Improving social welfare in non-cooperative games with different types of quantum resources

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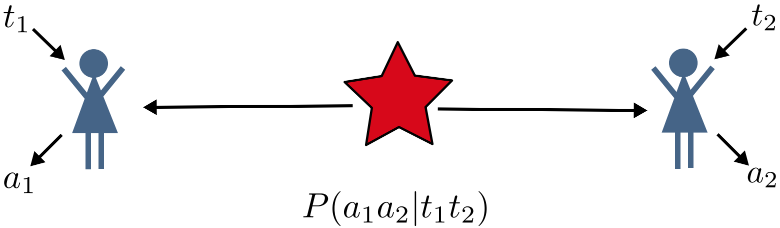
Warmup: Nonlocal Games

A familiear scenario:



- CHSH game: players win if $a_1 \oplus a_2 = t_1 t_2$
- How well can the players do given different resources?
  - Independent players; shared randomness; quantum resources; no-signalling boxes; communication; ...
- Cooperative game: all players win and lose together, goals are aligned

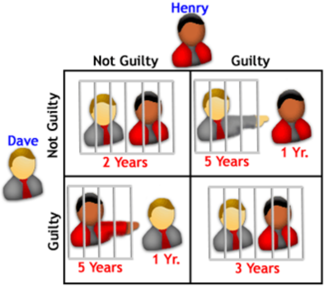
Outline

- Non-cooperative games and equilibria
- Two different quantum resources
  - Shared quantum correlations (classical "black box" access)
  - Shared quantum states (quantum access)
- Comparing different resources
  - Maximising the social welfare

Non-cooperative game theory

Reality: Players' objectives often not aligned:

- A player's payoff depends on the other players' actions
- Examples:
  - Zero-sum games
  - Prisoner's dilemma



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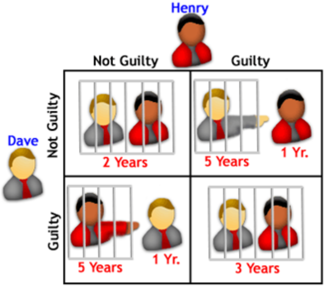
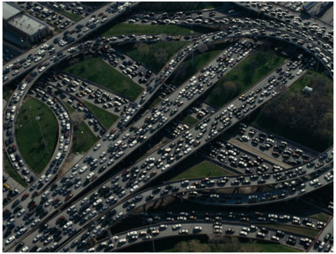
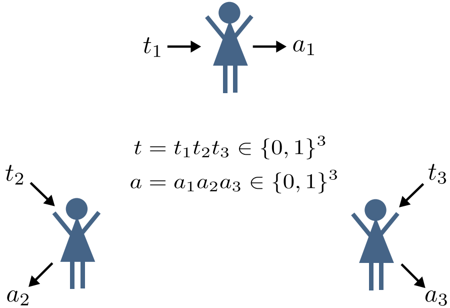


Extensively studied in game theory

- Complex behaviour, Nash equilibria, . . .
- Widely applicable



Example: A three-player game



$$t = t_1 t_2 t_3 \in \{0,1\}^3$$
$$a = a_1 a_2 a_3 \in \{0,1\}^3$$

Question $t_1 t_2 t_3$	Winning conditions
100	$a_1 \oplus a_2 \oplus a_3 = 0$
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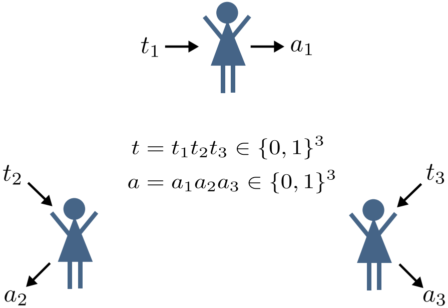

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Payoff function

$$u_i(a,t) = \begin{cases} 0 & \text{if } (a,t) \notin \mathcal{W} \\ v_0 & \text{if } a_i = 0 \text{ and } (a,t) \in \mathcal{W} \\ v_1 & \text{if } a_i = 1 \text{ and } (a,t) \in \mathcal{W}. \end{cases}$$

