+1)cn} G_t + \frac{a - z + cmR}{b + (K+1)cn}$$

The supervisor optimizes her own criterion, taking this reaction into account.

Optimal static choice

The supervisor gets to choose n and m once and for all.

Optimal pricing problem

The supervisor's problem is:

$$\max_{(n,m)\in[0,1]^2} \left\{ \sum_{t=0}^{\infty} \beta_L^t \sum_i \pi^i(u_t, G_t) \right\}$$

with the myopic agent reactions and stock dynamics

$$G_{t+1} = G_t + R - \sum_i u_t^i \ .$$

Optimal static choice (ctd.)

Thanks to the explicit solution for the dynamics:

$$V_{nm}(G_0) = \frac{b+2cn}{2}\left[\frac{\alpha^2(G_0-G_\infty)^2}{1-\beta(1-K\alpha)^2} + \frac{2R}{K}\,\frac{\alpha(G_0-G_\infty)}{1-\beta(1-K\alpha)} + \frac{R^2}{K^2(1-\beta)}\right]$$

with

$$G_\infty = \frac{R-K\gamma}{K\alpha}\,.$$

$$\alpha = \frac{c}{(K+1)cn+b} \qquad \gamma = \frac{mRc+a-z}{(K+1)cn+b}\,.$$

Optimal static choice (ctd.)

$\rightarrow$ optimization wrt (n, m) for each G_0.

Results:

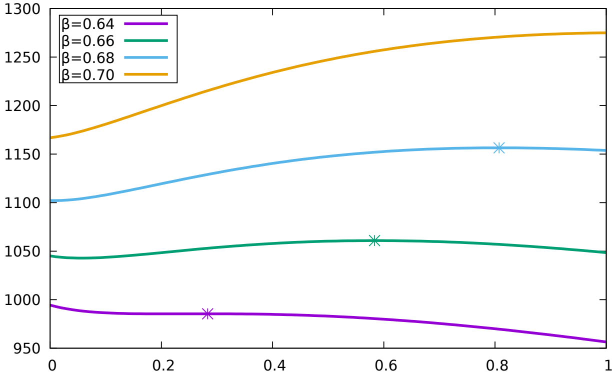
- $m^* = 1$ is always optimal if $K\alpha < 1$ (monotonous traj.)
- the optimal n is:

$$n^* = \begin{cases} 0 & \text{if } \beta_L \leq \underline{\beta}(G_0) \\ 1 & \text{if } \beta_L \geq \overline{\beta}(G_0) \\ n^*(\beta_L) & \text{otw.} \end{cases}$$

but $n^*(\beta_L)$ is the root of a 4^{th} degree polynomial.

Optimal static choice (ctd.)

A situation where $n^*(\beta_L) \notin \{0,1\}$:



Stackelberg with myopic followers

Followers being myopic, they apply the Nash controls. If symmetric:

$$u^i(G) = \frac{c}{b + (K+1)cn}\, G + \frac{a - z + cmR}{b + (K+1)cn}.$$

The benevolent regulator maximizes the total discounted profit on their behalf:

$$\max_{\{n_t, m_t\}} \left\{ \sum_{t=0}^{\infty} \beta_L^t \sum_i F(u^i(G_t)) - C(G_t + m_t R - n_t U(G_t)) u^i(G_t) \right\}$$

with the dynamics:

$$G_{t+1} \;=\; G_t \;+\; R \;-\; K u^i(G_t)\;.$$

$\Longrightarrow$ an optimal control problem.

First-order conditions

From the maximum principle:

$$
\begin{aligned}
0 &= \beta_L^t cU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K+1)cn)(q_t - q_{t-1}) \\
&\quad -Kcq_t \\
0 &= \beta_L^t cU(G_t, n_t, m_t)^2(b - (K+1)(b + cn_t)) \\
&\quad +(b + (K+1)cn_t)(\lambda_t^{(n)} - \mu_t^{(n)}) \\
&\quad +Kcq_t(K+1)U(G_t, n_t, m_t) \\
0 &= \beta_L^t cRU(G_t, n_t, m_t)(b + 2cn_t) + (b + (K+1)cn_t)(\lambda_t^{(m)} - \mu_t^{(m)}) \\
&\quad -KcRq_t.
\end{aligned}
$$

where:

- $\lambda_t^{(n)}$, $\mu_t^{(n)}$ multipliers for $n_t \geq 0$ and $n_t \leq 1$, *id* for $\lambda_t^{(m)}$, $\mu_t^{(m)}$.
- $U(\dots)$ is the feedback function of followers.

