

Ensemble controllability in qubits with a single control
via adiabatic and rotating wave approximations.

In collaboration with Nicolas Augier, Ugo Boscain and Mario Sigalotti

Rémi Robin (CAGE → QUANTIC)



November 30, 2022

défi EQIP, Strasbourg

Ensemble controllability : short introduction

$$\begin{cases} \dot{x}(t) &= f(x(t), w(t)) \\ x(0) &= x_0 \end{cases} \quad (1)$$

Definition : system (1) is approximatively controllable from x_0 toward $x_f \in E$ iif

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \|x(T) - x_f\| \leq \varepsilon$$

Ensemble controllability : short introduction

$$\begin{cases} \dot{x}(t) = f(x(t), w(t)) \\ x(0) = x_0 \end{cases} \quad (1)$$

$$\begin{cases} \dot{x}(t, \alpha) = f(x(t, \alpha), \alpha, w(t)) \\ x(0, \alpha) = x_0 \quad \alpha \in \mathcal{D} \end{cases} \quad (2)$$

Definition : system (1) is approximatively controllable from x_0 toward $x_f \in E$ iif

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \|x(T) - x_f\| \leq \varepsilon$$

Definition : system (2) is approximatively ensemble controllable from x_0 toward $x_f \in E$ iif

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \sup_{\alpha \in \mathcal{D}} \|x(T, \alpha) - x_f\| \leq \varepsilon$$

Ensemble controllability : short introduction

$$\begin{cases} \dot{x}(t) = f(x(t), w(t)) \\ x(0) = x_0 \end{cases} \quad (1)$$

$$\begin{cases} \dot{x}(t, \alpha) = f(x(t, \alpha), \alpha, w(t)) \\ x(0, \alpha) = x_0 \quad \alpha \in \mathcal{D} \end{cases} \quad (2)$$

Definition : system (1) is approximatively controllable from x_0 toward $x_f \in E$ iif

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \|x(T) - x_f\| \leq \varepsilon$$

Definition : system (2) is approximatively ensemble controllable from x_0 toward $x_f \in E$ iif

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \sup_{\alpha \in \mathcal{D}} \|x(T, \alpha) - x_f\| \leq \varepsilon$$

models :

- Uncertainties/fluctuations on the system
- set of systems that we want to control with a single control

Some seminal works : Li–Khaneja, Beauchard–Coron–Rouchon, Agrachev–Baryshnikov–Sarychev, Augier–Boscain–Sigalotti...

Quantum mechanics

Let V be \mathbb{C} vectorial space and $H(w)$ a self-adjoint operator.
The wave function $\psi \in V$ follows Schrödinger's equation :

$$i\partial_t\psi(t) = H(w(t))\psi(t)$$

For a qubit, $\dim_{\mathbb{C}} V = 2$ and

$$H(w(t)) = a(t)\sigma_z + b(t)\sigma_x + c(t)\sigma_y.$$

a quantum state is an element of the projective space $P(V)$.
In particular, qubit states can be represented on the Bloch sphere.

Let $E > 0$ and α be the dispersion parameter.

$$\forall \alpha \in [\alpha_0, \alpha_1], H^\alpha(w) = \begin{pmatrix} E + \alpha & w \\ w^* & -E - \alpha \end{pmatrix}.$$

Let $E > 0$ and α be the dispersion parameter.

$$\forall \alpha \in [\alpha_0, \alpha_1], H^\alpha(w) = \begin{pmatrix} E + \alpha & w \\ w^* & -E - \alpha \end{pmatrix}.$$

Population transfer : do we have ensemble controllability from the ground state (\bar{e}_2) towards the exited state (\bar{e}_1)?

Let $E > 0$ and α be the dispersion parameter.

$$\forall \alpha \in [\alpha_0, \alpha_1], H^\alpha(w) = \begin{pmatrix} E + \alpha & w \\ w^* & -E - \alpha \end{pmatrix}.$$

Population transfer : do we have ensemble controllability from the ground state (\bar{e}_2) towards the exited state (\bar{e}_1) ?

