

Quantum Rotor Codes

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Few Physical Systems are Qubits

Theoreticians like to work with **exact qubits**.

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Experimenters often need to work to **isolate a two level system** within a very large Hilbert space such as

- Bosonic modes
- Atom spectrum
- Ion spectrum + vibrational modes
- ...

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⚠ Leakage Errors

Need special care which can be costly using for instance “leakage reduction units” (more or less teleportation gadget with fresh ancilla).

Hardware Error Correction

Consider errors in the full Hilbert space (as full as you can handle).

¹Christophe Vuillot, Hamed Asasi, Yang Wang, Leonid P. Pryadko, and Barbara M. Terhal Phys. Rev. A 99, 032344

²Lisa Hänggli, Robert König, IEEE TIT, 68, 2, Feb 2022

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Continuous Variable Error Correction

n bosonic modes $\rightarrow k < n$ logical bosonic modes

- Symplectic (Gaussian) encoding \Rightarrow No protection¹
- Modular measurements \Rightarrow Protection limited by squeezing²

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Bosonic Codes: GKP, Cat, Binomial ...

1 bosonic mode \longrightarrow 1 qudit

- Very good qubits but no scaling
- Challenging realization

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This Work

We consider **quantum rotors** as our physical system

Features

- Naturally present in superconducting circuits for instance
- Possibly interesting even for symplectic encodings
- Links with integer (co)homology

Outline

Quantum Rotors

Homology for Quantum Rotor Codes

Definitions

Noise Models and Distances

Towards Physical Realizations

Protected Superconducting Qubits

Hilbert Space $\mathcal{H}_{\mathbb{Z}}$

- Orthonormal Basis

$$\forall l \in \mathbb{Z}, |l\rangle \in \mathcal{H}_{\mathbb{Z}}$$

$$\dots \quad | -2 \rangle \quad | -1 \rangle \quad | 0 \rangle \quad | 1 \rangle \quad | 2 \rangle \quad | 3 \rangle \quad \dots$$

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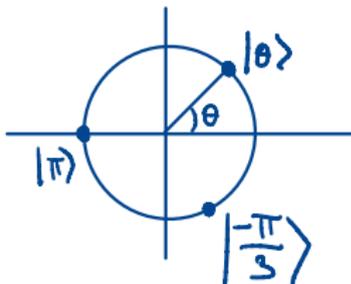
- States

$$|\psi\rangle = \sum_{l \in \mathbb{Z}} \alpha_l |l\rangle, \quad \sum_{l \in \mathbb{Z}} |\alpha_l|^2 = 1$$

Dual Space $\mathcal{H}_{\mathbb{T}}$

States

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad |\psi\rangle = \int_{\theta \in \mathbb{T}} d\theta \psi(\theta) |\theta\rangle, \quad \int_{\theta \in \mathbb{T}} d\theta |\psi(\theta)|^2 = 1$$



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Fourier Series

$$\forall |\psi\rangle \in \mathcal{H}_{\mathbb{Z}}, \forall \theta \in \mathbb{T}, \quad \tilde{\psi}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} \alpha_{\ell} e^{i\theta\ell}$$

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Phase States

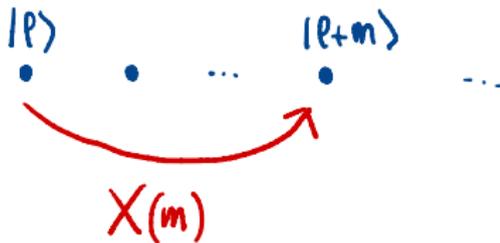
$$\forall \theta \in \mathbb{T}, \quad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathbb{Z}} e^{-i\theta\ell} |\ell\rangle$$