Stationary situations

If a stationary state and control exists, then:

- optimal m: $m^* = 1$,
- optimal n:

$$n^* = \begin{cases} 0 & \text{if } \beta_L \leq \dfrac{b}{b+c} \\ 1 & \text{if } \beta_L \geq \dfrac{K(b+c)+c}{K(b+2c)+c} \\ \dfrac{K}{K+1} \, \dfrac{\beta(b+c)-b}{(1-\beta)c} & \text{otw.} \end{cases}$$

But we don't know if a stationary state exists.

Stationary situations (ctd.)

Example with corner solution:

$$a = 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{95}{100}$$

gives a candidate optimal strationary solution:

$$G^\infty = \frac{9}{2}, \qquad n^* = 1.$$

Satisfies all first-order conditions.

Stationary situations (ctd.)

Example with interior solution:

$$a = 1, \quad b = 1, \quad c = \frac{1}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = 0.919$$

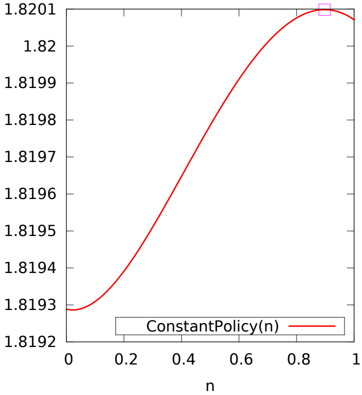
gives a candidate optimal strationary solution:

$$G^\infty \simeq 4.35, \qquad n^* = \frac{218}{243}.$$

Satisfies all first-order conditions.

Stationary situations (ctd.)

The constant-policy value (starting from the expected steady state) is maximum at n^*:



Non-stationary situations

However, convexity plays us tricks.
Example:

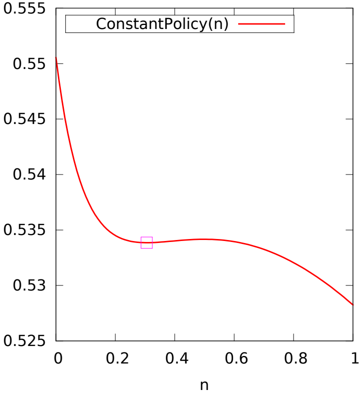
$$a = 1, \quad b = 1, \quad c = \frac{6}{10}, \quad z = \frac{9}{10}, \quad R = 1, \quad \beta_L = \frac{68}{100}$$

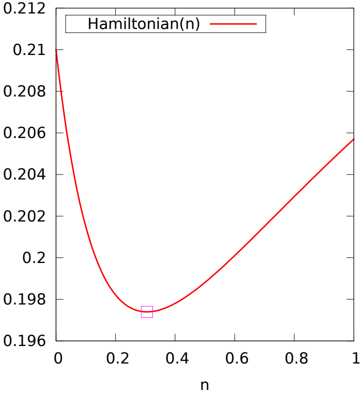
gives a candidate optimal strationary solution:

$$G^\infty = \frac{1}{8}, \qquad n^* = \frac{11}{36}.$$

But...
the optimal constant policy is not there!
And the Hamiltonian has actually a *local minimum* there wrt n.