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \forall \alpha \in \mathcal{D}, \exists \theta \in \mathbb{R},$$
$$\|\psi^\alpha(T) - e^{i\theta} e_1\| \leq \varepsilon$$

with

$$i\partial_t \psi^\alpha(t) = H^\alpha(w(t))\psi^\alpha(t), \quad \psi^\alpha(0) = e_2$$

Let $E > 0$ and α be the dispersion parameter.

$$\forall \alpha \in [\alpha_0, \alpha_1], H^\alpha(w) = \begin{pmatrix} E + \alpha & w \\ w^* & -E - \alpha \end{pmatrix}.$$

Population transfer : do we have ensemble controllability from the ground state (\bar{e}_2) towards the excited state (\bar{e}_1)?

$$\forall \varepsilon > 0, \exists T > 0, \exists w \in L^\infty([0, T], U), \forall \alpha \in \mathcal{D}, \exists \theta \in \mathbb{R},$$
$$\|\psi^\alpha(T) - e^{i\theta} e_1\| \leq \varepsilon$$

with

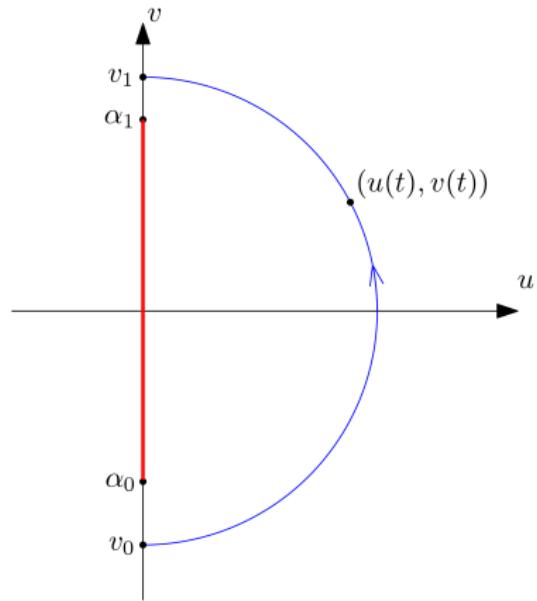
$$i\partial_t \psi^\alpha(t) = H^\alpha(w(t))\psi^\alpha(t), \quad \psi^\alpha(0) = e_2$$

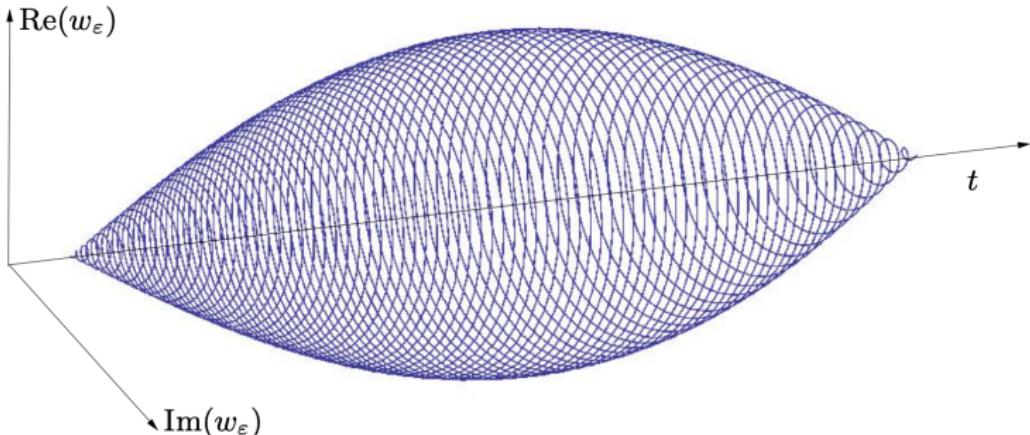
- Yes with a bounded complex control
- In most experiments, people 'duplicate' the real control thanks to the RWA.