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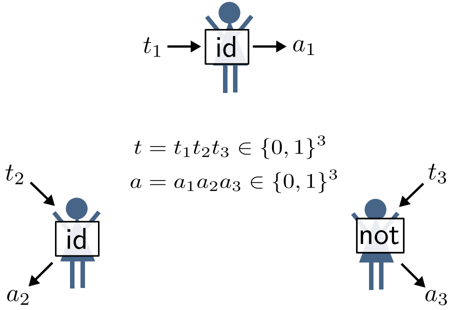

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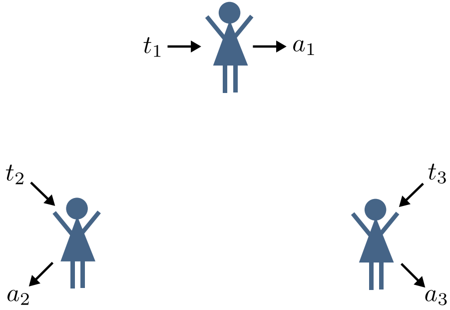
- The strategy $(\text{id}, \text{id}, \text{not})$ wins $3/4$ of the time
- Can a player increase their expected gain, potentially at the expense of the others?
- What strategy maximises the overall (or average) payoff?

Different types of resources



- Base scenario: independent local strategies

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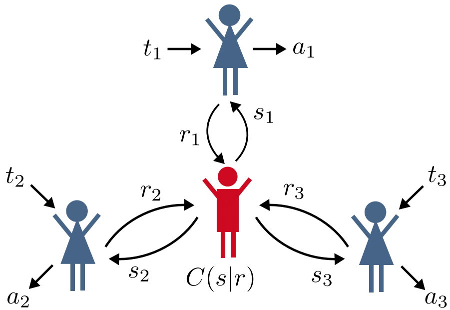

- Base scenario: independent local strategies
- Shared resources: correlated advice

Different class of correlations $\mathcal{C}$:

- Classical shared random variables
- n-partite quantum correlations ($\mathcal{C}_Q$)
- Belief-invariant (non-signalling) correlations
- Full communication

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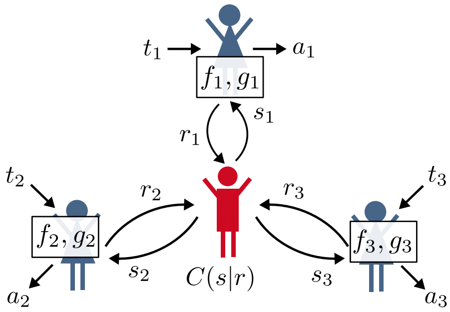

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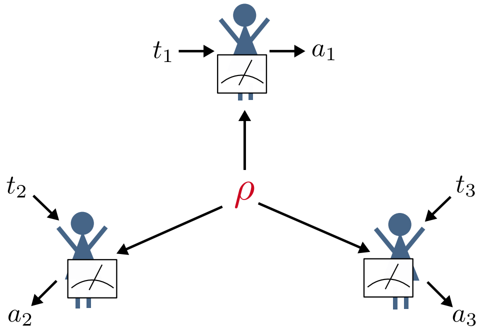
Definition (Solution)

A solution is a tuple $(f_1, \ldots, f_n, g_1, \ldots, g_n, C)$ and induces a correlation

$$P(a|t) = \sum_s C(s|f(t))\delta_{g(t,s),a}$$

Quantum resources: states as advice



Players receive part of a shared quantum state as "advice", and can measure it directly.

Definition (Quantum solution)

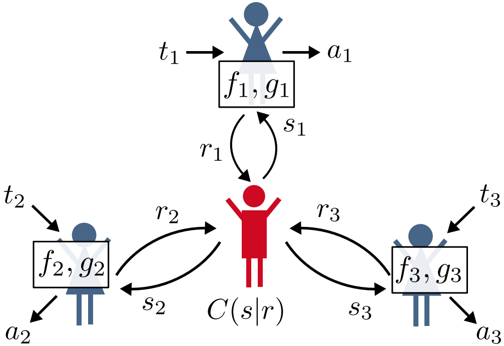
A quantum solution is a tuple $\left(\rho, \mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(n)}\right)$, with $\mathcal{M}^{(i)}$ sets of POVMs $\{M^{(i)}_{a_i|t_i}\}_{a_i,t_i}$.
It induces a correlation:

$$P(a|t) = \mathrm{Tr}\Big[\rho\,\Big(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\Big)\Big].$$

[Auletta, Ferraioli, Rai, Scarpa, Winter, JTCS (2021)]

Nash equilibria

In game theory, we are interested in equilibrium solutions, where no player can increase their payoff by unilaterally deviating from a solution.



