An Adaptive Sublinear Time Block Sparse Fourier Transform

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Given $x \in \mathbb{C}^N$, compute the Discrete Fourier Transform (DFT) of x:

$$\widehat{x}_i = \frac{1}{N} \sum_{j \in [N]} x_j \cdot \omega^{-ij},$$

where $\omega = e^{2\pi i/N}$ is the N-th root of unity.

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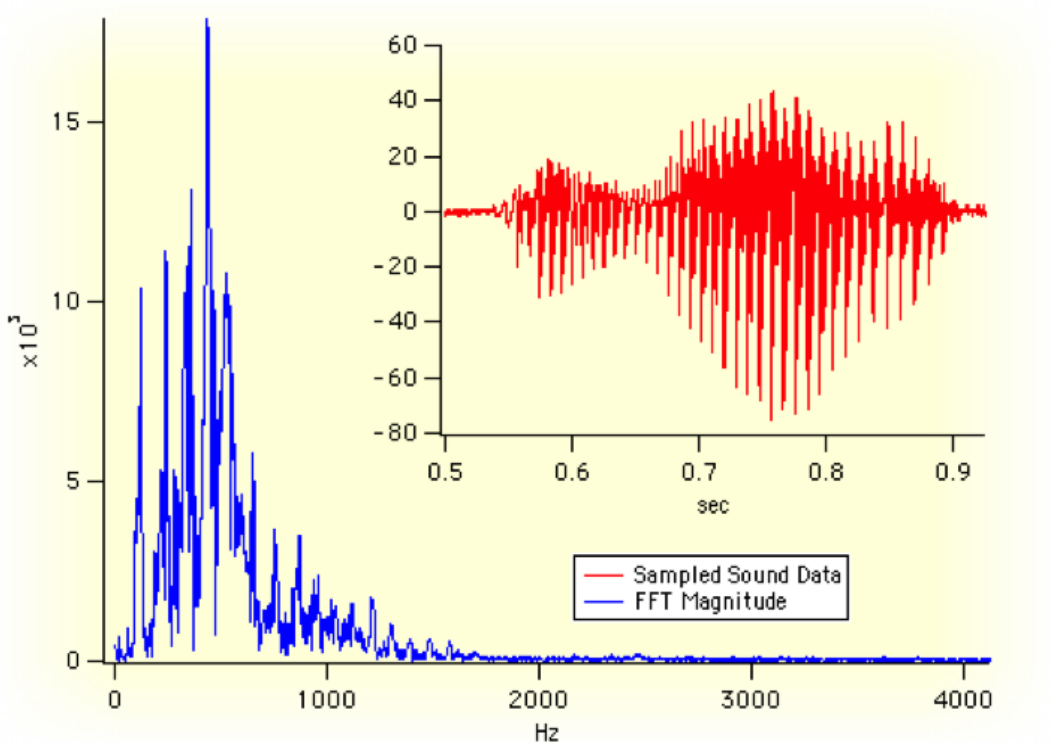

compression schemes (JPEG, MPEG)
signal processing
data analysis
imaging (MRI, NMR)

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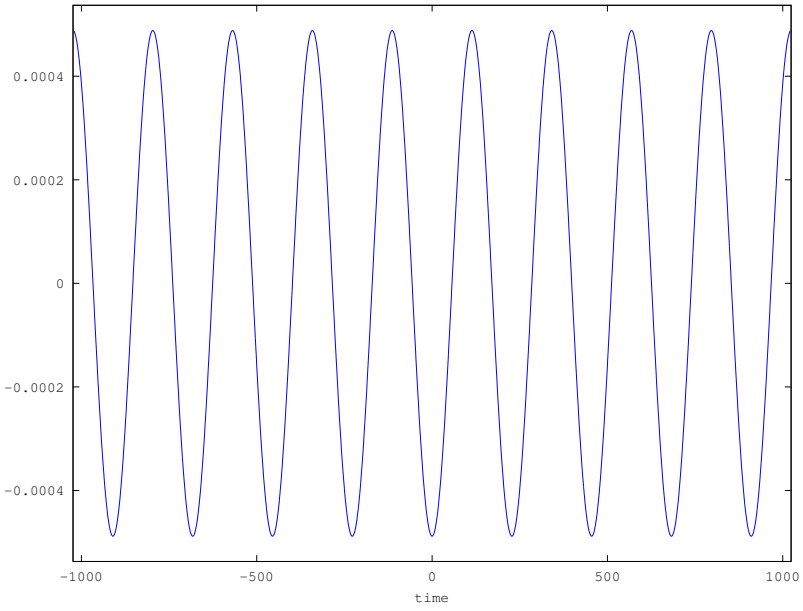
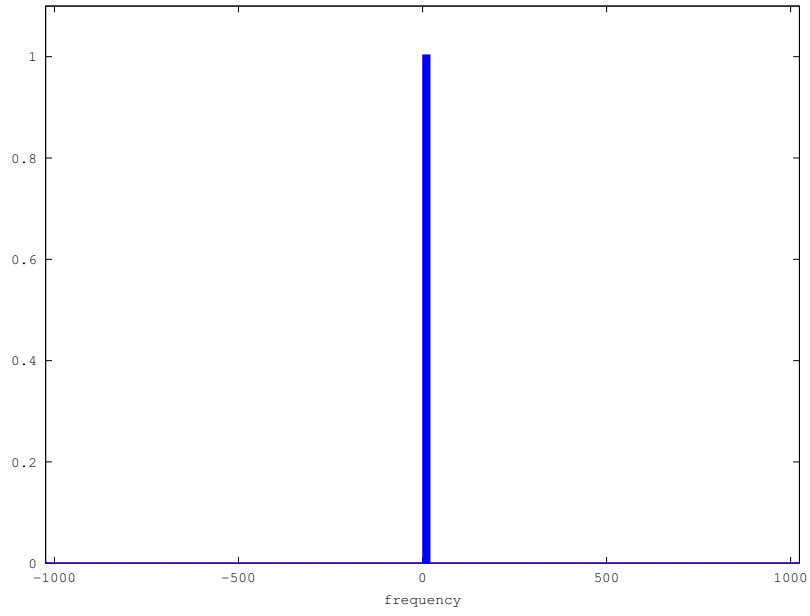
Sparse FFT

Say that $\hat{x}$ is *k*-sparse if $\hat{x}$ has k nonzero entries

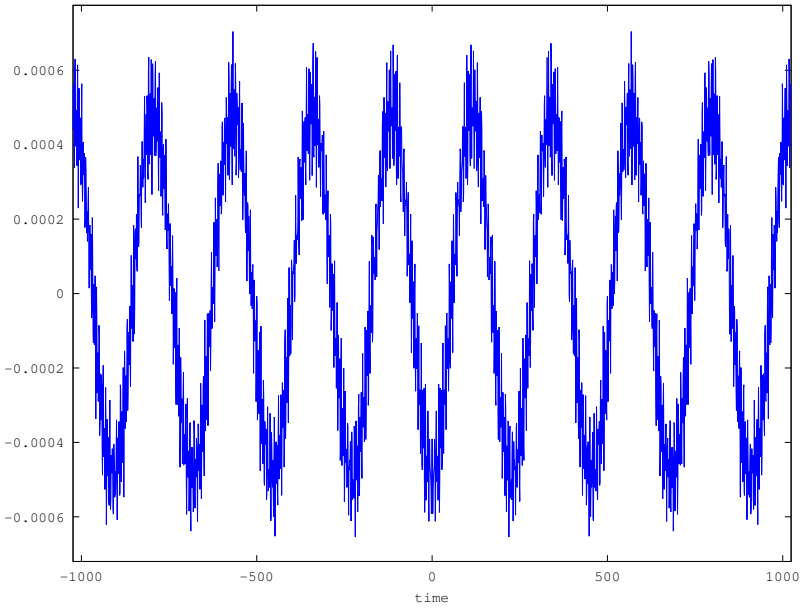


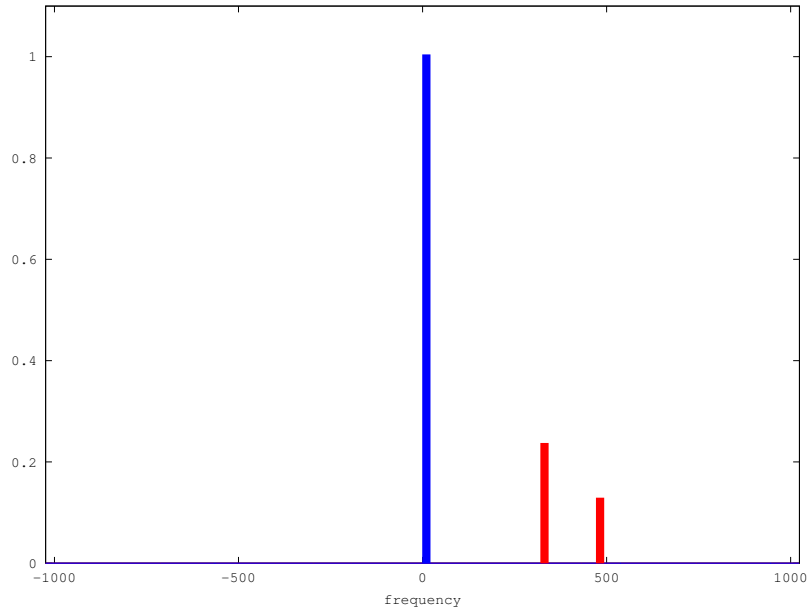


Sparse FFT

Say that $\hat{x}$ is *k*-sparse if $\hat{x}$ has *k* nonzero entries

Say that $\hat{x}$ is approximately *k*-sparse if $\hat{x}$ is close to *k*-sparse in some norm





Sparse approximations

JPEG
$\Longrightarrow$

Image and video compression schemes
(e.g. JPEG, MPEG)

Compute top k coefficients of $\hat{x}$ faster than FFT?

Sample complexity

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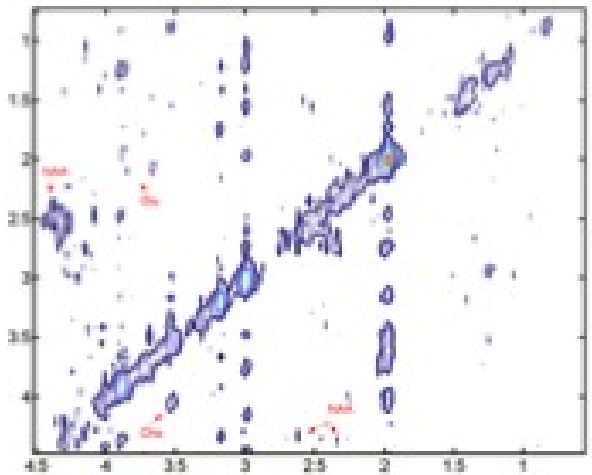
In medical imaging (MRI, NMR), one measures Fourier coefficients $\hat{x}$ of imaged object x (which is often sparse)



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Minimize runtime and sample complexity

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Formally, want to find $\widehat{y}$ such that

$$||\widehat{x} - \widehat{y}||^2 \leq (1+\varepsilon) \cdot \min_{k-\text{sparse } \widehat{z}} ||\widehat{x} - \widehat{z}||^2$$

(ℓ_2/ℓ_2 sparse recovery guarantees)

Uniform bounds (for all):

Candes-Tao'06
Rudelson-Vershynin'08
Cheraghchi-Guruswami-Velingker'12
Bourgain'14
Haviv-Regev'15

Deterministic, $\Omega(N)$ runtime

$O(k \log^2 k \log N)$

Non-uniform bounds (for each):

Goldreich-Levin'89
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Gilbert-Guha-Indyk-Muthukrishnan-Strauss'02
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Indyk-K.-Price'14 $(k \log n (\log\log n)^C$, but only in 1d)
Indyk-K.'14
K'16
K'??

Randomized, $O(k \cdot \mathrm{poly}(\log N))$ runtime

$O(k \log N)$

Lower bound: $k \log(N/k)$ for non-adaptive algorithms Do-Ba-Indyk-Price-Woodruff'10

(Also Boufounos-Cevher-Gilbert-Li-Strauss'12, Prince-Song'14 on continuous Sparse FFT)

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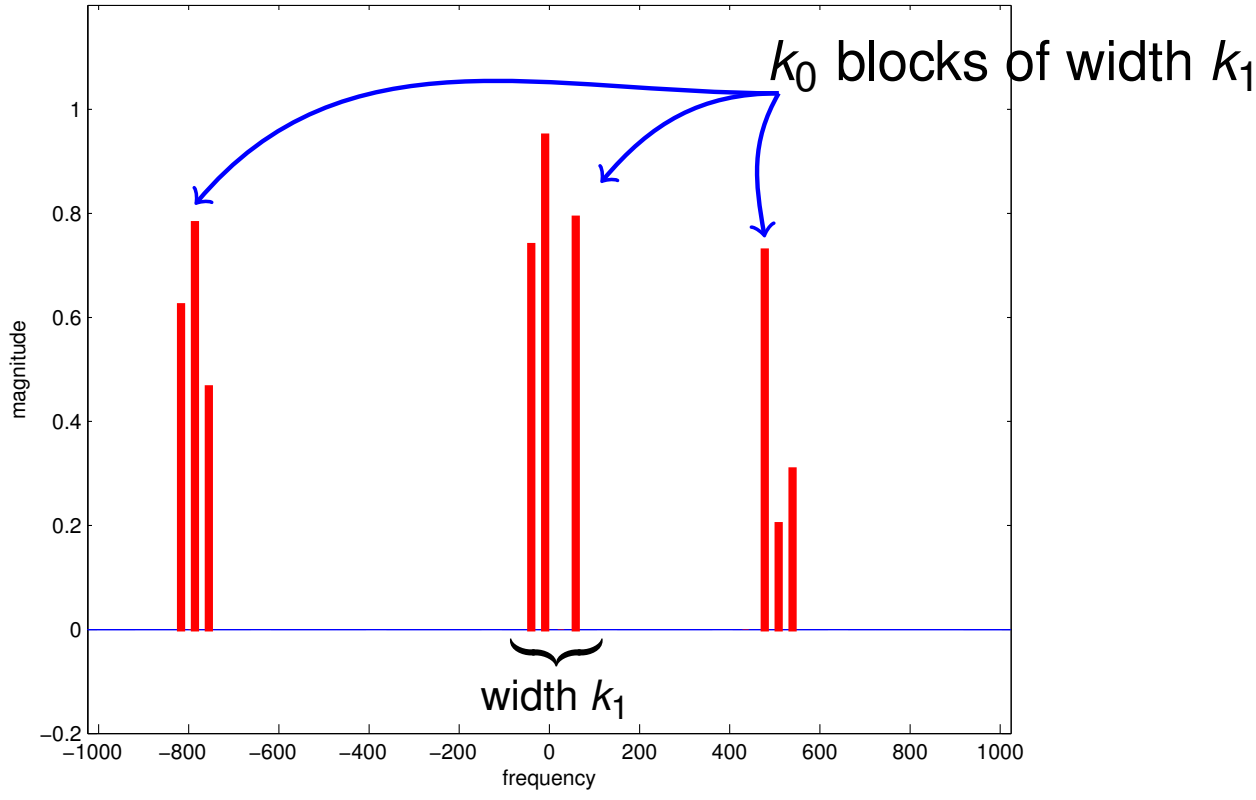
Nearly optimal results under sparsity assumption alone – can we exploit structure beyond sparsity?