Adiabatic control

$$w(t) = u(t)e^{-2i(Et + \int_0^t v(s)ds)}$$

Chirp:
frequency modulation between
 $2(E + v_0)$ and $2(E + v_1)$ together
with an amplitude modulation





Adiabatic theorem

There exists $C > 0$ (depending on the path) such that, for every $\alpha \in [\alpha_0, \alpha_1]$ and $\varepsilon > 0$, the solution ψ_ε^α of

$$i \frac{d\psi_\varepsilon^\alpha}{dt} = H^\alpha(u(\varepsilon t)e^{-2i(Et + \frac{1}{\varepsilon} \int_0^{\varepsilon t} v(s)ds)})\psi_\varepsilon^\alpha, \quad \psi_\varepsilon^\alpha(0) = e_2$$

satisfies $|\psi_\varepsilon^\alpha(1/\varepsilon) - (e^{i\theta}, 0)| \leq C\varepsilon$ for some θ .

Rotating wave approximation

Motivation

$$w_\varepsilon(t) = 2\varepsilon u(\varepsilon t) \cos \left(2Et + 2 \int_0^{\varepsilon t} v(s) ds \right)$$

$$w_\varepsilon^R(t) = \varepsilon u(\varepsilon t) \exp \left(-i(2Et + 2 \int_0^{\varepsilon t} v(s) ds) \right)$$

In the case $\alpha = 0$, the time evolution associated to the complex control in the interaction frame is:

$$i \frac{d\hat{\psi}_{w_\varepsilon}^R}{dt} = \varepsilon \begin{pmatrix} -v(\varepsilon t) & u(\varepsilon t) \\ u(\varepsilon t) & v(\varepsilon t) \end{pmatrix} \hat{\psi}_{w_\varepsilon}^R$$

Rotating wave approximation

Motivation

$$w_\varepsilon(t) = 2\varepsilon u(\varepsilon t) \cos \left(2Et + 2 \int_0^{\varepsilon t} v(s) ds \right)$$

$$w_\varepsilon^R(t) = \varepsilon u(\varepsilon t) \exp \left(-i(2Et + 2 \int_0^{\varepsilon t} v(s) ds) \right)$$

In the case $\alpha = 0$, the time evolution associated to the real control in the interaction frame is:

$$i \frac{d\hat{\psi}_{w_\varepsilon}}{dt} = \left[\varepsilon \begin{pmatrix} -v(\varepsilon t) & u(\varepsilon t) \\ u(\varepsilon t) & v(\varepsilon t) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & e^{4i(Et + \int_0^{\varepsilon t} v(s) ds)} u(\varepsilon t) \\ e^{-4i(Et + \int_0^{\varepsilon t} v(s) ds)} u(\varepsilon t) & 0 \end{pmatrix} \right] \hat{\psi}_{w_\varepsilon}$$

Rotating wave approximation

$$w_\varepsilon(t) = 2\varepsilon u(\varepsilon t) \cos \left(2Et + 2 \int_0^{\varepsilon t} v(s) ds \right),$$
$$w_\varepsilon^R(t) = \varepsilon u(\varepsilon t) \exp \left(-i(2Et + 2 \int_0^{\varepsilon t} v(s) ds) \right).$$

averaging

Let $\alpha = 0$, ψ_{w_ε} and $\psi_{w_\varepsilon}^R$ be the evolution of ψ_0 with controls w_ε and w_ε^R .
Then, there exists $C > 0$ such that for every $\varepsilon > 0$

$$\forall t \in [0, 1/\varepsilon], |\psi_{w_\varepsilon}(t) - \psi_{w_\varepsilon^R}(t)| < C\varepsilon.$$

Compatibility between the two approximations

RWA

$$w_\varepsilon(t) = 2\varepsilon u(\varepsilon t) \cos \left(2Et + 2 \int_0^{\varepsilon t} v(s) ds \right)$$
$$w_\varepsilon^R(t) = \varepsilon u(\varepsilon t) \exp \left(-i(2Et + 2 \int_0^{\varepsilon t} v(s) ds) \right)$$

have trajectories close at ε during a time $1/\varepsilon$.