Definition (Nash equilibrium (informal))

A solution is a Nash equilibrium if no player can increase their payout $\sum_{a,t} u_i(a,t)P(a|t)\Pi(t)$ by changing their local strategy (f_i, g_i) to (ν_i, μ_i).

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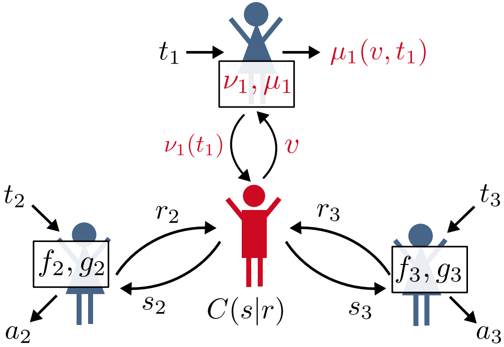

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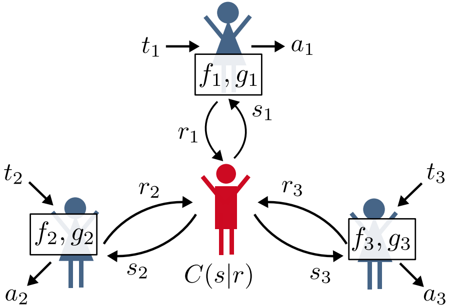
Simplifying things

It turns out that for most classes of correlations $\mathcal{C}$, we can restrict ourselves to canonical solutions:

- Each player sends t_i to the mediator and outputs what they receive as a_i
- $P(a|t) = C(a|t)$



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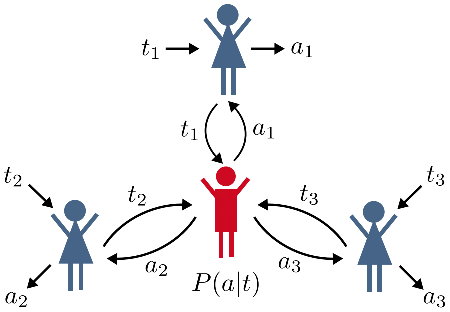

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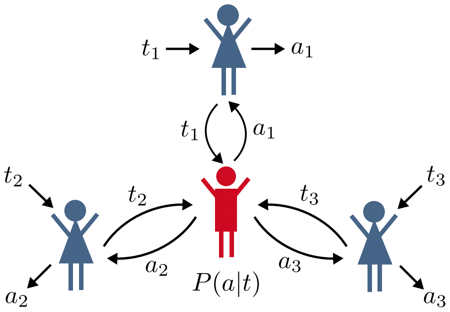

Definition (Nash equilibrium)

A solution is a Nash equilibrium if, for all players i, all $t_i, r_i \in T_i$, and all functions $\mu_i : T_i \times A_i \to A_i$:

$$\sum_{t_{-i},a} u_i(a,t)P(a|t) \geq \sum_{t_{-i},a} u_i(\mu_i(a_i,t_i)a_{-i}, t_i t_{-i})P(a|r_i t_{-i}).$$

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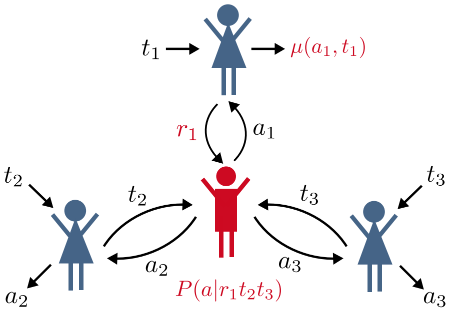

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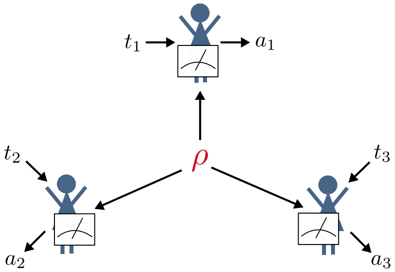
Quantum equilibria



Definition (Quantum equilibrium)