Sparse FFT beyond the sparsity assumption

Signals arising in applications often have structure beyond sparsity

Can we exploit further structure to reduce sample complexity and runtime?

Block sparsity in Fourier domain



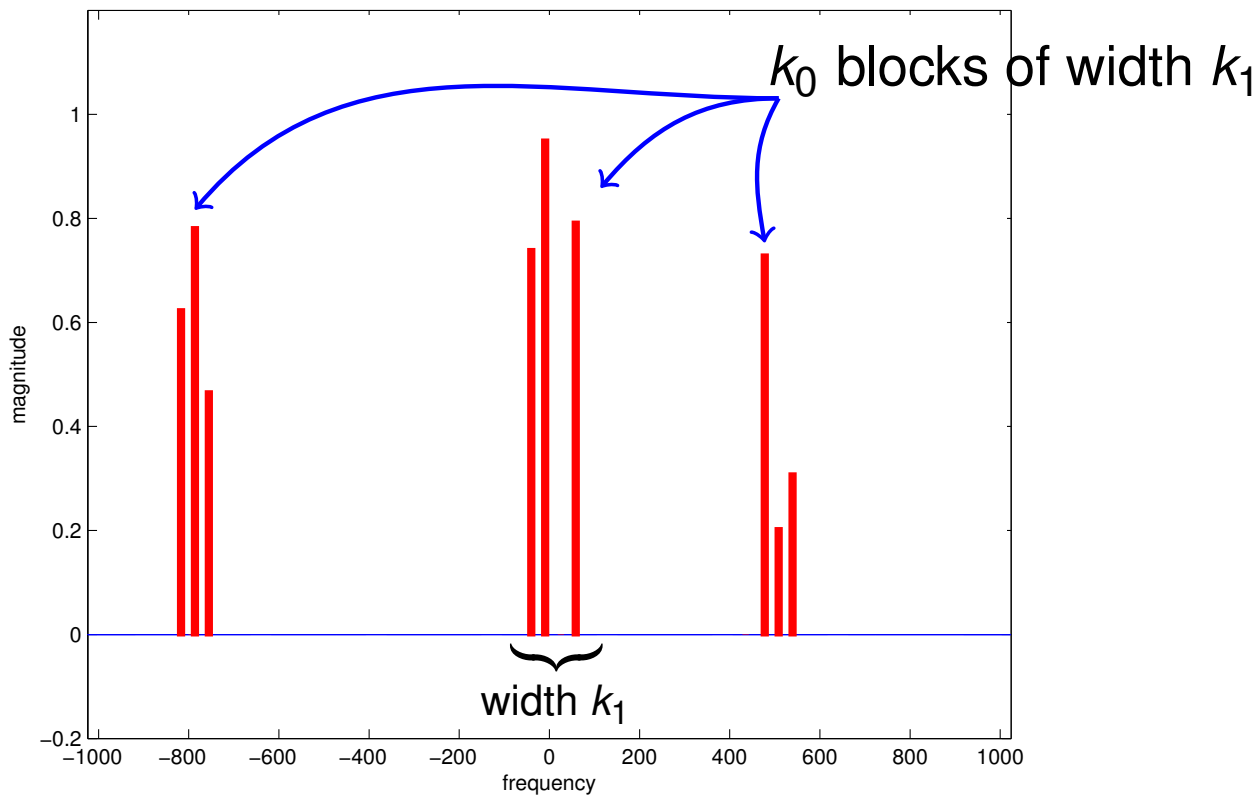
This work: suppose that dominant frequencies are contained in k_0 clusters (intervals) of length k_1 each?

$O(k_0 \log n + k_0 k_1)$ sample complexity in sublinear time?

Model based compressed sensing

Framework for exploiting structured sparsity to reduce sample complexity, introduced by Baraniuk, Cevher, Duarte and Hegde'10

Baraniuk et al'10: optimal $O(k_0 \log n + k_0 k_1)$ sample complexity using Gaussian measurements in $\widetilde{O}(k_0 k_1 n)$ time



Adaptivity in compressed sensing

An algorithm is adaptive its signal access pattern is guided by samples taken

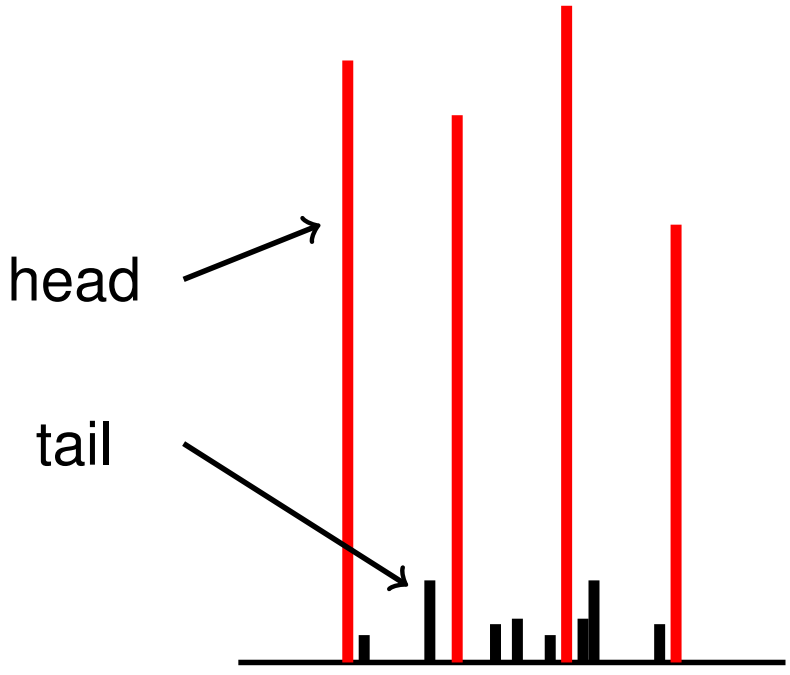
All Sparse FFT and compressed sensing results mentioned so far are non-adaptive

Main result

Theorem

There exists an adaptive (k_0, k_1)*-block sparse Fourier transform with sample complexity* $O^*(k_0 \log(1+k_0) \log n + k_0 k_1) \cdot \log SNR$ *samples and* $O^*(k_0 k_1 \log^3 n) \cdot \log SNR$ *runtime.*

Sample complexity optimal up to lower order terms for constant SNR



SNR=ratio of total signal energy to total noise energy

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No improvement upon vanilla sparsity possible with non-adaptive Fourier measurements

First result along the following directions:

- First sublinear time model based compressed sensing primitive
- Separation between adaptive vs non-adaptive Sparse FFT
- Structured (Fourier) vs unstructured (Gaussian) measurements in model based compressed sensing

1. Sparse recovery from Fourier measurements

2. Main result

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Summary of techniques from

Gilbert-Guha-Indyk-Muthukrishnan-Strauss'02, Akavia-Goldwasser-Safra'03,
Gilbert-Muthukrishnan-Strauss'05, Iwen'10, Akavia'10,
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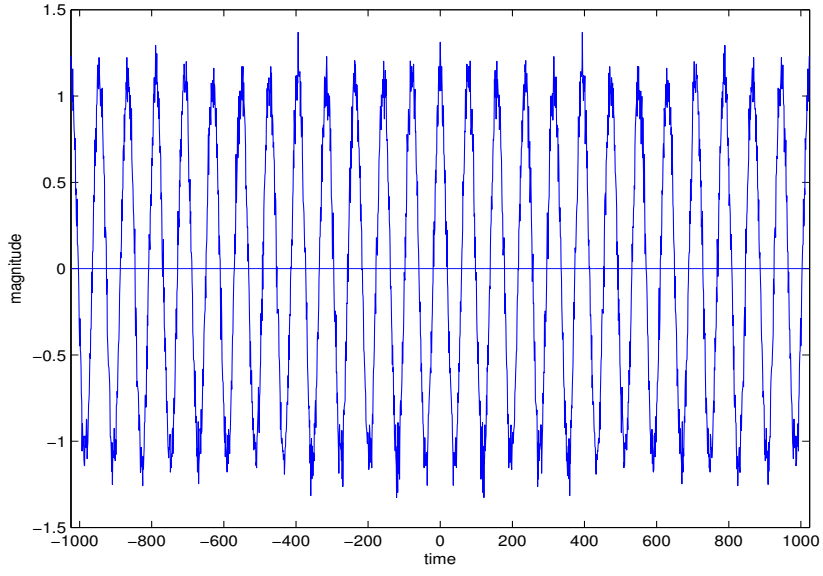
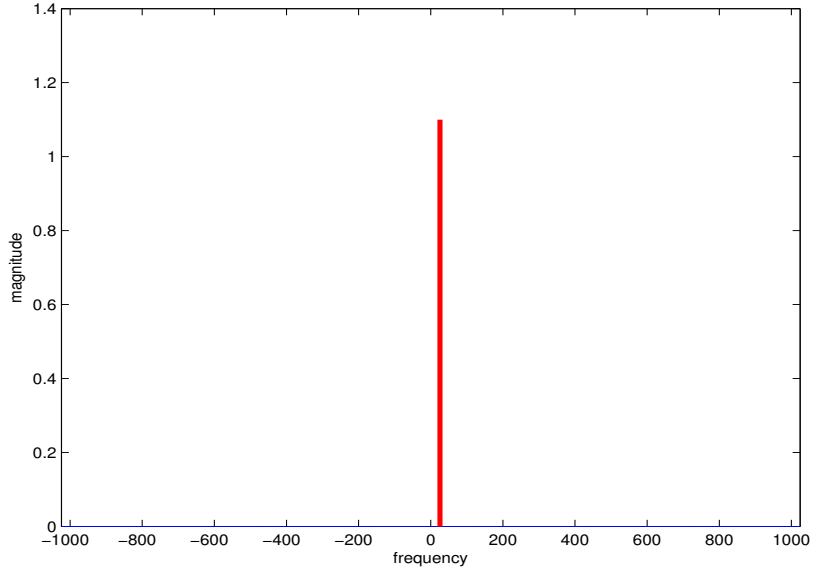
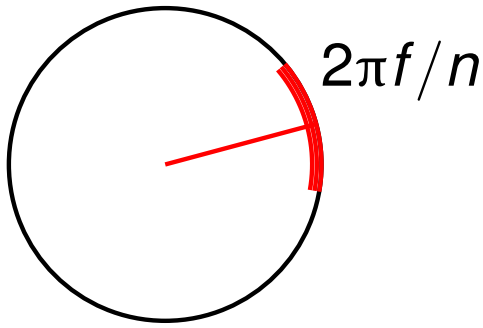
1-sparse recovery from Fourier measurements





$$x_a = \omega^{a \cdot f} + \text{noise}$$

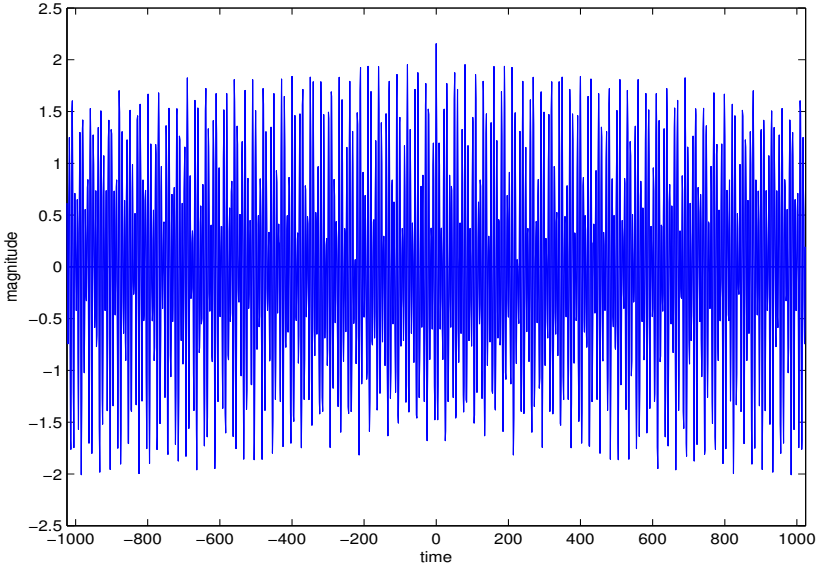


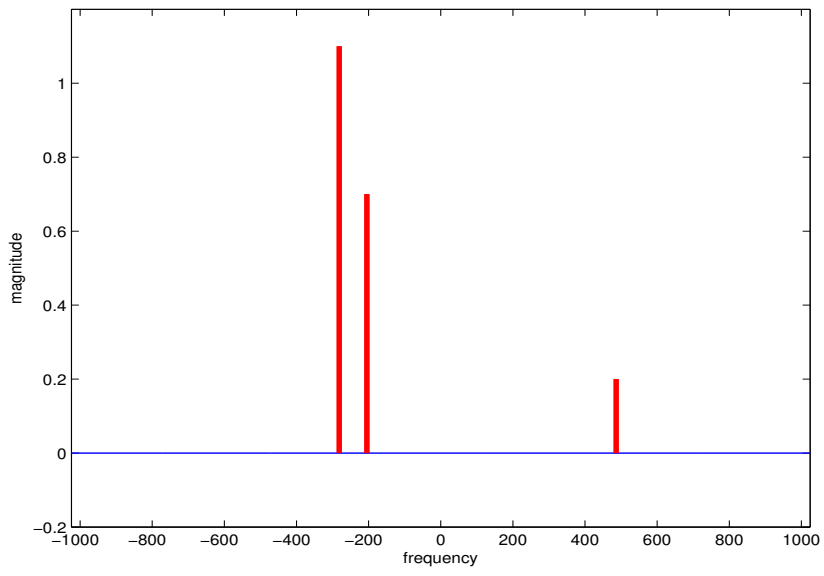
$O(\log n)$ measurements

for random a

Reducing k-sparse recovery to 1-sparse recovery

Permute with a random linear transformation and phase shift





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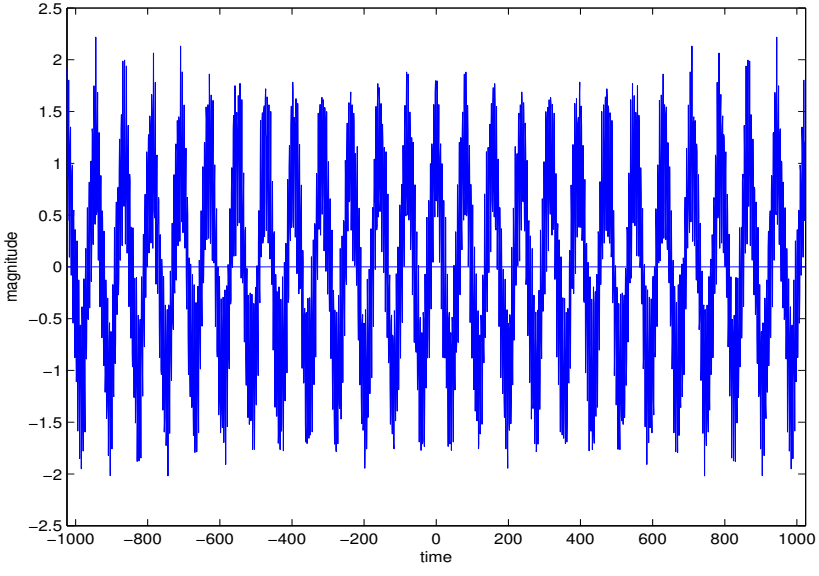