AA

$$w_\varepsilon(t) = u(\varepsilon t) \exp \left(-i(2Et + \frac{2}{\varepsilon} \int_0^{\varepsilon t} v(s) ds) \right)$$

ensures a population transfer for all α up to an order ε in time $1/\varepsilon$.

First idea:

Take u small and use

$$w_\varepsilon(t) = u(\varepsilon t) 2 \cos \left(2Et + \frac{2}{\varepsilon} \int_0^{\varepsilon t} v(s) ds \right)$$

to simulate

$$u(\varepsilon t) \exp \left(-i(2Et + \frac{2}{\varepsilon} \int_0^{\varepsilon t} v(s) ds) \right)$$

Pb: Does not work in general

First idea:

Take u small and use

$$w_\varepsilon(t) = u(\varepsilon t) 2 \cos \left(2Et + \frac{2}{\varepsilon} \int_0^{\varepsilon t} v(s) ds \right)$$

to simulate

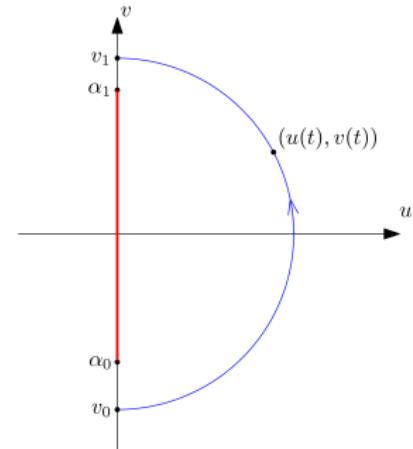
$$u(\varepsilon t) \exp \left(-i(2Et + \frac{2}{\varepsilon} \int_0^{\varepsilon t} v(s) ds) \right)$$

Pb: Does not work in general

dilemma

- RWA: small control and finite time not too long.
- AA: the smallest the control is, the longer the finite time should be.

$$w_{\varepsilon_1, \varepsilon_2}(t) = 2\varepsilon_1 u(\varepsilon_1 \varepsilon_2 t) \cos \left(2Et + \frac{2}{\varepsilon_1 \varepsilon_2} \int_0^{\varepsilon_1 \varepsilon_2 t} v(s) ds \right)$$



Theorem [R., Augier, Boscain, Sigalotti, *JDE*, 2022]

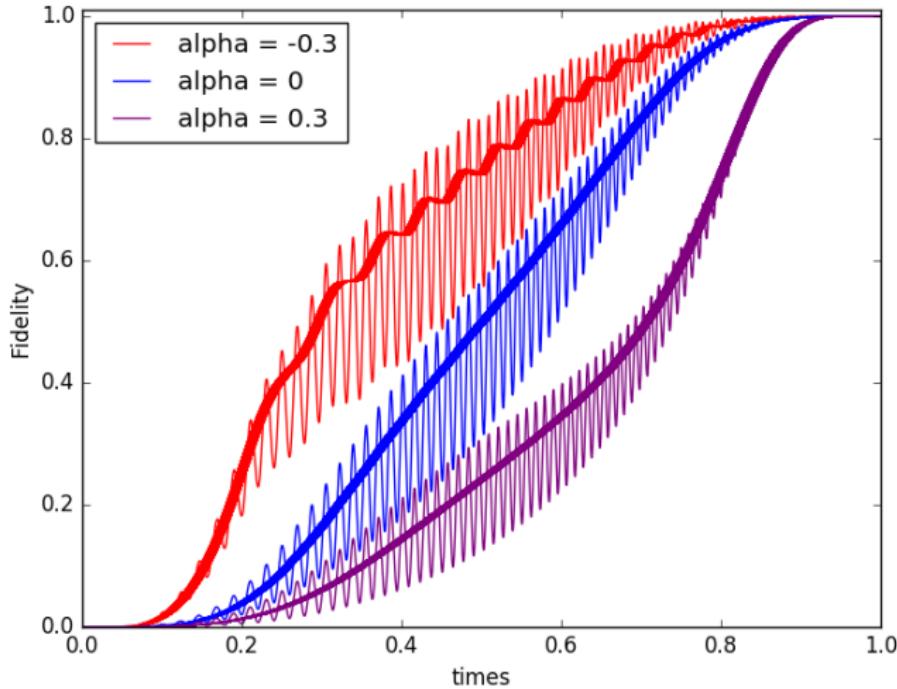
Assume $3(E + \alpha_0) > E + \alpha_1$. Then for every $N_0 \in \mathbb{N}$, there exists C_{N_0} such that