A quantum solution $\left(\rho, \mathcal{M}^{(1)}, \dots, \mathcal{M}^{(n)}\right)$, is a *quantum equilibrium* if, for every player i, for any type t_i and any POVM $N^{(i)} = \{N_{a_i}^{(i)}\}_{a_i \in A_i}$:

$$\sum_{t_{-i},a} u_i(a,t) \operatorname{Tr}\left[\rho\left(M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}\right)\right]\Pi(t)$$

$$\geq \sum_{t_{-i},a} u_i(a,t) \operatorname{Tr}\left[\rho\left(M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_{i-1}|t_{i-1}}^{(i-1)} \otimes N_{a_i}^{(i)} \otimes M_{a_{i+1}|t_{i+1}}^{(i+1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}\right)\right]\Pi(t).$$

Comparing equilibria

- What equilibria can we obtain with a given resource?
- How to compare correlation vs quantum resources?
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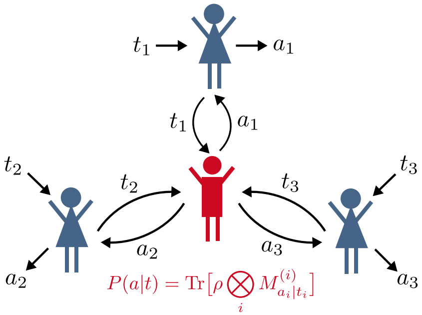
Definition (Social welfare)

For a game G, the *social welfare* of a solution inducing a distribution P is

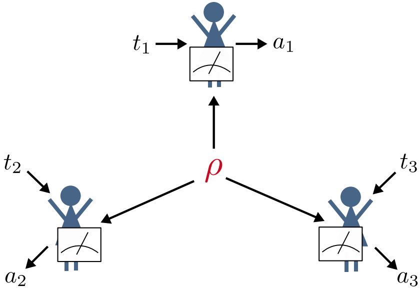
$$SW(P) = \frac{1}{n} \sum_i \sum_{a,t} u_i(a,t) P(a|t) \Pi(t).$$

Two types of quantum resources

Classical access: advice $P \in \mathcal{C}_Q$



Quantum access



- Two different levels of access to quantum resources leads to two different notions of equilibria
- Two corresponding sets of equilibrium correlations:

$$Q_{\mathrm{corr}}(G) = \{P \mid P \text{ defines a canonical Nash equilibrium and } P \in \mathcal{C}_Q\} \subseteq \mathcal{C}_Q$$
$$Q(G) = \{P \mid \text{there exists } (\rho, \mathcal{M}) \text{ a quantum equilibrium inducing } P\} \subseteq \mathcal{C}_Q$$

Quantum access restricts equilibria

Counter-intuitively, allowing the players more control restricts the equilibriums they can reach

Theorem

For any game G, $Q(G) \subseteq Q_{\text{corr}}(G)$.

Proof idea.

Any modification on the classical output of a quantum correlation could also be represented by changing the POVMs used to obtained the correlations. □

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The quantum families fit within a hierarchy of equilibrium correlations:

$$\text{Nash}(G) \subset \text{Corr}(G) \subset Q(G) \subseteq Q_{\text{corr}}(G) \subset \text{B.I.}(G) \subset \text{Comm}(G)).$$

[Auletta, Ferraioli, Rai, Scarpa, Winter, JTCS (2021)]

Is the separation strict? Can we obtain *better* equilibria?

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Optimising the social welfare

- Comparing the sets of equilibria is challenging:
  - No restriction on dimension of systems
  - Many solutions may give equivalent equilibria
- Relevant proxy: investigate achievable social welfare

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Maximising social welfare

$$\max_P SW_G(P) = \frac{1}{n}\sum_{a,t}\sum_i u_i(a,t)P(a|t)\Pi(t),$$

where the maximisation is either over $Q_{\mathrm{corr}}(G) \subseteq \mathcal{C}_Q$ or $Q(G) \subseteq \mathcal{C}_Q$

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- Question: how to characterise these sets of equilibria?
- Use numerical and SDP methods to compute upper and lower bounds on the maximum social welfare.