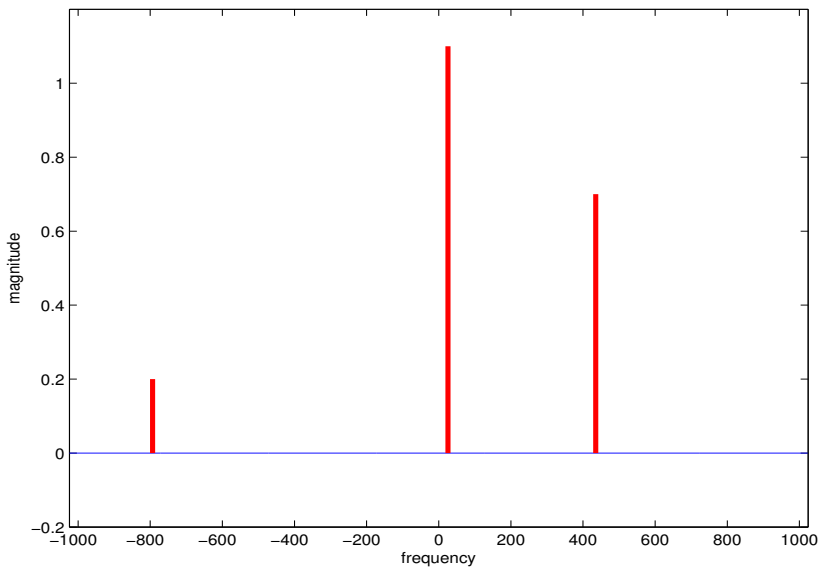

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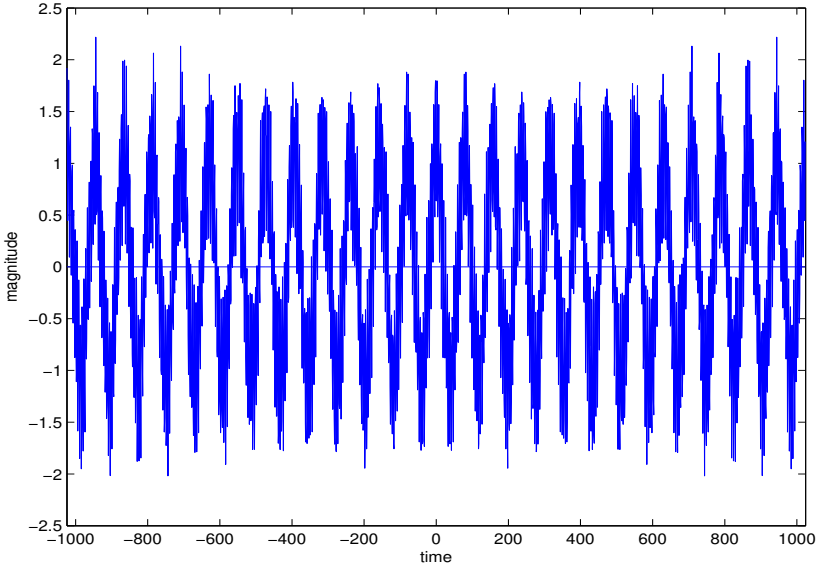

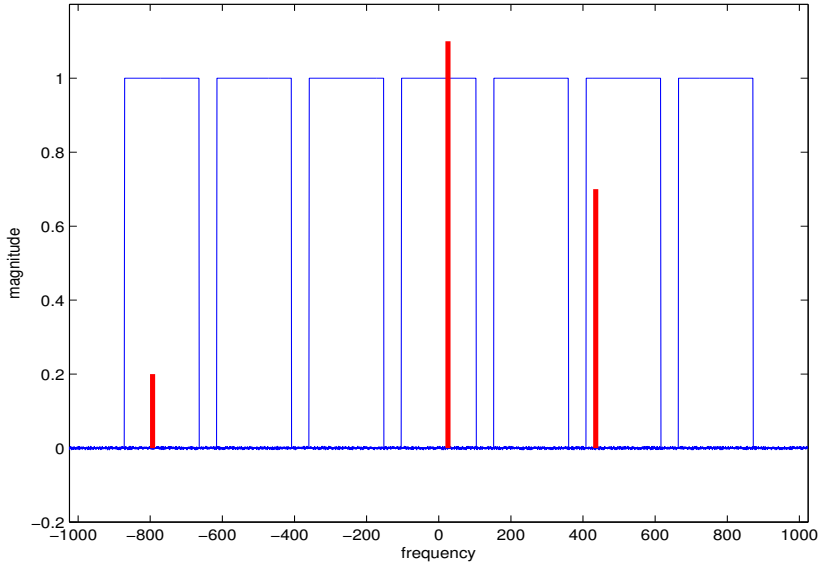

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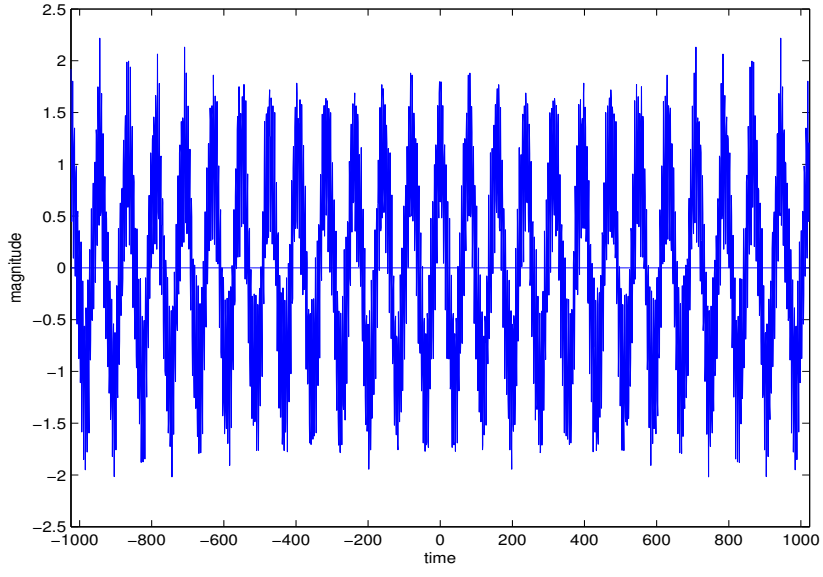

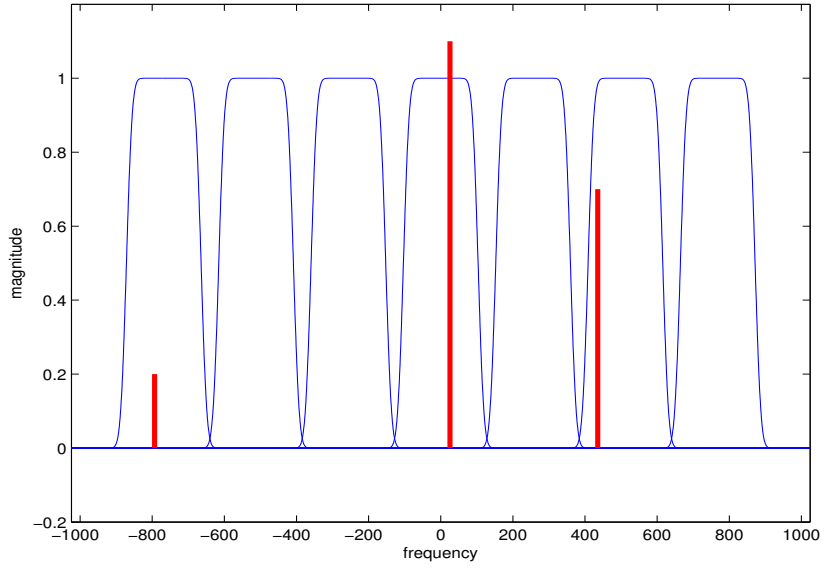

Choose a filter $G, \widehat{G}$ such that

- $\widehat{G} \approx$ rectangle
- G has support $\approx k$

Compute $\widehat{x} * \widehat{G} = \widehat{(x \cdot G)}$

Sample complexity=supp G!

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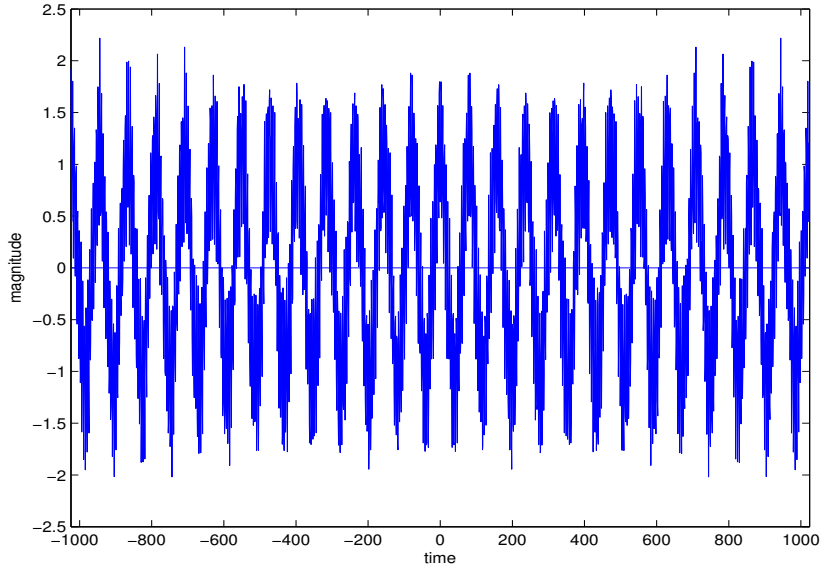

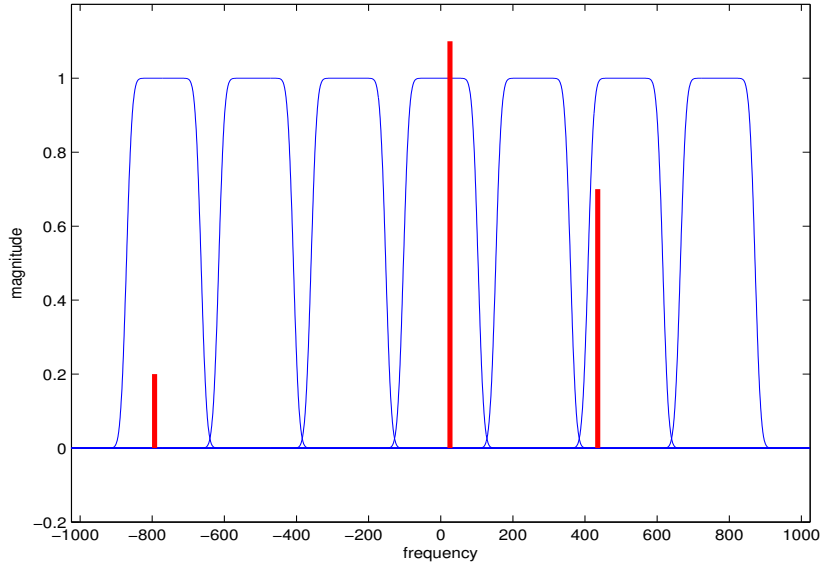

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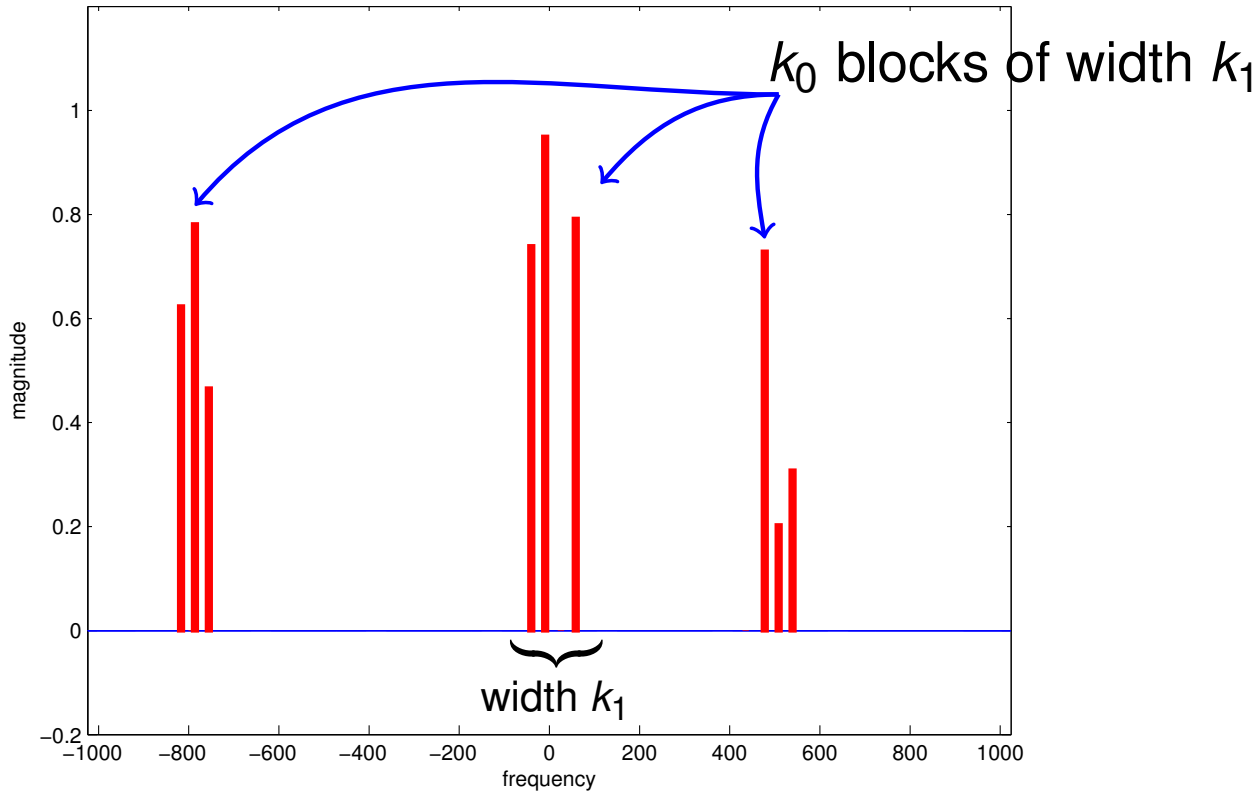
Block sparsity in Fourier domain