Generalized Pauli Operators

Pauli X: Jumps

$$\forall m \in \mathbb{Z}, \quad X(m) |\ell\rangle = |\ell + m\rangle$$

$$X(m) |\theta\rangle = e^{i\theta m} |\theta\rangle$$

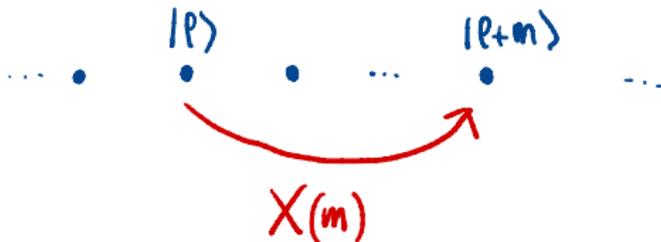


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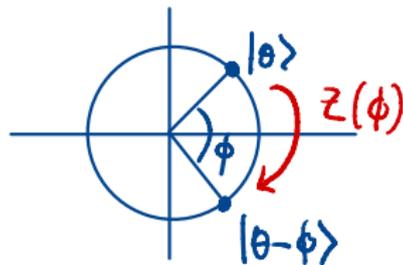
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$$\forall \phi \in \mathbb{T}, \quad Z(\phi) |\ell\rangle = e^{i\phi \ell} |\ell\rangle$$

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- $X(m)Z(\phi) = e^{-i\phi m} Z(\phi)X(m)$

Phase and Number Operators

Number Operator $\hat{\ell}$

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⚠ \mathbb{T} has no multiplication law: $\theta_1(\theta_2 + 2k\pi) = ?$

The operator $\hat{\theta}$ can only appear in 2π -periodic functionals

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Multi-Rotor Pauli Operators

$$\mathbf{m} \in \mathbb{Z}^n, \quad X(\mathbf{m}) = \prod_{j=1}^n X_j(m_j)$$

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Quantum Rotors in Real Life ?³

- Diatomic molecule trapped in a plane
- trapped 2-ions cristal
- orbital angular momentum of light
- **phase of a superconducting island**

³Robust encoding of a qubit in a molecule Victor V. Albert, Jacob P. Covey, John Preskill, Phys. Rev. X 10, 031050

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Definition

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The corresponding quantum rotor code is defined as

$$\mathcal{C}^{\text{rot}}(H_X, H_Z) = \{|\psi\rangle \mid \forall P \in \mathcal{S}, P|\psi\rangle = |\psi\rangle\}$$

Commutation and Small Example

Stabilizers Commute

$$S_X(\mathbf{s})S_Z(\phi) = e^{-i\phi \cancel{H_Z} H_X^T \mathbf{s}^T} S_Z(\phi)S_X(\mathbf{s})$$

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4-Rotors Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \quad H_Z = (1 \ 1 \ 1 \ 1)$$

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$$\mathcal{S} = \left\langle X_1(m)X_2^\dagger(m), X_3(m)X_4^\dagger(m), X_1^\dagger(m)X_2^\dagger(m)X_3(m)X_4(m), \right. \\ \left. Z_1(\phi)Z_2(\phi)Z_3(\phi)Z_4(\phi) \right\rangle_{m \in \mathbb{Z}, \phi \in \mathbb{T}}$$

Code States

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Z Constraints

$$\begin{aligned}\forall \phi, |\bar{\psi}\rangle &= S_Z(\phi) |\bar{\psi}\rangle \\ \Rightarrow \sum_{\ell \in \mathbb{Z}^n} \alpha_{\ell} |\ell\rangle &= \sum_{\ell \in \mathbb{Z}^n} e^{i\phi H_Z \cdot \ell^T} \alpha_{\ell} |\ell\rangle \\ \Rightarrow \forall \ell, \alpha_{\ell} \neq 0 &\Rightarrow \ell \in \ker(H_Z).\end{aligned}$$

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Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \quad H_Z = (1 \quad 1 \quad 1 \quad 1)$$

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$$\mathbf{x} = (0 \ -1 \ +1 \ 0) \in \ker(H_Z)$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} = (0 \ -2 \ +2 \ 0) \in \text{im}(H_X)$$

Homology

Chain Complex

$$\mathcal{C} : \quad C_2 \quad \xrightarrow{\partial} \quad C_1 \quad \xrightarrow{\sigma} \quad C_0 \quad \text{with } \sigma \circ \partial = 0$$

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$$\begin{array}{ccc} & H_X & \\ & & H_Z^T \end{array}$$

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Homology Group = X Logical Operators

$$\begin{aligned}
 H_1(\mathcal{C}, \mathbb{Z}) &= \ker \sigma / \text{im} \partial = \ker (H_Z) / \text{im} (H_X) \\
 &= F \oplus T = \mathbb{Z}^{k'} \oplus (\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_{k''}}) \\
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$$\forall \mathbf{m} \in \mathcal{L}_X, \quad \bar{X}(\mathbf{m}) = X(\mathbf{m}L_X + \mathbf{s}H_X), \quad L_X \in \mathbb{Z}^{(k'+k'') \times n}$$

