Mathematical modelling and numerical simulations applied to cardiac diffusion MRI

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4^{th} Workshop INRIA-EPFL
February 16, 2018

Motivation

Diffusion MRI:

- Measures the diffusion of water molecules in the biological tissues.
- Allows the characterization of the microscopic structure of tissues.
- Provides assistance in the diagnosis of brain, cardiac abnormalities...

[1] S. Rapacchi et al. Investigative Radiology, 46(12):751–758 (2011)

Motivation

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Cardiac diffusion MRI:

Figure: Cardiac motion (Tagged MRI).

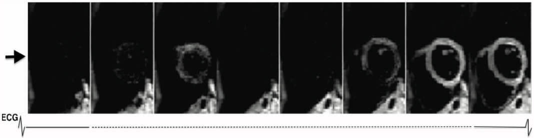

Figure: Diffusion-weighted MRI Images of the left ventricule during the cardiac cycle[1].

Tissue motion $\Longrightarrow$ Signal loss and bad reconstruction of diffusion MRI images.

[1] S. Rapacchi et al. Investigative Radiology, 46(12):751–758 (2011)

Motivation

Diffusion MRI: Bloch-Torrey equation

– The Bloch-Torrey equation is given by:

$$\partial_t M - \operatorname{div}(\mathbf{D}\nabla M) + i\gamma \mathbf{x} \cdot \mathbf{G}(t) M = 0. \tag{1}$$

M is the magnetization, $\mathbf{D}$ is a diffusion tensor of second order, and $\mathbf{G}$ is a diffusion encoding magnetic field gradient.

Motivation

Diffusion MRI: Bloch-Torrey equation

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M is the magnetization, $\mathbf{D}$ is a diffusion tensor of second order, and $\mathbf{G}$ is a diffusion encoding magnetic field gradient.

– The diffusion MRI signal is:

$$S(t) := \int_V M(\mathbf{x}, t) d\mathbf{x}.$$

Motivation

Diffusion MRI: Diffusion measurement

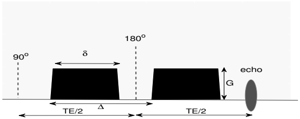

Figure: Spin echo (ES) diffusion encoding gradient ($\mathbf{G}(t)$).

Figure: (Left) Signal without diffusion S_0. (Right) Signal with diffusion S.

Motivation

Diffusion MRI: diffusion measurements

– The diffusion can be estimated at each point of the MRI image by the *apparent diffusion coefficient* (ADC) :

$$ADC := -\frac{1}{b} \ln\left(\frac{S_b}{S_0}\right). \quad (2)$$

S_b (resp. S_0) corresponds to the magnetization M measured with diffusion encoding gradient ($G(t)$) (resp. without diffusion encoding gradient), and

$$b = \gamma^2 \int_0^{TE} \left(\int_0^{t'} G(s)ds\right)^2 dt'$$

is the diffusion weighting factor.

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is the diffusion weighting factor.

– Anisotropic diffusion: $\ln\left(\frac{S}{S_0}\right) := -\mathbf{B} : \mathbf{D}^{\mathbf{A}}$.

$\mathbf{D}^A$ is the apparent diffusion tensor. $\mathbf{B}$ is the diffusion weighted matrix:

$$\mathbf{B} = \gamma^2 \int_0^{TE} \left(\int_0^{t'} \mathbf{G}(s)ds \right) \otimes \left(\int_0^{t'} \mathbf{G}(s)ds \right) dt'.$$

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Bloch-Torrey equation with motion: Mathematical model

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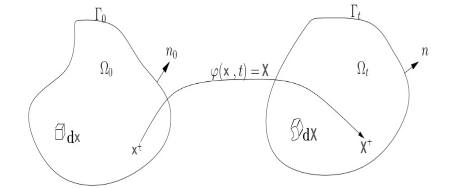


Figure: Reference and deformed configurations. φ is the deformation field.

Bloch-Torrey equation with motion: Mathematical model

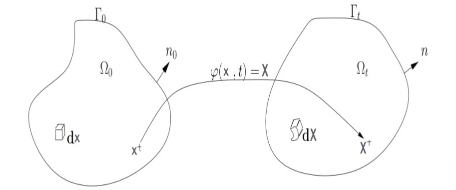


Figure: Reference and deformed configurations. φ is the deformation field.