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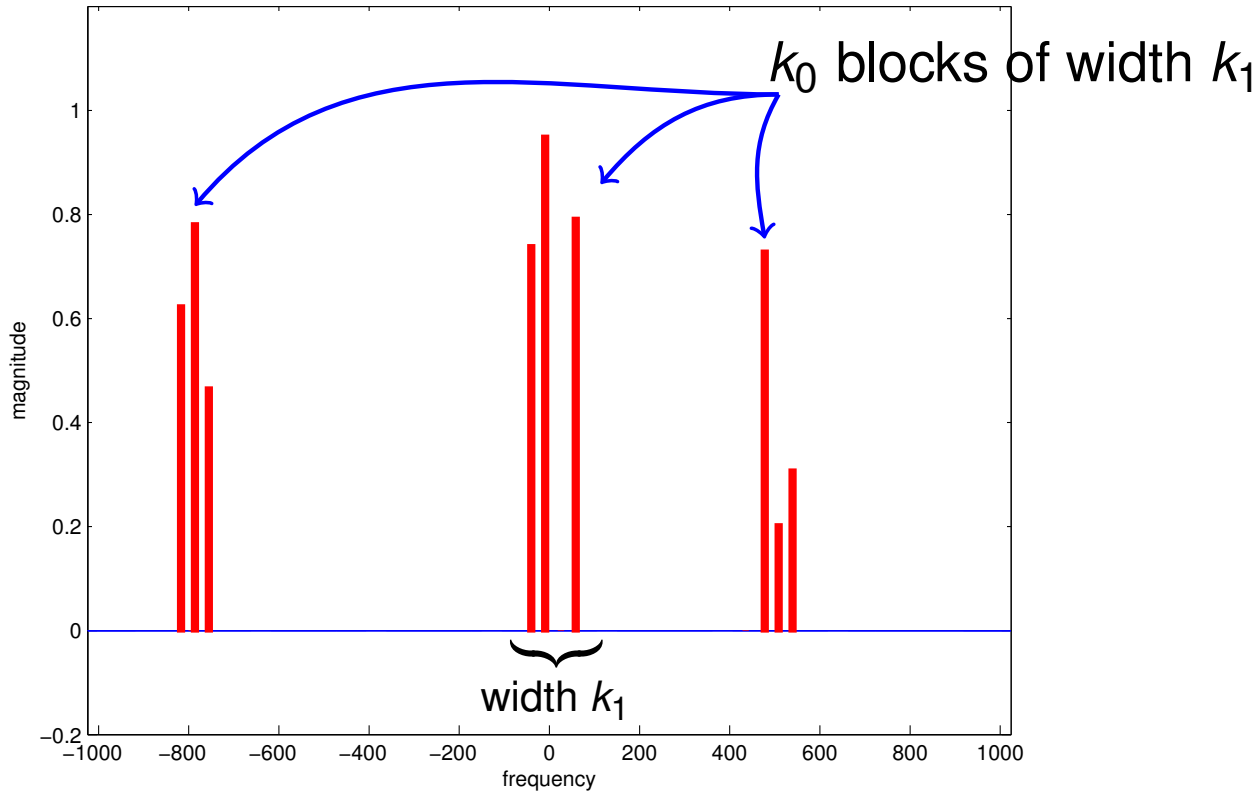

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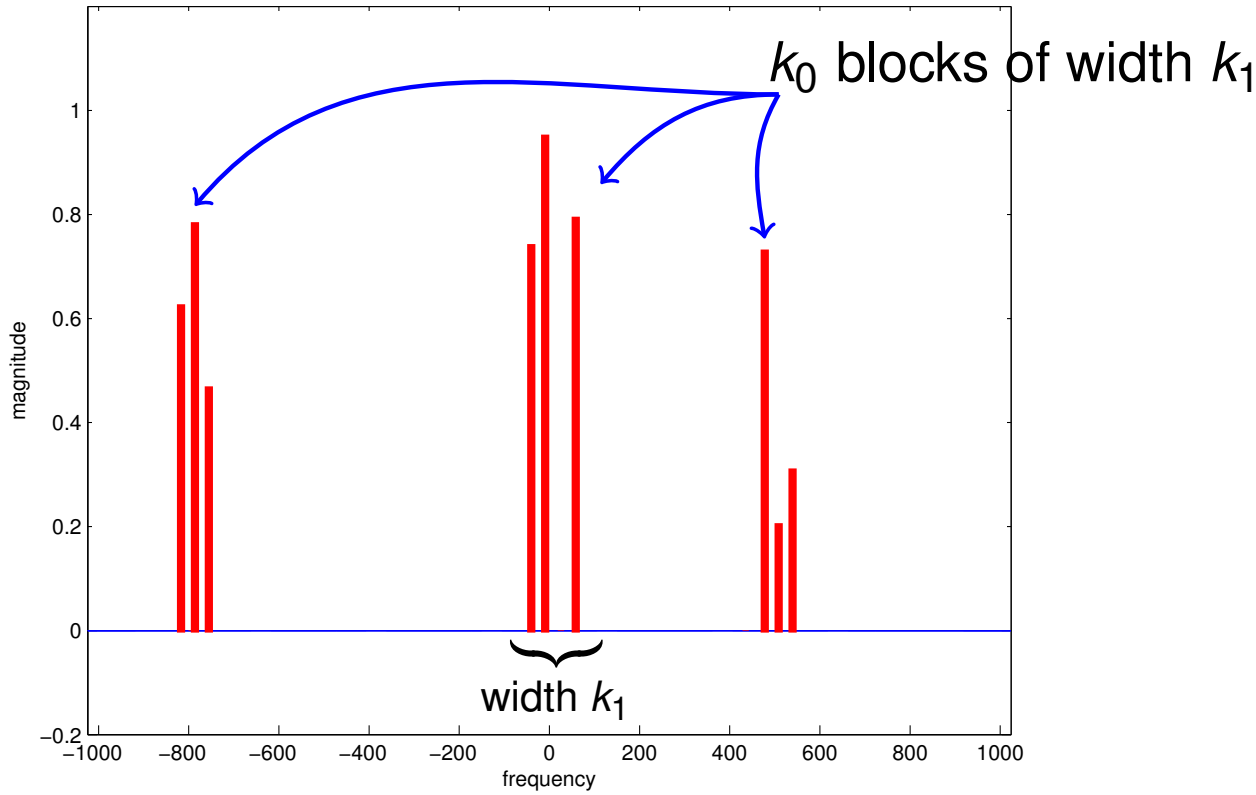

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$O(k_0 \log n + k_0 k_1)$ sample complexity in sublinear time?

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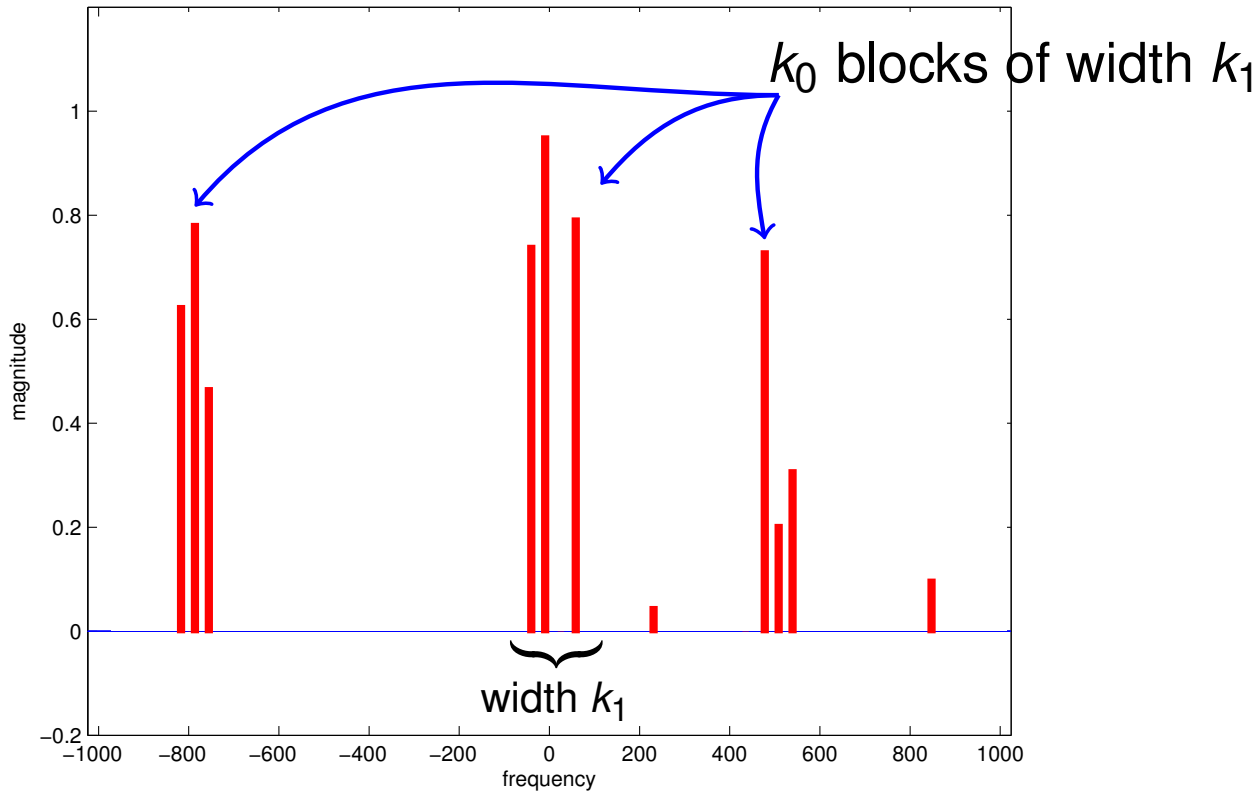
Spectrum is well approximated by k_0 blocks of length k_1:



$k_0 = 3$ in the example

Natural idea: can we treat blocks as individual frequencies, and thus reduce to standard k_0-sparse recovery?

YES, to some extent...

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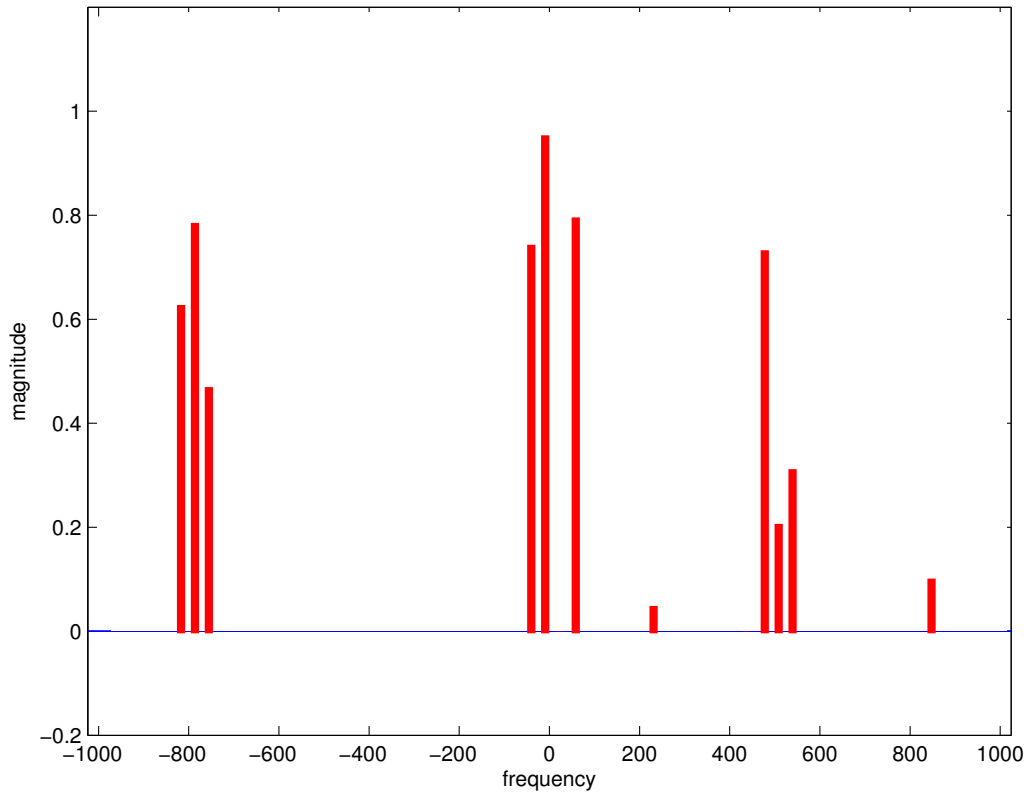

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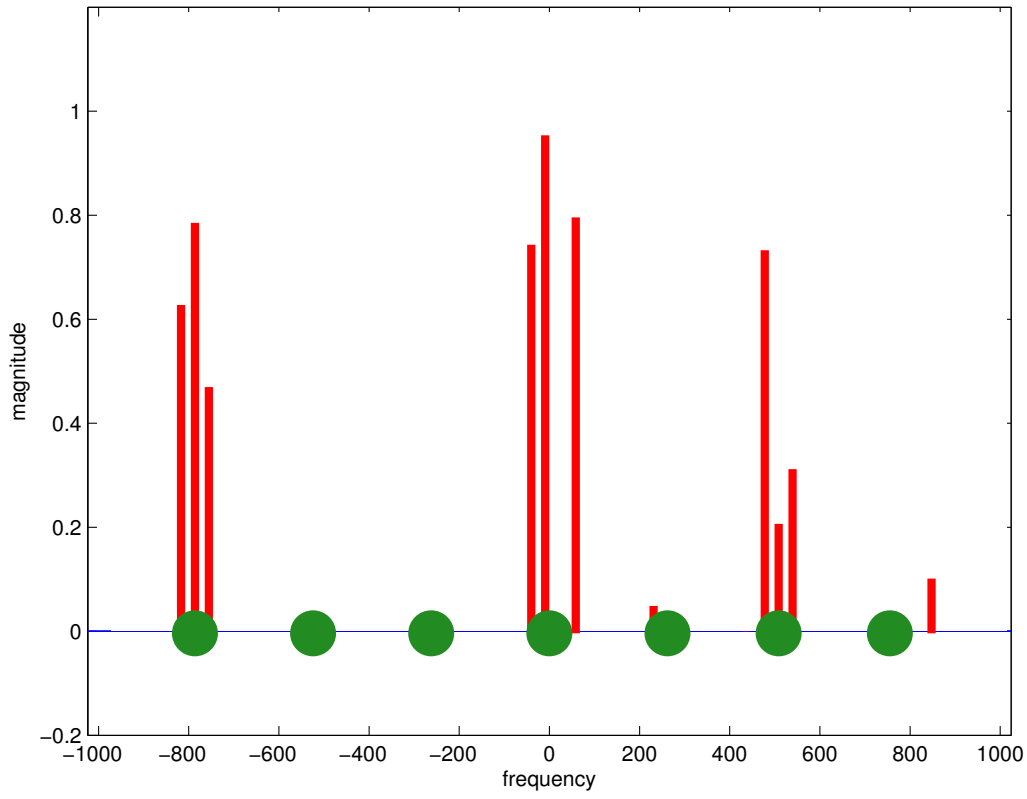