Lower bounds: See-saw optimisation

- Key observation: checking if $(\rho, \mathcal{M})$ is a quantum equilibrium is an SDP
- Constructive method by iterating over each party

See-saw iteration over $\mathcal{C}_Q$

$$\max_{\mathcal{M}^{(N)}} \cdots \max_{\mathcal{M}^{(1)}} \max_{\rho} SW(P) = \frac{1}{N} \sum_{a,t} \sum_{i} u_i(a,t) \operatorname{Tr}\left[\rho \left(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\right)\right] \Pi(t)$$

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To converge to an equilibrium, we then add:

Quantum equilibria: $Q(G)$

Each player tries to optimise their own payoff

$$\max_{\mathcal{M}^{(N)}} \cdots \max_{\mathcal{M}^{(1)}} \sum_{a,t} u_i(a,t) \operatorname{Tr}\left[\rho \left(M^{(1)}_{a_1|t_1} \otimes \cdots \otimes M^{(n)}_{a_n|t_n}\right)\right] \Pi(t).$$

Nash equilibria: $Q_{\mathrm{corr}}(G)$

The (finite) inequalities constraining Nash equilibria.

Upper bounds: NPA hierarchy

Main difficulty computing upper bounds: there is no easy way to characterise the set of quantum correlations $\mathcal{C}_Q$.

NPA hierarchy

Convergent hierarchy of SDP constraints to test if a distribution is in $\mathcal{C}_Q$, approximating it from the outside (upper bounds).

$$+$$

Nash equilibrium

Finite number of linear constraint to test if a probability distribution is a Nash equilibrium.

$$\underbrace{\max_{P \in \widetilde{Q}_{\text{corr}}(G)}}SW(P) = \frac{1}{N}\sum_{a,t}\sum_i u_i(a,t)P(a|t)\Pi(t).$$

[Navascues, Pironio, Acin, NJP (2008)]

Example revisited

Recall the following family of three-player $\mathrm{NC}(C_3)$ games:

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[Groisman, McGettrick, Mhalla, Pawlowski, IEEE JSAIT (2020)]

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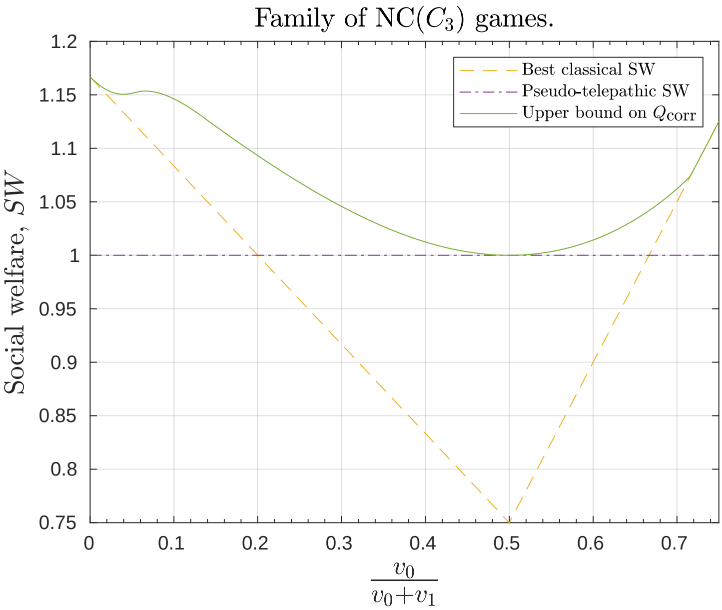
We take $v_0, v_1 > 0$, $v_0 + v_1 = 2$.

- The best classical (correlated) strategy wins $3/4$ of the time
- Graph state and σ_x, σ_z measurements give pseudotelepathic solution
  - Both a quantum correlated and a quantum equilibrium
- But is it the *best* equilibrium in terms of social welfare?
- Is there a difference between types of quantum resources in this game?

[Groisman, McGettrick, Mhalla, Pawlowski, IEEE JSAIT (2020)]