– The Bloch-Torrey equation in the deformed configuration:

$$\partial_t M(X,t) - \mathrm{div}_X(\mathbf{D}(X)\nabla_X M(X,t)) + \mathrm{div}_X(M(X,t)\mathbf{v}(X,t)) + i\gamma X\mathbf{G}(t)M(X,t) = 0, \; X \in \Omega(t), \; t > 0. \quad (3)$$

Bloch-Torrey equation with motion: Mathematical model

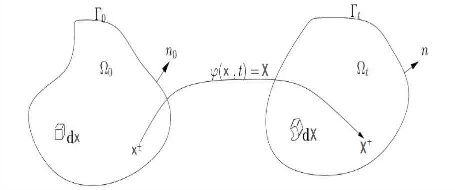


Figure: Reference and deformed configurations. φ is the deformation field.

– The Bloch-Torrey equation in the deformed configuration:

$$\partial_t M(X,t) - \text{div}_X(\mathbf{D}(X)\nabla_X M(X,t)) + \text{div}_X(M(X,t)\mathbf{v}(X,t)) + i\gamma X\mathbf{G}(t)M(X,t) = 0,\; X \in \Omega(t),\; t > 0. \quad (3)$$

– The Bloch-Torrey equation in the reference configuration:

$$\partial_t M(\mathbf{x},t) - \text{div}\big(\mathbf{F}^{-1}(t,\mathbf{x})\mathbf{D}(\mathbf{x})\mathbf{F}^{-t}(t,\mathbf{x})\nabla M(\mathbf{x},t)\big) + i\gamma\varphi(\mathbf{x},t)\cdot\mathbf{G}(t)M(\mathbf{x},t) = 0,\; \mathbf{x} \in \Omega(0),\; t > 0. \quad (4)$$

$\mathbf{F}$ is the Jacobian matrix of φ, $\mathbf{F}^{-t} = (\mathbf{F}^{-1})^t$. $\mathbf{v}$ is the velocity field of φ.

Contents

Application to cardiac diffusion MRI: Analytical cardiac motion model in 2D

The heart deformation function φ is given by:

$$\varphi(t,(R_0,\theta_0)) = \begin{cases} R(t) = \left(\frac{R_0^2 - R_{int}^2}{\lambda(t)} g(\theta_0) + R_{int}(\theta_0,t)^2\right)^{1/2} \\ \theta(t) = \theta_0 + \psi(t,R_0) + \chi(t) \end{cases}$$

with

$$\lambda(t) = 1 - 0.2S(t)$$

$$R_{int}(\theta_0,t) = R_{int} - (R_{int} - R_{min}) g(\theta_0) S(t)$$

$$g(\theta_0) = 1 + 0.1(\cos(\theta_0 + 3\pi/4) + 1)$$

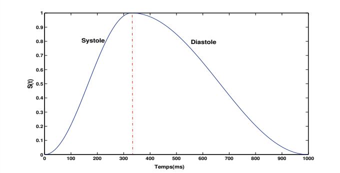
$$R_{min} = R_{int} - 5(R_{ext} - R_{int})/6$$

$$\psi(t,R_0) = -(0.2\pi/180) S(t) R_0$$

$$\chi(t) = (10\pi/180) S(t).$$

– The cardiac deformation is controlled by the function S.

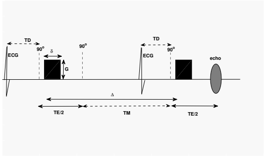
– The simulated cardiac cycle $\sim$ 1000ms, with a systole time $\sim$ 330ms and a diastole time $\sim$ 660ms.



The function S

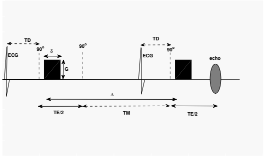
Figure: Simulated cardiac motion.

Application to cardiac diffusion MRI: Numerical simulations



Unipolar Stimulated Echo (STEAM) diffusion encoding gradient ($\mathbf{G}(t)$).

Application to cardiac diffusion MRI: Numerical simulations



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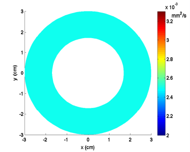
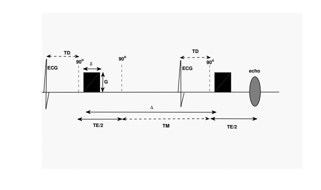


Exact diffusion.