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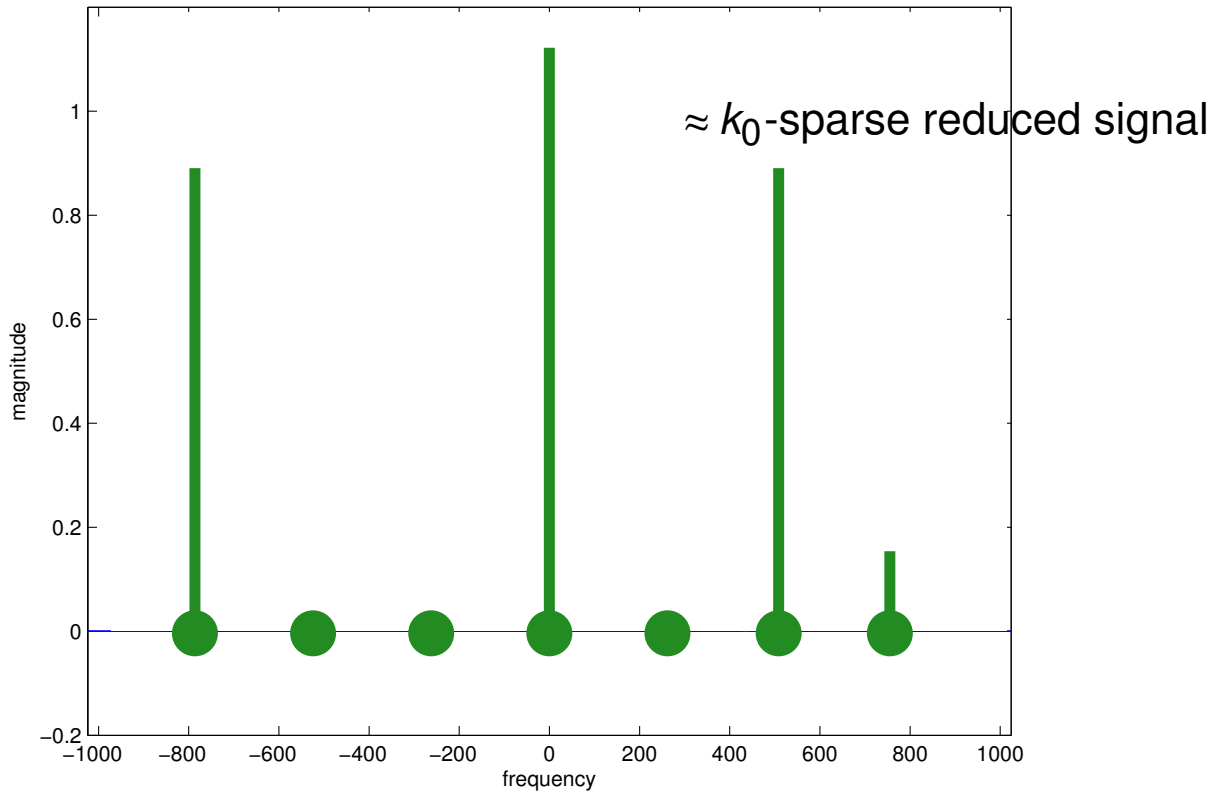

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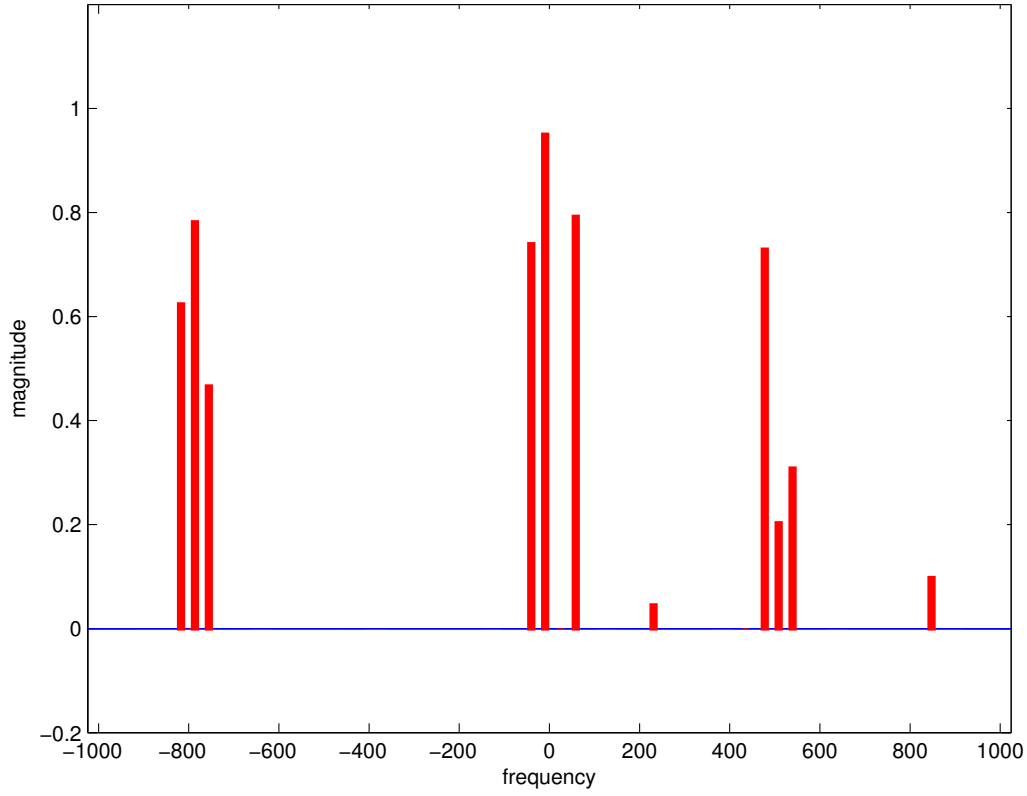

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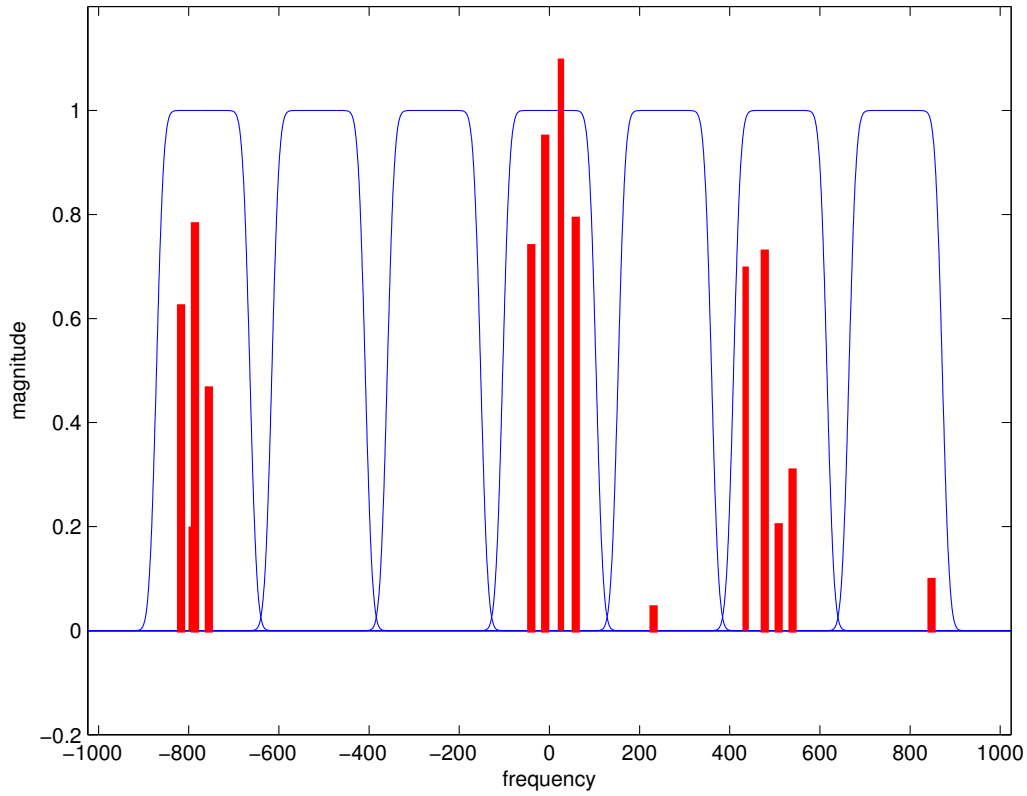

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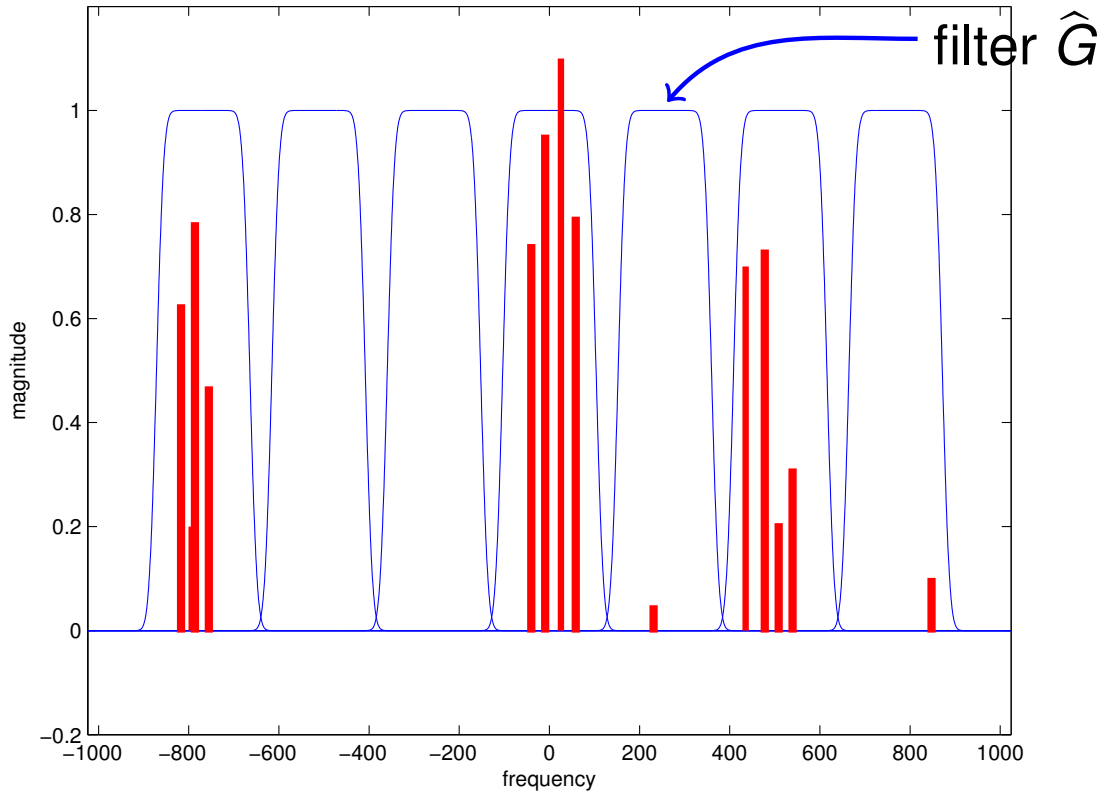

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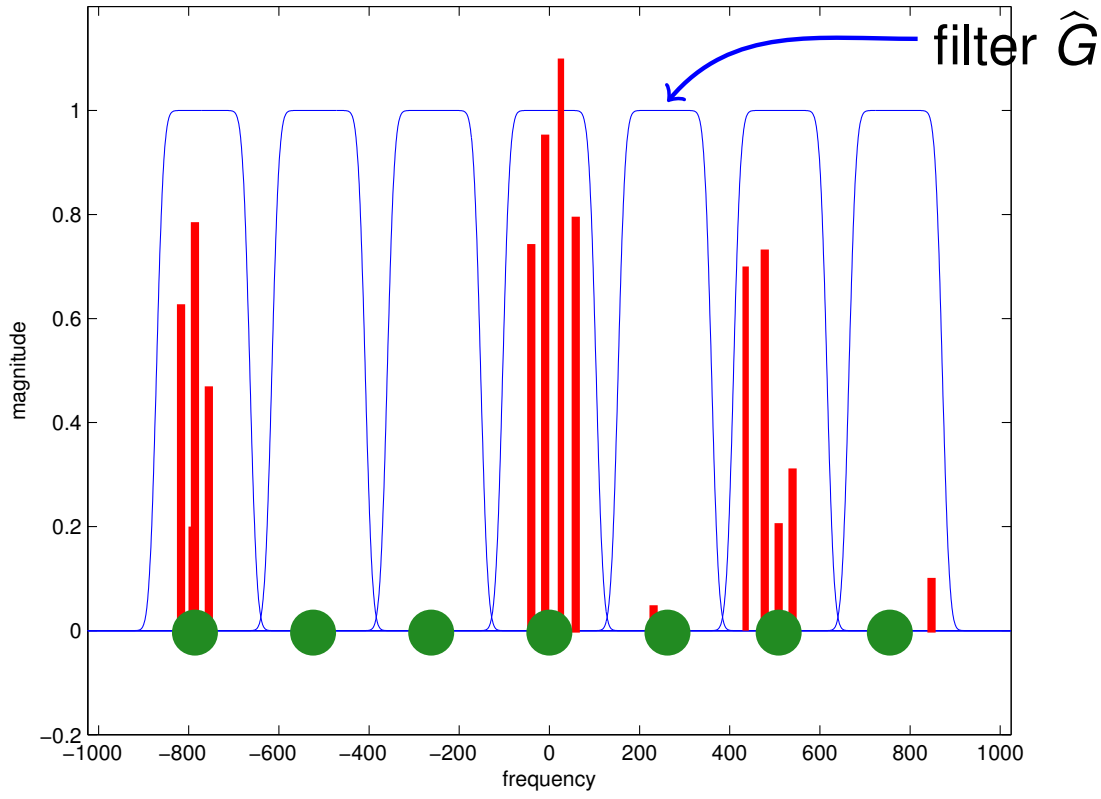

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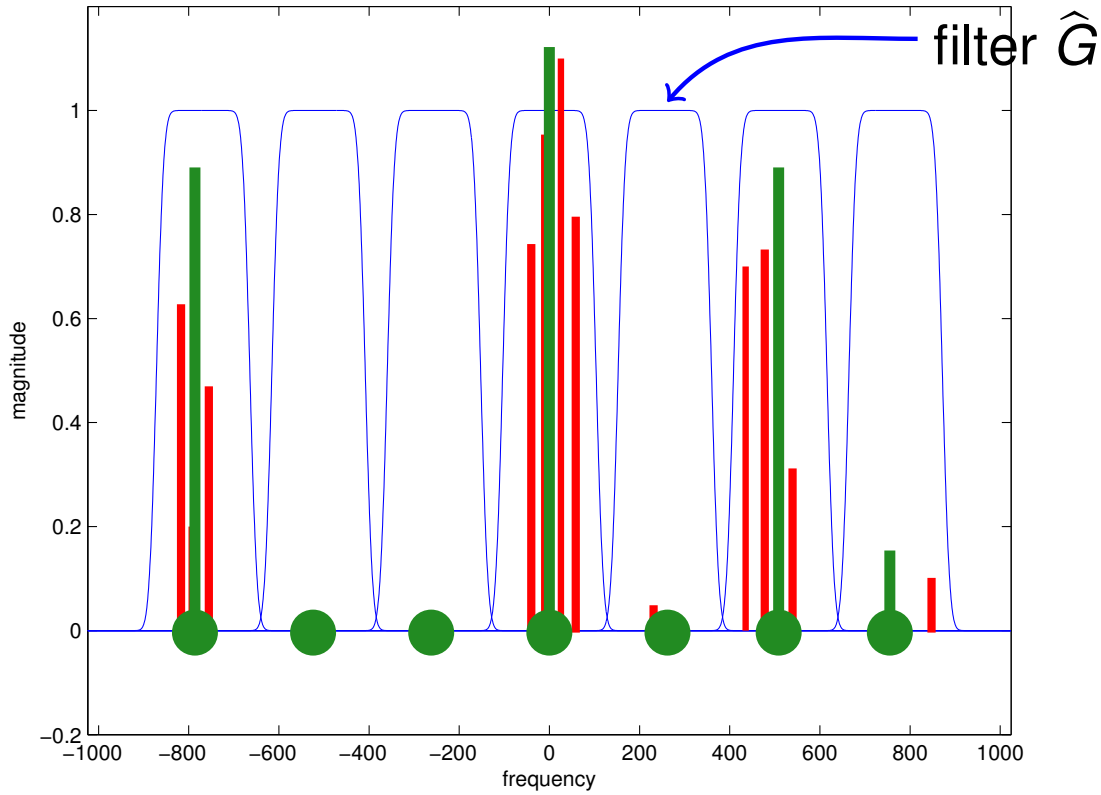