$$|\psi_{\varepsilon_1, \varepsilon_2}^\alpha(1/\varepsilon_1 \varepsilon_2) - (e^{i\theta}, 0)| < C_{N_0} \max(\varepsilon_2/\varepsilon_1, \varepsilon_1^{N_0-1}/\varepsilon_2), \quad \theta \in \mathbb{R}$$

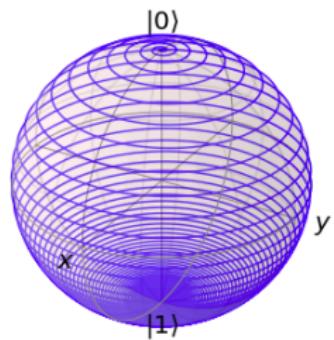
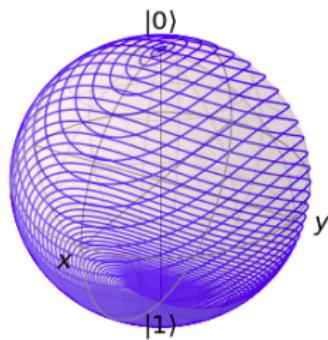
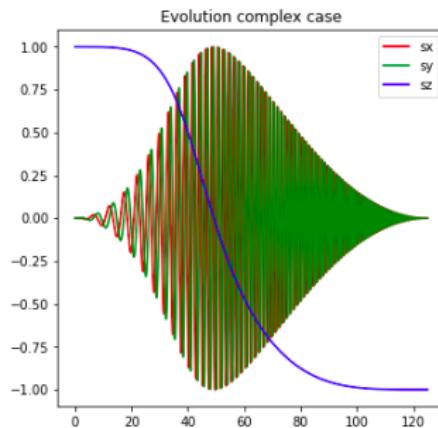
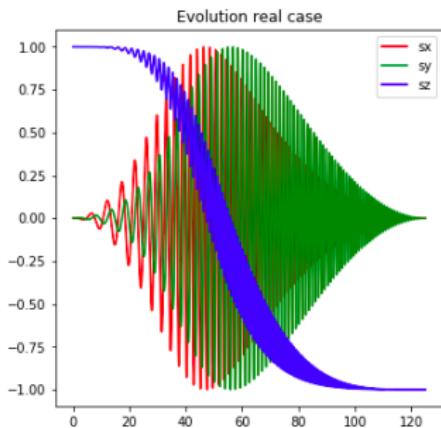
where $\psi_{\varepsilon_1, \varepsilon_2}^\alpha$ is the solution with control $w_{\varepsilon_1, \varepsilon_2}$.

Sketch of proof

- ① Using $3(E + \alpha_0) > E + \alpha_1$, we introduce an algorithm of change of variables corresponding to a kind of RWA up to N_0^{th} order
- ② Prove that the obtained Hamiltonian induces a dynamics close to first order RWA.
- ③ Prove that the first order RWA Hamiltonian ensures population transfer under the assumption $\varepsilon_2 \ll \varepsilon_1$



$\varepsilon_1 = 0.5$ and $\varepsilon_2 = 0.1$. Adiabatic trajectories are plotted in thick lines.



Bloch's sphere trajectories.

A controllability result

Let $\mathcal{D} = [\alpha_0, \alpha_1] \times [\delta_m, \delta_M]$ be a compact set and define
 $\mathcal{F} := C^0(\mathcal{D}, \mathrm{SU}_2)$. $d_{\mathcal{F}}(f, g) := \max_{d \in \mathcal{D}} \|f(d) - g(d)\|$.