Application to cardiac diffusion MRI: Numerical simulations



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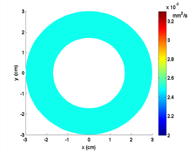

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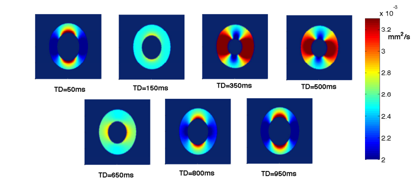
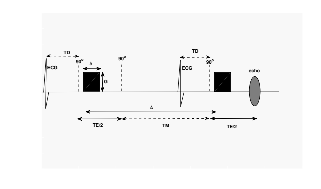


ADC images at different times of the cardiac cycle.

Application to cardiac diffusion MRI: Numerical simulations



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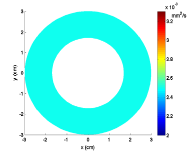

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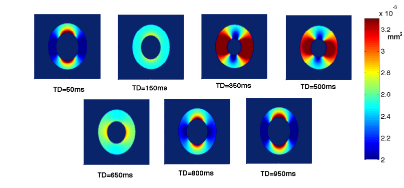

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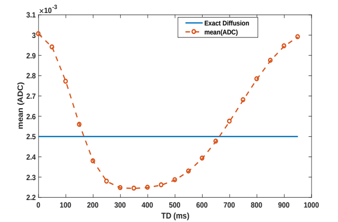

Mean ADC during the cardiac cycle.

Application to cardiac diffusion MRI: Comparison with experimental results

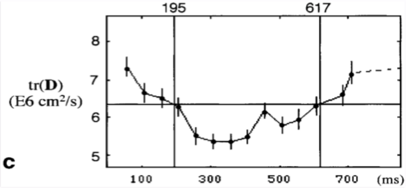

Figure: Time curve of mean trace of the diffusion tensor (tr(D)) obtained from a normal subject with a heart rate of approximately 72 bpm and locations of two sweet spots, at mid-systolic and mid-diastolic moments, indicated by the intersections of the curve with $< \mathrm{tr}(\mathbf{D}(t)) >_t$. (Figure extracted from[2]).

[2] W-Y I. Tseng et al. Magn. Reson. Med. 42:393–403 (1999).

Contents

Diffusion correction using an asymptotic model

– Transformation of the Bloch-Torrey equation:

$$M(\mathbf{x}, t) = m(\mathbf{x}, t) \exp\Big(- i\Phi(\mathbf{x}, t)\Big) \tag{5}$$

$$\Phi(\mathbf{x}, t) = \gamma \int_0^t \varphi(\mathbf{x}, t') \cdot \mathbf{G}(t') dt'$$

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– Asymptotic expansion of m:

$$m = m_0 + \varepsilon m_1 + \cdots$$

$m \to m_0$ formally with ε, where m_0 satisfies:

$$\begin{aligned} &\partial_t m_0(\mathbf{x},t) + (\mathbf{K}\nabla\Phi\cdot\nabla\Phi)m_0(\mathbf{x},t) = 0 \quad \text{in } \Omega\times(0,T)\\ &m_0(\mathbf{x},0) = m^0(\mathbf{x}) \text{ on } \Omega\times\{0\}, \end{aligned} \tag{6}$$

$$\mathbf{K}(\mathbf{x},t) = \mathbf{F}^{-1}(t,\mathbf{x})\mathbf{D}(\mathbf{x})\mathbf{F}^{-t}(t,\mathbf{x}).$$

Diffusion correction using an asymptotic model: Diffusion correction formula

– Then the magnetization M is:

$$M(\mathbf{x}, t) \approx m^0(\mathbf{x}) \exp\Big(- \int_0^t (\nabla\Phi(\mathbf{x}, t'))^t \mathbf{K}(\mathbf{x}, t')\nabla\Phi(\mathbf{x}, t')dt' \Big) \exp\Big(- i\Phi(\mathbf{x}, t)\Big).$$

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– Recovery of the exact diffusion:

We have $\mathbf{K} = \mathbf{F}^{-1}\mathbf{D}\mathbf{F}^{-t}$, the diffusion attenuation can be written as:

$$\ln\Big(\Big|\frac{M(\mathbf{x},TE)}{M(\mathbf{x},TE)|_{\mathbf{G}=0}}\Big|\Big) = -\int_0^{TE} (\nabla\Phi(\mathbf{x},t'))^t \mathbf{K}(\mathbf{x},t')\nabla\Phi(\mathbf{x},t')dt'$$