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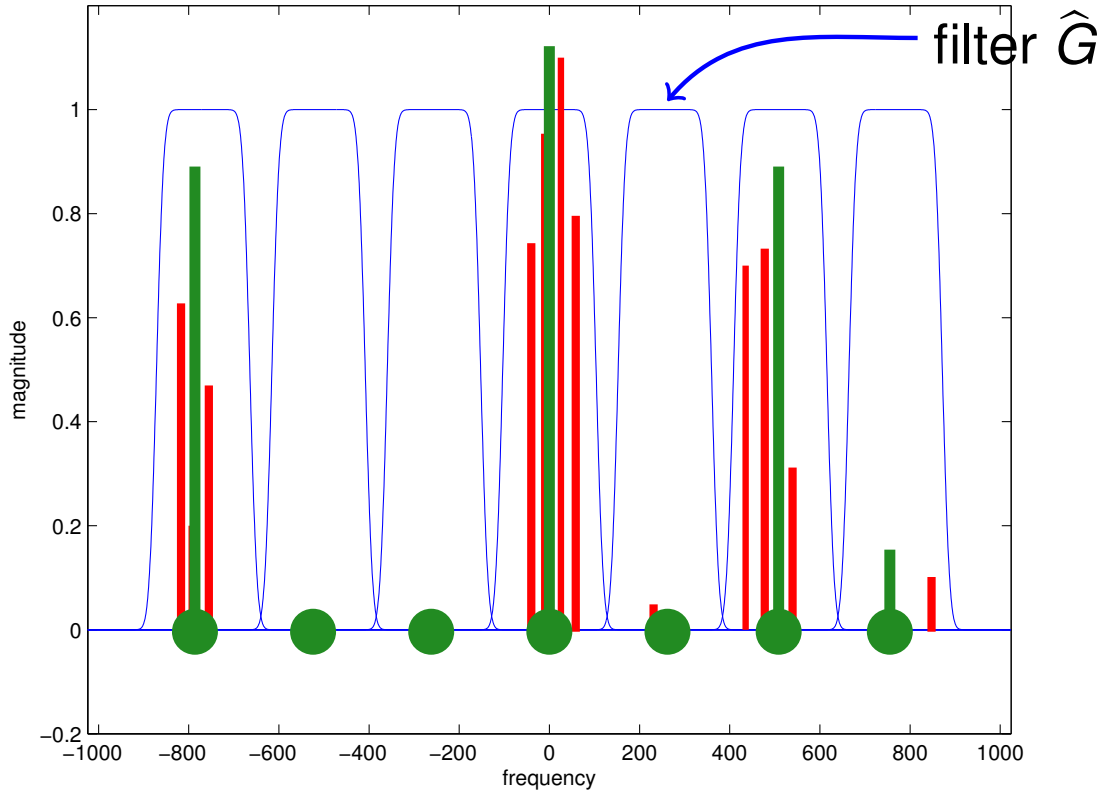
Good news: if X is approximately (k_0, k_1)-block sparse, then Z is approximately $O(k_0)$-sparse.

Reduced signals



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Major problem: some blocks of X might have essentially zero contribution to Z due to cancellations!

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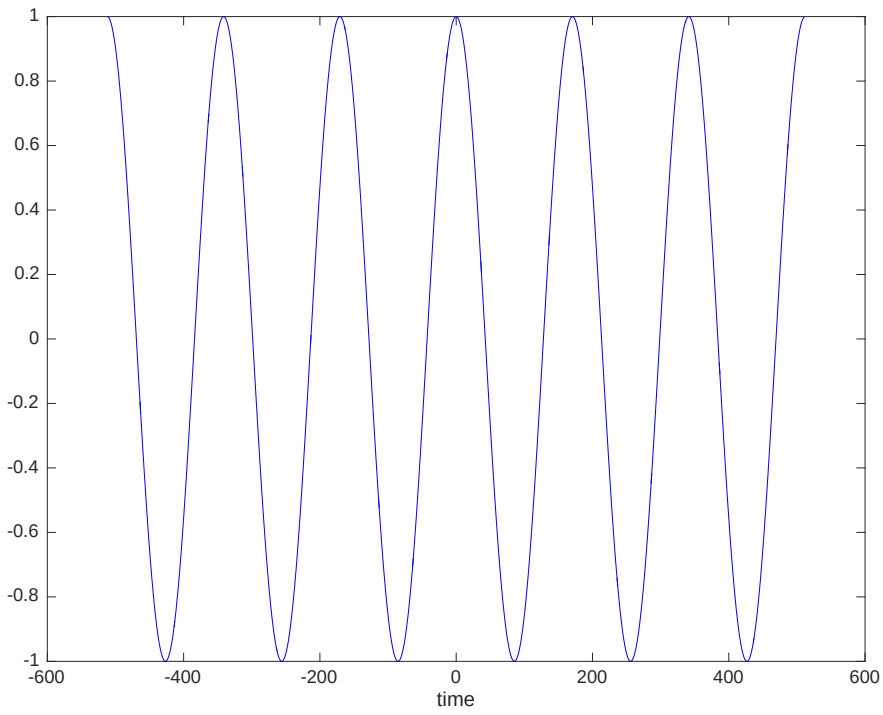

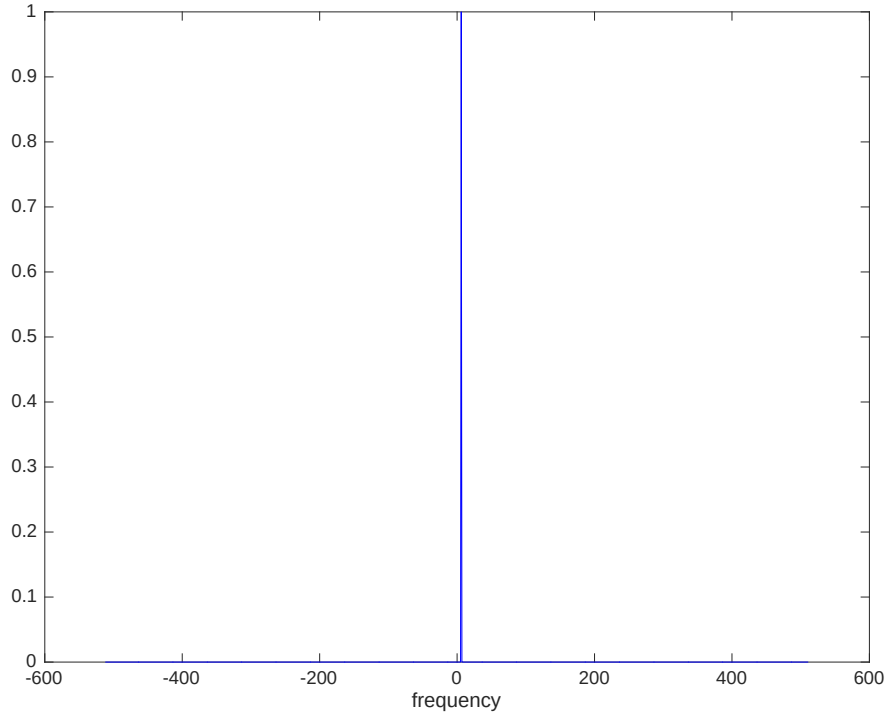

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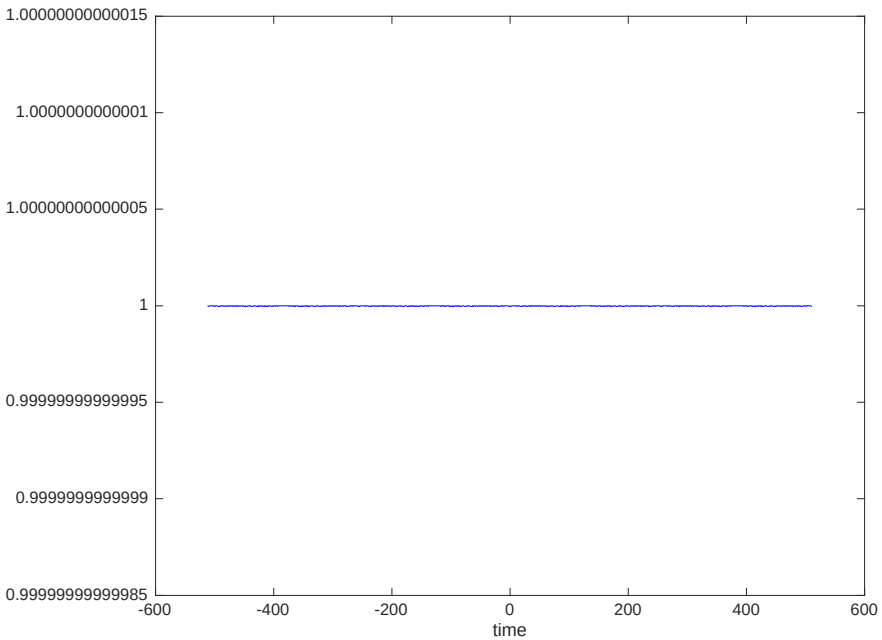

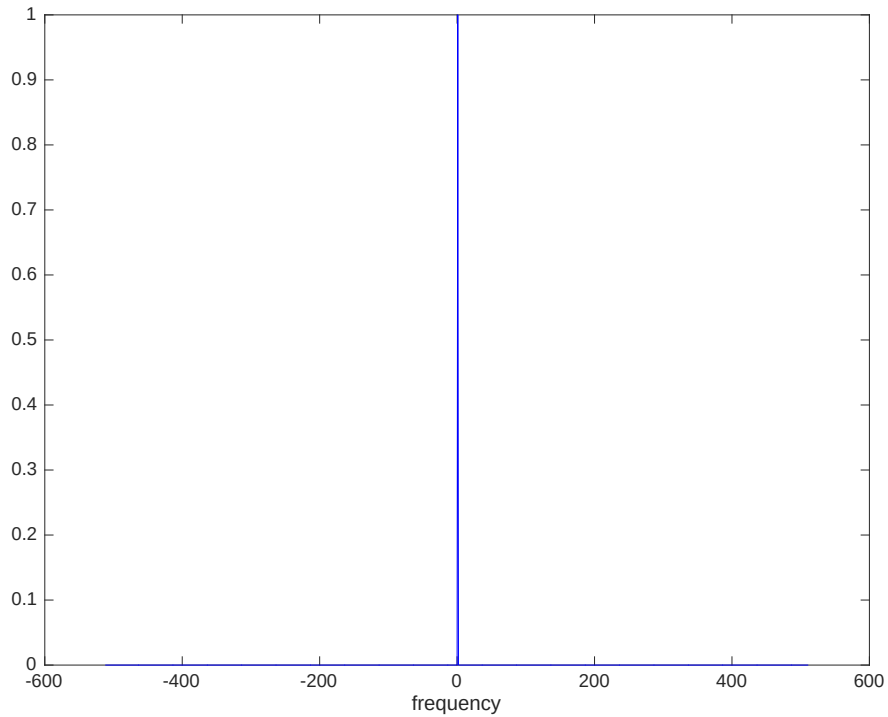

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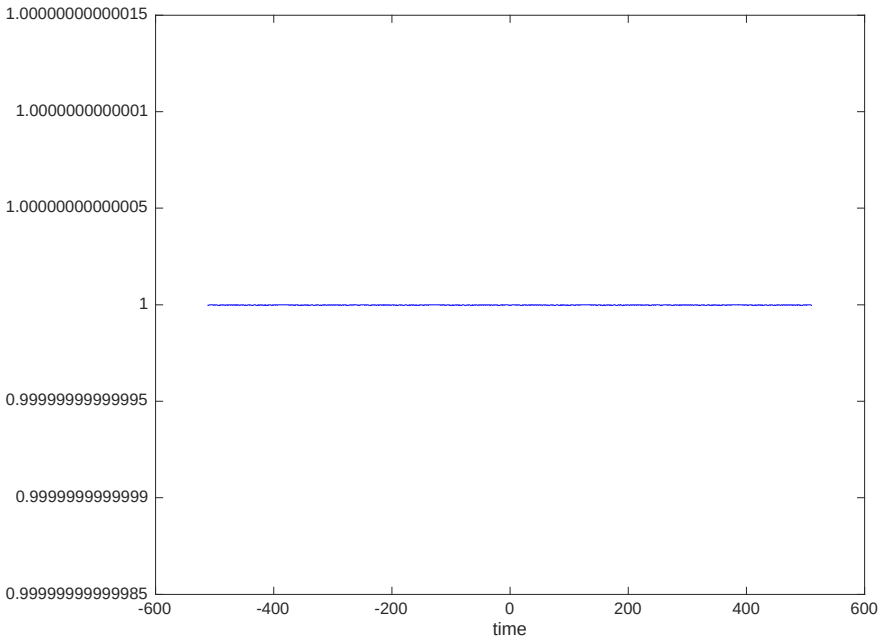