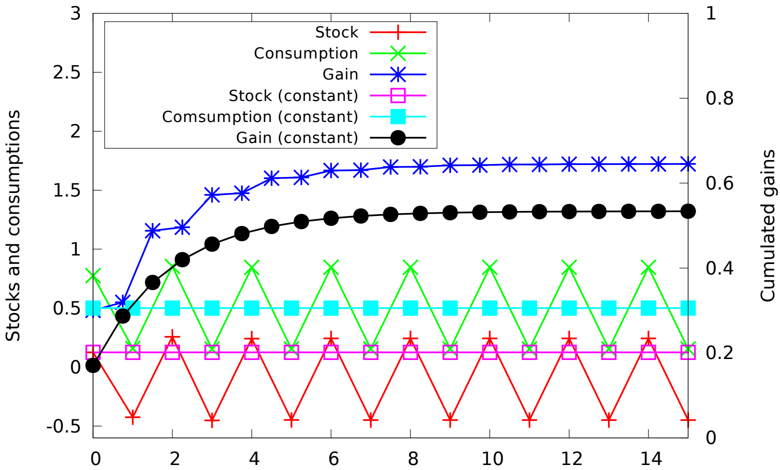
Non-stationary situations (ctd.)





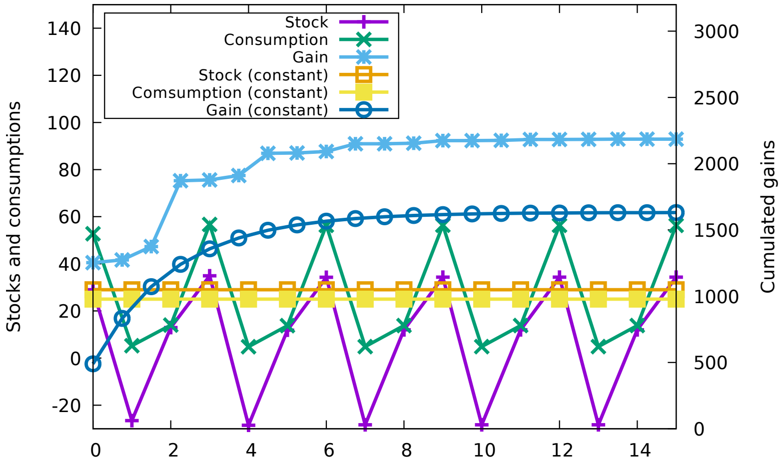
Hamiltonian is for constant trajectory from the expected steady state.

Non-stationary situations (ctd.)



Non-stationary situations (ctd.)

Same values except $R = 50$, $\beta = 7/10$:



A heuristic policy

These periodic policies were identified by using the following heuristic:

$$\begin{aligned} m(G) &= 1 \\ n(G) &= \text{argmax}\{V_{n1}(G) \mid 0 \leq n \leq 1\}. \end{aligned}$$

i.e. use the best static policy... but as state feedback.

Stackelberg with non-myopic followers

Followers being non-myopic, at time t they react to the sequence of "announced" regulations:

$$\{(n_t, m_t), (n_{t+1}, m_{t+1}), \ldots\}$$

while playing Nash!

$\Longrightarrow$ very complicated control law...

$\Longrightarrow$ not time-consistent?

Stackelberg with non-myopic followers (ctd)

A reasonable formulation: the supervisor announces feedback laws

$$G \mapsto n(G), \qquad G \mapsto m(G).$$

Followers play Nash Feedback with criterion:

$$\max_{\{u_t^i\}} \sum_0^\infty \beta_F^t [F_i(u_t^i) - C_i(G_t + m(G_t)R - n(G_t)\sum_k u_t^k)\ u_t^i],$$

and dynamics

$$G_{t+1} = G_t + R - \sum_k u_t^k, \quad G_0, \quad \text{given.}$$

Again an optimal control problem... to be studied.

Conclusions and extensions

Conclusions:

- regulator charges users in function of their behavior, not just in function of the level of resource
- introduce strategic interaction where there was none, in case of myopic agents ($n \neq 0$)
- difficult optimal control problem

Extensions under investigation

- simulations for the optimal control problem of the regulator
  $\rightarrow$ numerical methods, value iteration, policy iteration...
- non-myopic followers in the case of constant regulator policies
- stochastic case
  $\rightarrow$ learning algorithms under test
  $\rightarrow$ importance of m