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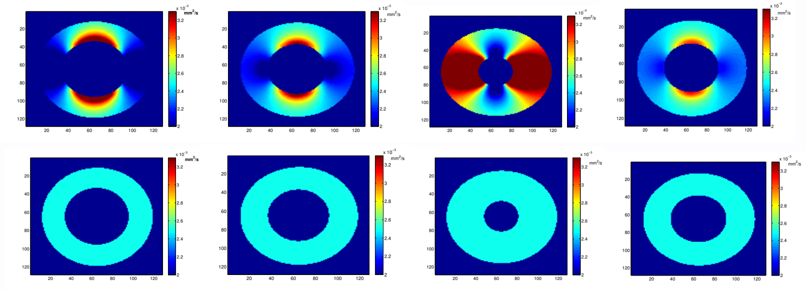
with

$$\mathbf{B}_\varphi = \Big(\int_0^{TE} \mathbf{w}(\mathbf{x},t') \otimes \mathbf{w}(\mathbf{x},t')dt'\Big)$$

and $\mathbf{w}(\mathbf{x},t) = \mathbf{F}^{-t}(\mathbf{x},t)\nabla\Phi(\mathbf{x},t)$, and $\mathbf{D}$ is the exact diffusion tensor.

Diffusion correction using an asymptotic model: Simulations



TD=0ms TD=100ms TD=350ms TD=750ms

Figure: ADC images simulated at different times in the cardiac cycle. Before correction (Top). After correction by the asymptotic model (Bottom).

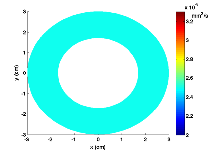


Figure: Exact diffusion.

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Microscopic-scale numerical simulations of cardiac diffusion MRI

Equipe DEFI, INRIA-Saclay:

- Contributing expertise to the project: diffusion MRI, inverse problems, mathematical homogenization.
- Under the supervision of: *Jing-Rebecca Li* and *Houssem Haddar*.

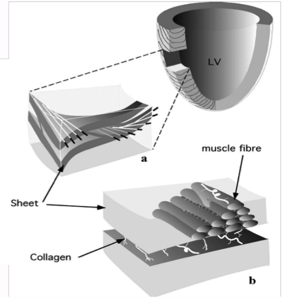


Figure: Cardiac microstructure: schematic presentation of heart muscle fibers.

Microscopic-scale numerical simulations of cardiac diffusion MRI

Multiple compartments Bloch-Torrey equation:

Integrate sub-compartments with interface conditions: $\Omega \in \mathbf{R}^3$ a volume of biological tissue.

– $\Omega \equiv \Omega_i \cup \Omega_e$,

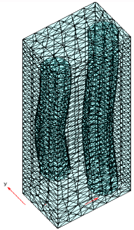
– Ω_e: extra-cellular space,

– Ω_i: union biological cells. Γ: union of the boundaries of biological cells.

– κ: cellular permeability.

$$\begin{cases} \partial_t M - \mathrm{div}(\mathbf{F}^{-1}\mathbf{D}\mathbf{F}^{-t}\nabla M) + i\gamma \mathbf{G}(t)\cdot\varphi(\mathbf{x},t)M = 0 \quad \text{in } \Omega\times(0,T) \\ \mathbf{F}^{-1}\mathbf{D}\mathbf{F}^{-t}\nabla M\cdot \mathbf{n}|_\Gamma = \kappa[\![M]\!] \quad \text{on } \Gamma\times(0,T) \\ [\![\mathbf{F}^{-1}\mathbf{D}\mathbf{F}^{-t}\nabla M\cdot\mathbf{n}]\!] = 0 \quad \text{on } \Gamma\times(0,T) \\ M(\mathbf{x},0) = M_{ini}(\mathbf{x}) \quad \text{on } \Omega\times\{0\}. \end{cases} \tag{7}$$

Figure: Example of computational domain Ω with intra-cellular space (the cylinders) and extra-cellular space (extra-cylinder region).



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Parameters estimation for cardiac diffusion MRI

Chair of Computational Mathematics and Simulation Science (MCSS), EPFL.

- Contributing expertise to the project: high order and adaptive numerical solution of PDEs, reduced models.
- With: *Pr. Jan Hesthaven*.

Estimation of macroscopic apparent diffusion coefficient (ADC) after mathematically cancelling the effects of cardiac motion and deformation:

- *Data set library from simulations*: Generate large number of ADC images in the presence of heart motion at different times of cardiac cycle by solving the Bloch-Torrey equation with prescribed motion to make a library.
- Generate reduced model to map the distorted ADC image to the original ADC image.

Thank you !