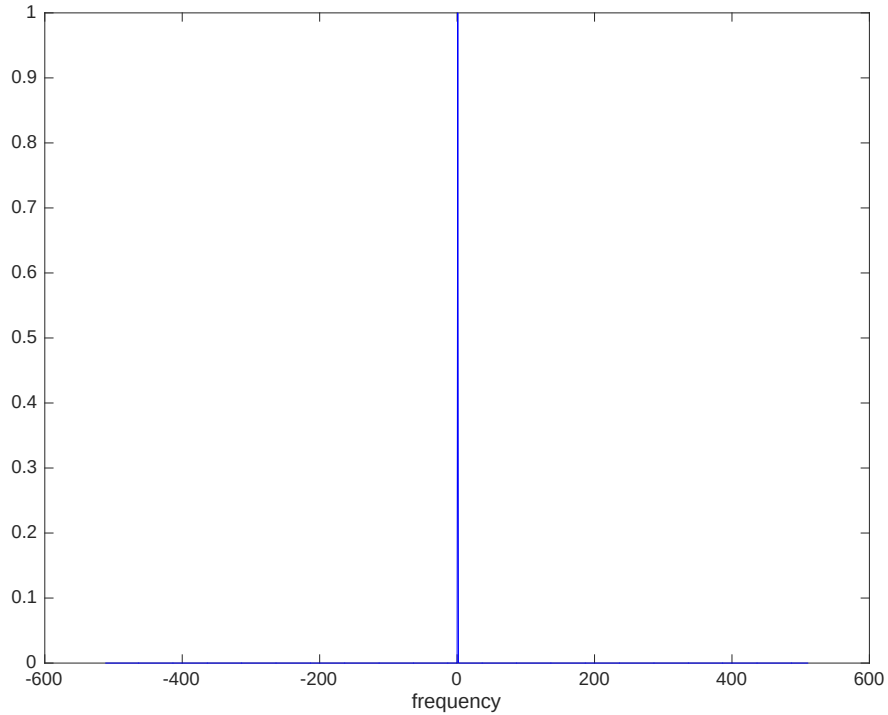

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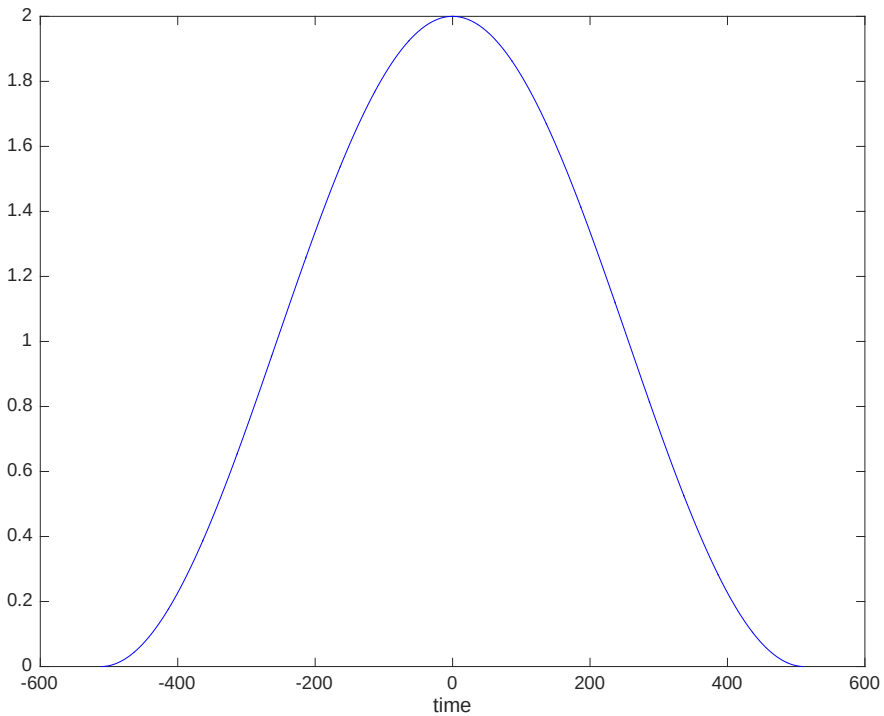

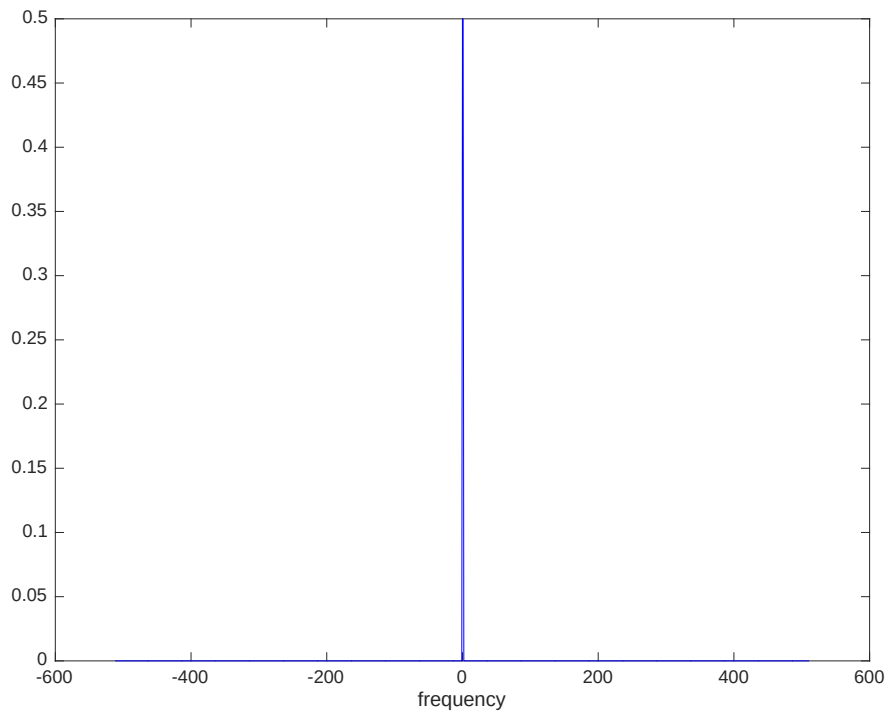

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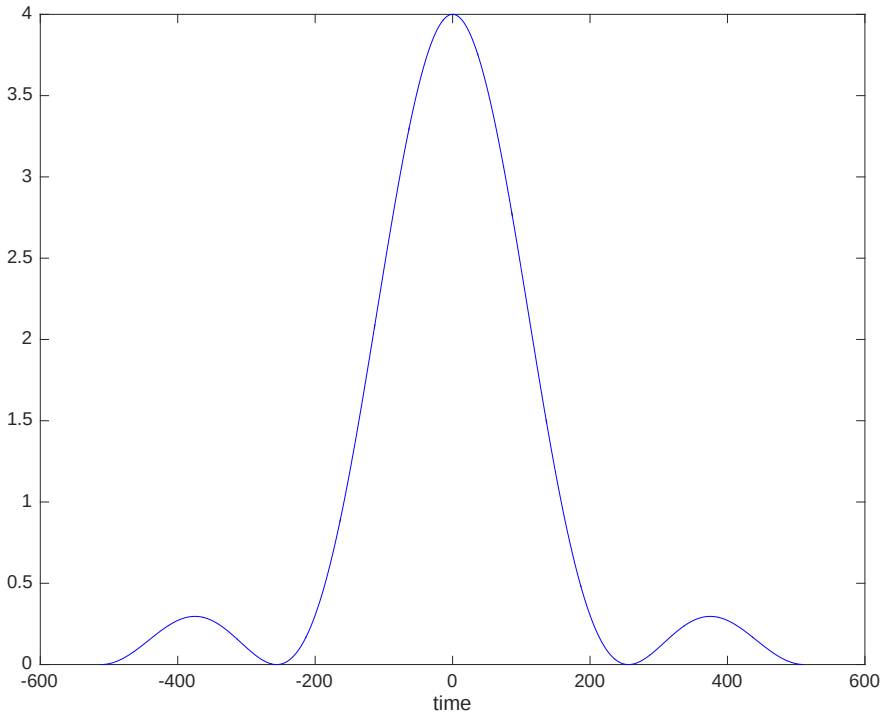

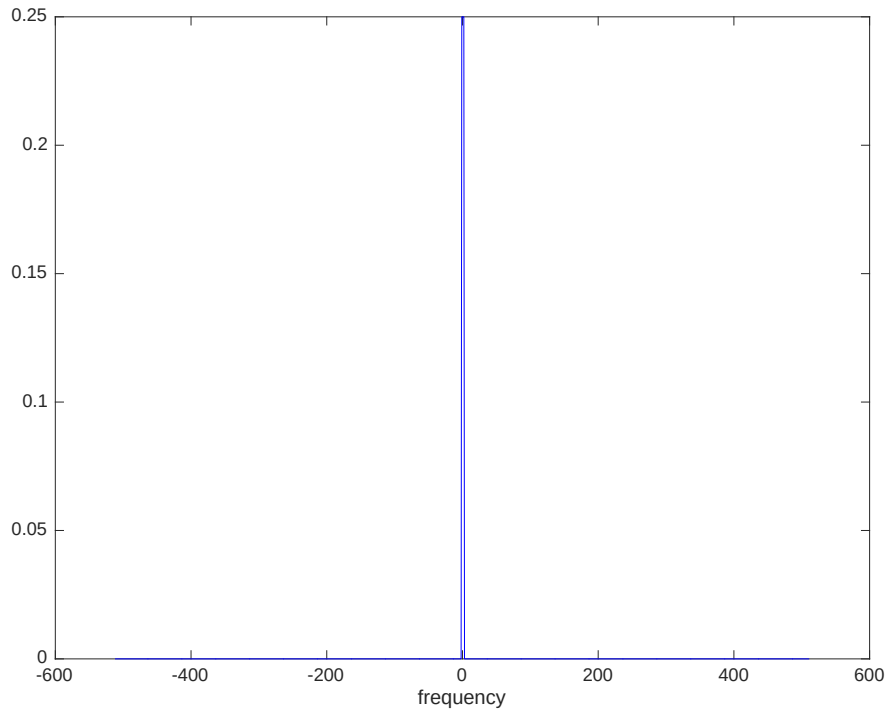

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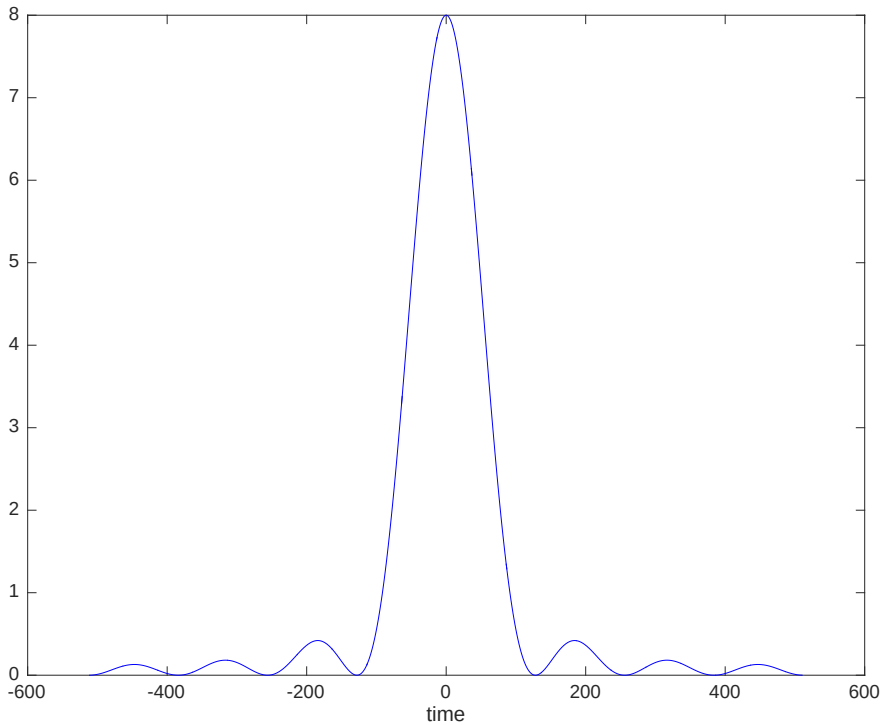

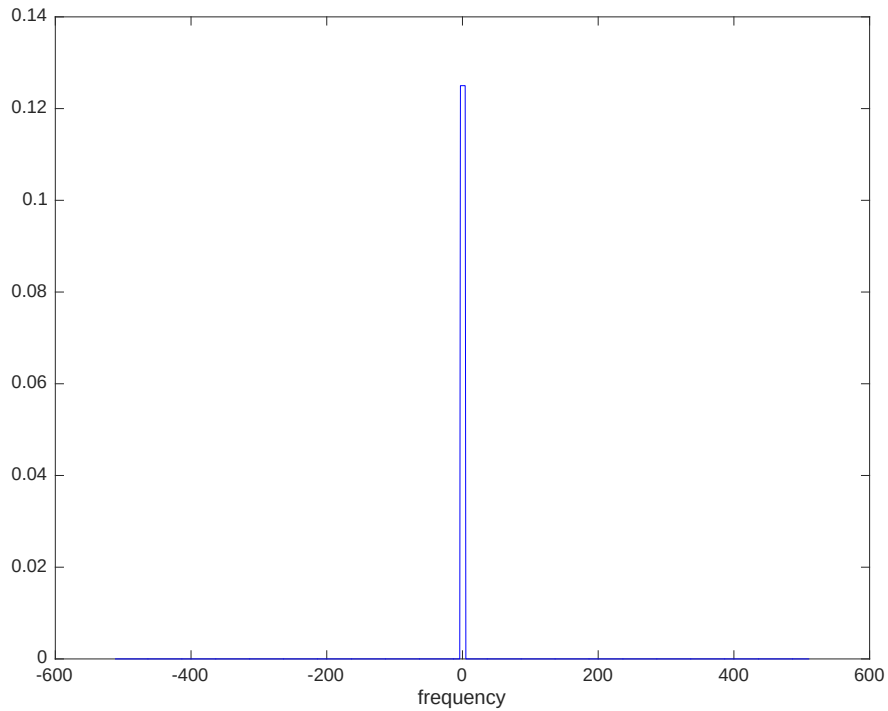

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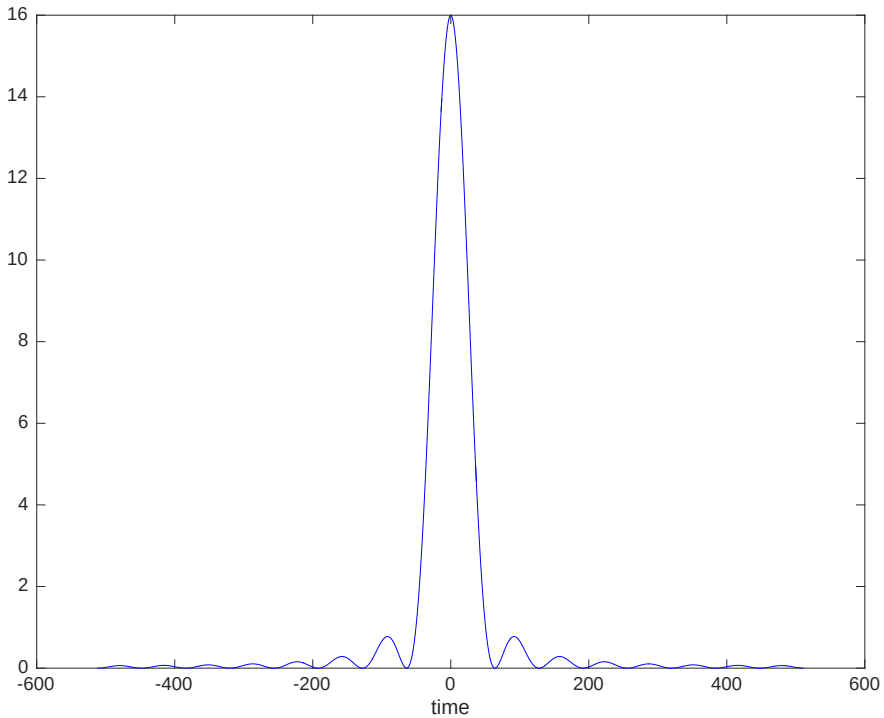