Theorem [Li–Khaneja 2006, see also Beauchard–Coron–Rouchon 2010]

For every control bound $K > 0$, target distribution $M_F \in \mathcal{F}$ and precision $\varepsilon > 0$, there exist $T > 0$ and some controls $u, v \in L^\infty([0, T], [-K, K])$ such that the solution of

$$\begin{cases} i \frac{d}{dt} M(\alpha, \delta, t) = ((E + \alpha)\sigma_z + \delta u(t)\sigma_x + \delta v(t)\sigma_y)M(\alpha, \delta, t) \\ M(\alpha, \delta, 0) = I_2, \quad (\alpha, \delta) \in \mathcal{D} \end{cases}$$

satisfies $d_{\mathcal{F}}(M(\cdot, \cdot, T), M_F(\cdot, \cdot)) < \varepsilon$.

A generalisation of Li–Khaneja

Theorem [R., Augier, Boscain, Sigalotti, *JDE*, 2022]

Assume $3(E + \alpha_0) > E + \alpha_1$, set

$$\mathcal{D} = [\alpha_0, \alpha_1] \times [\delta_m, \delta_M] \subset (-E, +\infty) \times \mathbb{R}_+^*.$$

Fix $\epsilon > 0$, $M_F \in \mathcal{F}$ and $K > 0$. Then, there exist $T > 0$ and $u \in L^\infty([0, T], [-K, K])$ such that the solution of

$$\begin{cases} i \frac{d}{dt} M(\alpha, \delta, t) = ((E + \alpha)\sigma_z + \delta u(t)\sigma_x)M(\alpha, \delta, t) \\ M(\alpha, \delta, 0) = I_2, \quad (\alpha, \delta) \in \mathcal{D} \end{cases}$$

satisfies $\|M(\cdot, \cdot, T) - M_F(\cdot, \cdot)\|_{\mathcal{F}} < \epsilon$.