Social Welfare in NC(C_3) games



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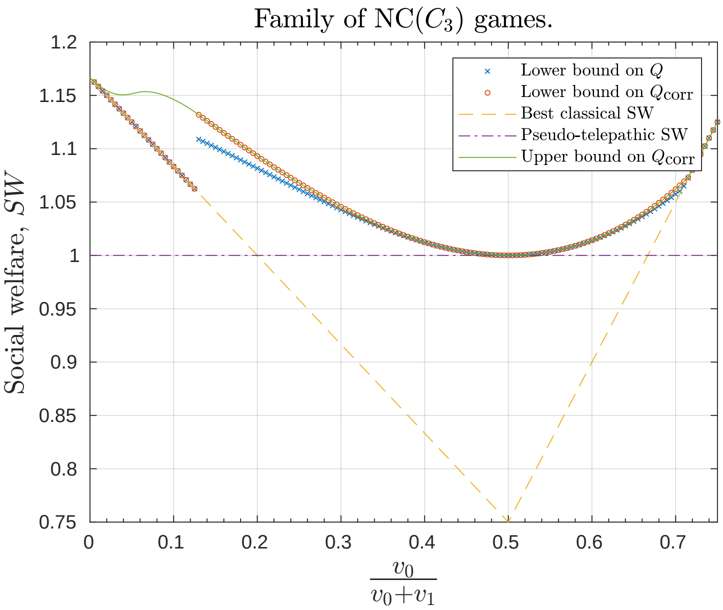
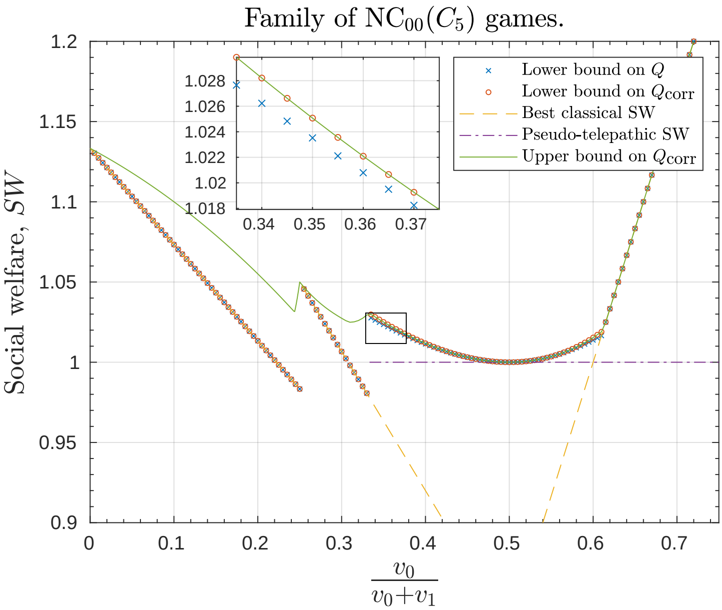


Social Welfare in some five-player games



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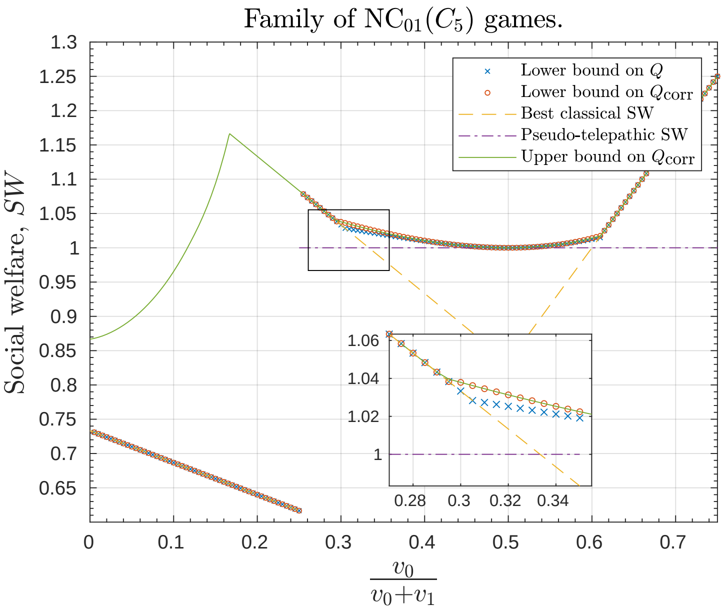

Summary

- Non-cooperative games as a portal to adress different types of quantum resources:
  - **Classical access** to a quantum resources: $Q_{\mathrm{corr}}(G)$
  - **Quantum access** to a quantum resource: $Q(G)$
- Counterintuitively, quantum access gives less equilibria: $Q(G) \subseteq Q_{\mathrm{corr}}(G)$
- Evidence of a strict separation in terms of social welfare

Open questions and ongoing work:

- How to prove a strict separation?
  - Can the NPA hierarchy be adapted to give upper bounds on $Q(G)$?
  - Use techniques from self-testing to prove a distribution in $Q_{\mathrm{corr}}(G)$ is not in $Q(G)$?
- Intermediate settings (with classical or quantum access for different players)