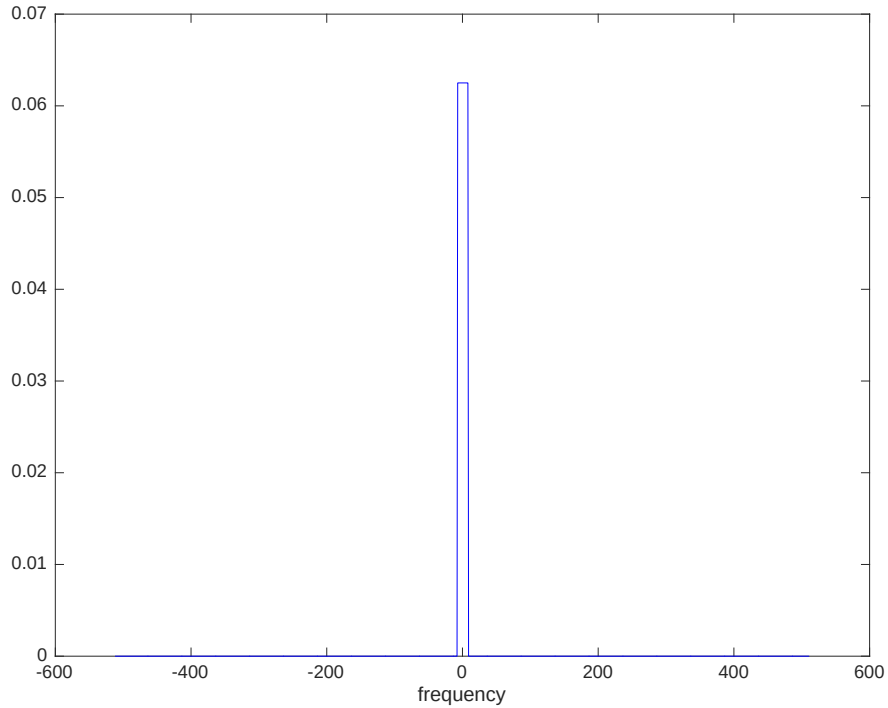

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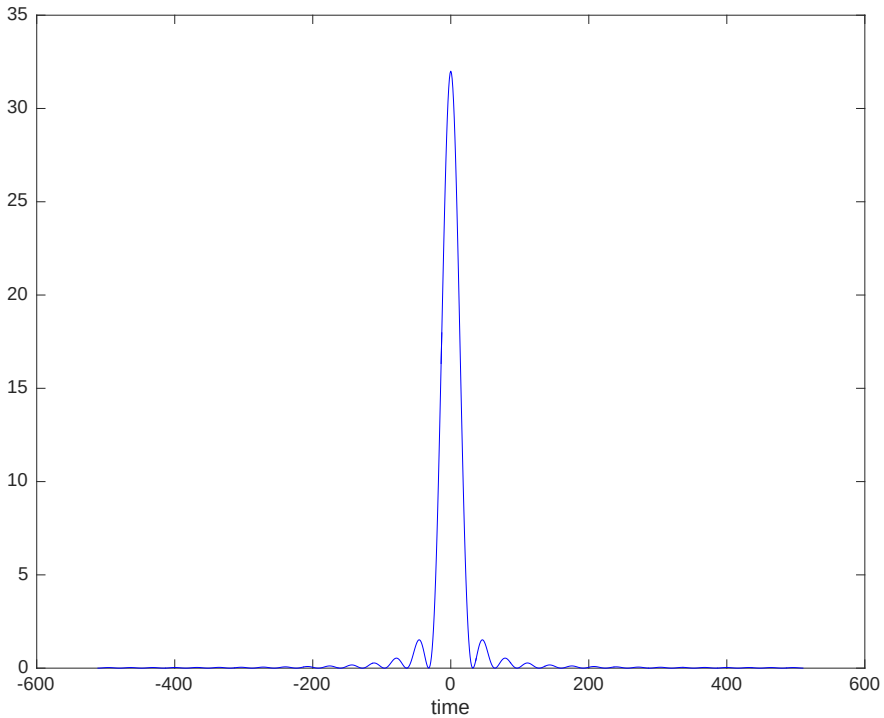

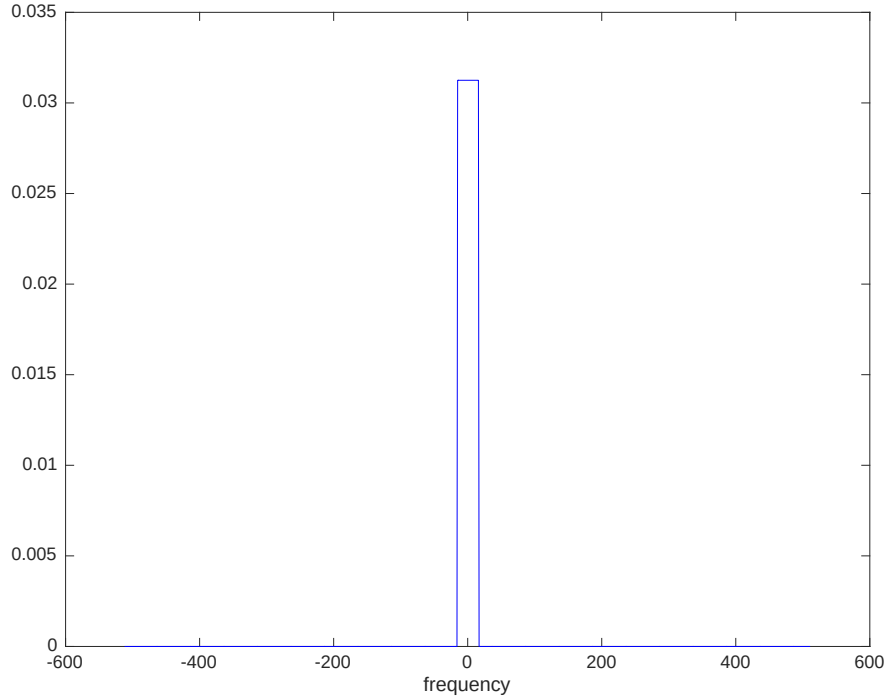

Recovery of block sparse signals

For each $r = 1, \ldots, 2k_1$ define reduced signal $Z^{\mathbf{r}}$ by

$$\widehat{Z^{\mathbf{r}}_j} := \left(X \widehat{\left(\cdot - r \cdot \frac{n}{2k_1} \right)} * \widehat{G} \right)_{j \cdot k_1}$$

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An adaptive solution can first probe which signals carry most energy, then allocate hashing budgets carefully

Importance sampling

AllocateBudgets(X, k_0, k_1)
For $t = 1, \ldots, O(k_0)$
  Sample $r \sim ||Z^r||_2^2$, $q \sim Geom$
  Set $s^t \leftarrow (r, 2^q)$
End For

$O(k_0 k_1)$ samples

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BudgetedRecovery$(X, \{s^t\})$
For $t = 1, \ldots, O(k_0)$
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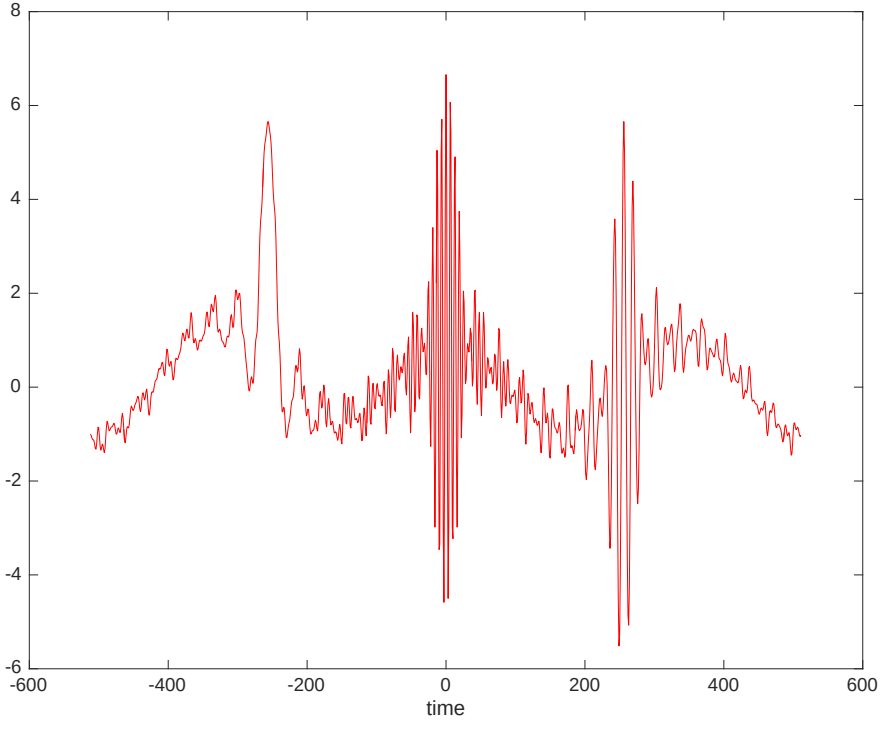

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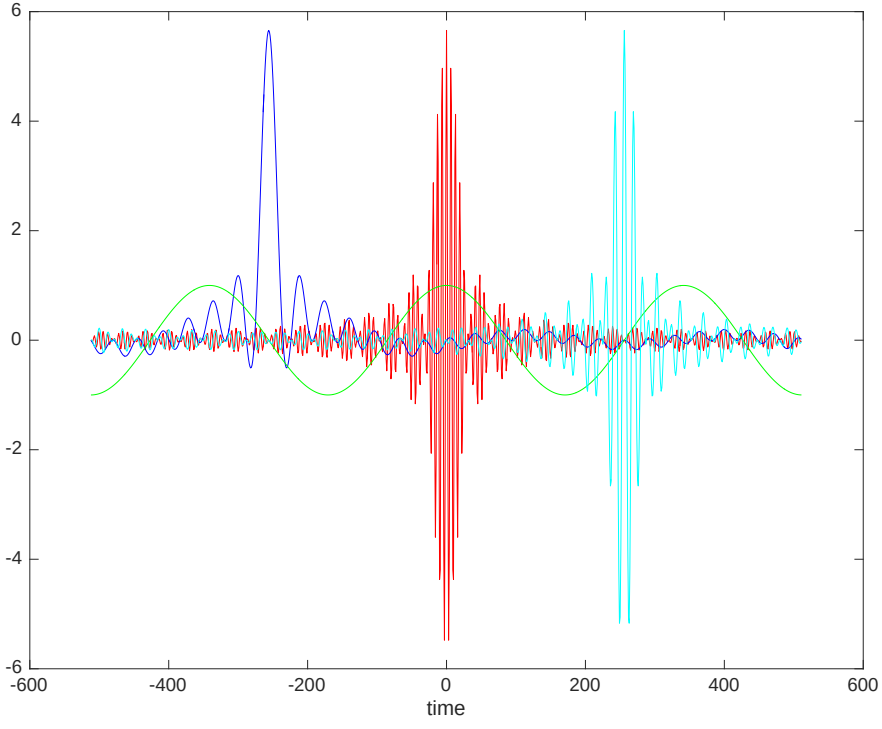

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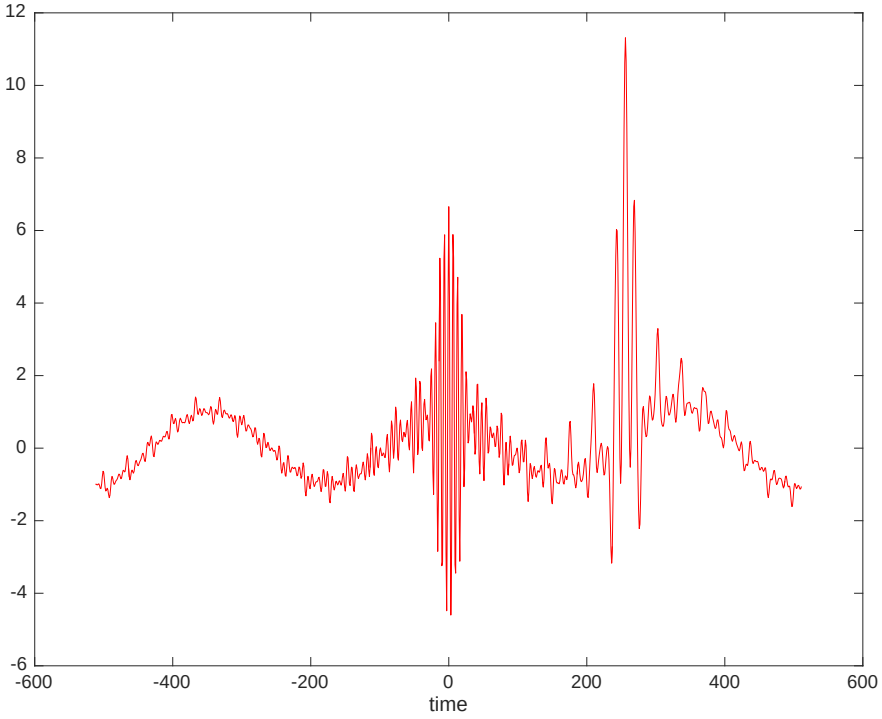

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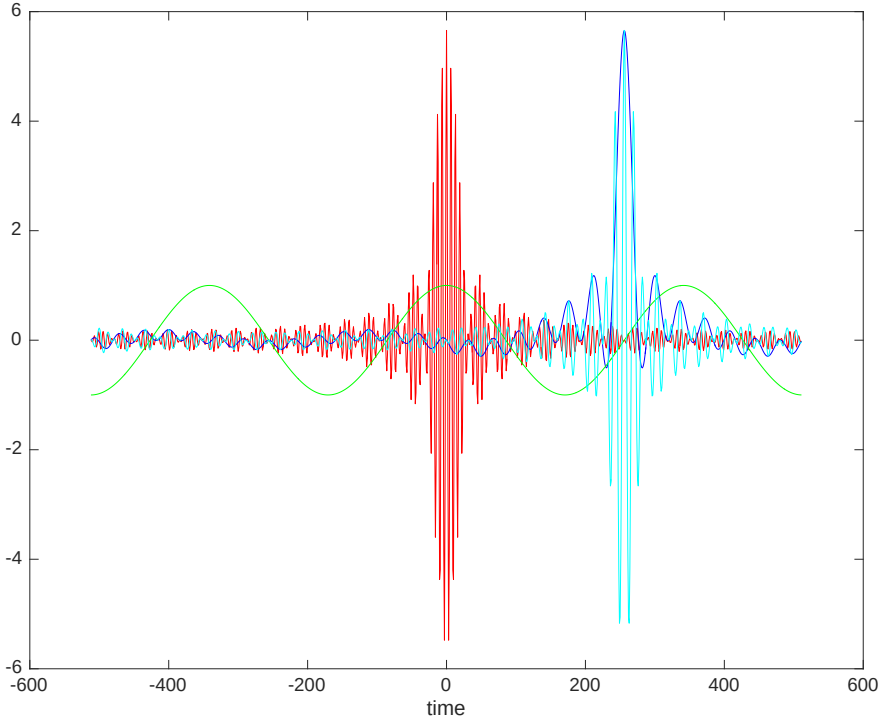

Adaptivity is necessary

Algorithm crucially uses adaptivity: sample a few points first to find a non-uniform sampling distribution for second stage

Theorem

Any non-adaptive Sparse FFT algorithm must take $\Omega(k_0 k_1 \log \frac{n}{k_0 k_1})$ *samples.*

Our results:

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Thank you!