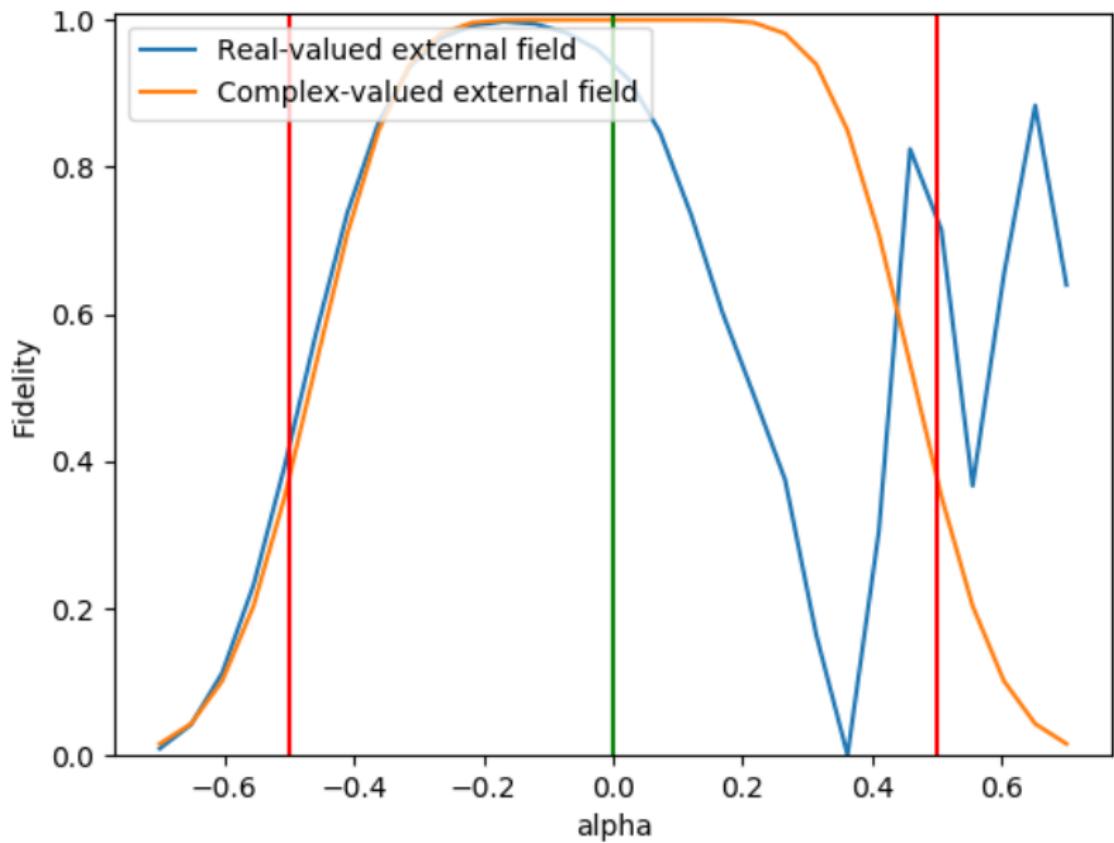
A few open questions

- Under the assumption $3(E + \alpha_0) > E + \alpha_1$, can we prove the population transfer with ε_1 fixed?
- Can we extend the compatibility between RWA and AA to higher dimensions ?
- Can we extend the controllability result without the assumption $3(E + \alpha_0) > E + \alpha_1$?
- Is there more efficient methods to ensure ensemble controllability (better controllability time...)?

A few open questions

- Under the assumption $3(E + \alpha_0) > E + \alpha_1$, can we prove the population transfer with ε_1 fixed?
- Can we extend the compatibility between RWA and AA to higher dimensions ?
- Can we extend the controllability result without the assumption $3(E + \alpha_0) > E + \alpha_1$?
- Is there more efficient methods to ensure ensemble controllability (better controllability time...)?

Thank you for your attention



Reachable propagators :

$$\mathcal{R} = \{M(\cdot, \cdot, T) \mid T > 0, M \text{ solution pour } u \in L^\infty([0, T], [-K, K])\}$$

- $\forall t \in \mathbb{R}, (\alpha, \delta) \mapsto e^{-it(E+\alpha)\sigma_z} \in \bar{\mathcal{R}}$
- $\forall u \in \mathbb{R}, (\alpha, \delta) \mapsto e^{u\delta i\sigma_x} \in \bar{\mathcal{R}}$

An (infinite dimensional) Lie Algebra

$$\mathfrak{g} = \{X \in \mathcal{C}^0(\mathcal{D}, \mathfrak{su}_2) \mid \forall t \in \mathbb{R}, e^{tX} \in \bar{\mathcal{R}}\}$$

\mathfrak{g} is a linear subspace stable under brackets.

A corollary of AA+RWA

Assume $3(E + \alpha_0) > E + \alpha_1$, then for every $K > 0$ and $\varepsilon > 0$, there exist $T > 0$ and a control $u \in L^\infty([0, T], [-K, K])$ such that the solution of

$$\begin{cases} i \frac{d}{dt} M(\alpha, \delta, t) = ((E + \alpha)\sigma_z + \delta u(t)\sigma_x)M(\alpha, \delta, t) \\ M(\alpha, \delta, 0) = I_2, \quad (\alpha, \delta) \in \mathcal{D} \end{cases}$$

satisfies $\max_{(\alpha, \delta) \in \mathcal{D}} \min_{\theta \in [0, 2\pi]} \|M(\alpha, \delta, T)(0, 1)^T - (e^{i\theta}, 0)^T\| < \varepsilon$.

Adiabatic Approximation

Adiabatic Approximation

Let $H_\alpha(u, v) = (\alpha - v)\sigma_z + u\sigma_x$. The eigenvalues of $H_\alpha(u, v)$ are $\pm\sqrt{(\alpha - v)^2 + u^2}$.

Initial and finite time Hamiltonian

$$H_\alpha(0, v_0) = \begin{pmatrix} \alpha - v_0 & 0 \\ 0 & -\alpha + v_0 \end{pmatrix}$$

$$H_\alpha(0, v_1) = \begin{pmatrix} \alpha - v_1 & 0 \\ 0 & -\alpha + v_1 \end{pmatrix}$$