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where

$$\begin{aligned} C_j^* &= \text{Hom}(C_j, \mathbb{T}) \\ \partial_j^* &: \begin{array}{ccc} C_{j-1}^* & \longrightarrow & C_j^* \\ \phi & \longmapsto & \phi \circ \partial \end{array} \end{aligned}$$

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$$\text{Hom}(\mathbb{Z}, \mathbb{T}) \simeq \mathbb{T}, \quad \partial^* = H_X^T, \quad \sigma^* = H_Z$$

Our Case

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Cohomology Group = Z Logical Operators

$$\begin{aligned}
 H^1(\mathcal{C}, \mathbb{T}) &= \ker \partial^* / \text{im} \sigma^* = \ker (H_X) / \text{im} (H_Z) \\
 &= \mathbb{T}^{k'} \oplus \left(\mathbb{Z}_{d_1}^* \oplus \cdots \oplus \mathbb{Z}_{d_{k''}}^* \right) \\
 &= \mathcal{L}_Z
 \end{aligned}$$

$$\forall \phi \in \mathcal{L}_Z, \bar{Z}(\phi) = Z(\phi L_Z + \nu H_Z), \quad L_Z \in \mathbb{Z}^{(k'+k'') \times n}$$

Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \quad H_Z = (1 \quad 1 \quad 1 \quad 1)$$

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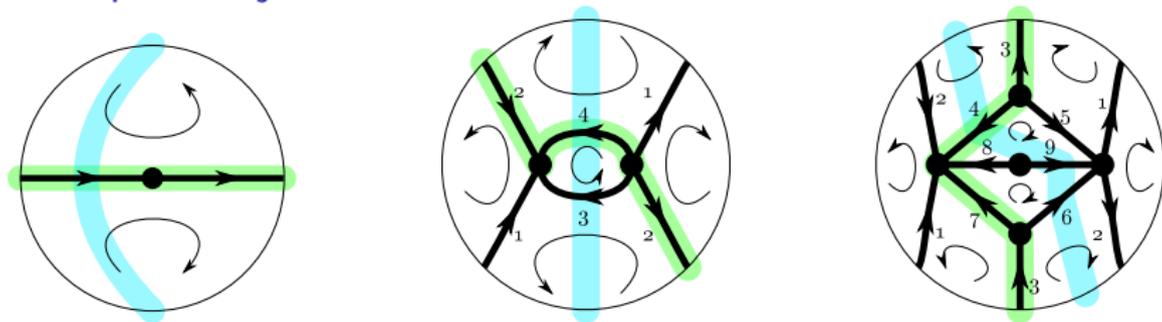
A Logical Qubit

$$\bar{X} = X((0 \ -1 \ +1 \ 0)), \quad \bar{Z} = Z(\pi(1 \ 1 \ 0 \ 0))$$

Codes from Cellular Homology

$$\begin{array}{ccccccc}
 \mathcal{C} : & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\sigma} & C_0 & \text{with } \sigma \circ \partial = 0 \\
 & \parallel & & \parallel & & \parallel & \\
 & \mathbb{Z}^F & & \mathbb{Z}^E & & \mathbb{Z}^V & \\
 & \parallel & & \parallel & & \parallel & \\
 & \text{faces} & & \text{edges} & & \text{vertices} &
 \end{array}$$

Example: Projective Plane



Projective Plane (Co)Homology

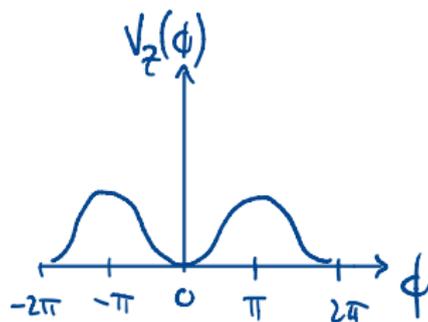
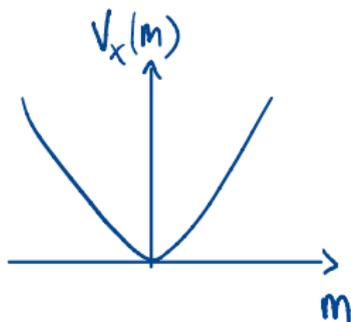
Coefficients	Homology			Cohomology		
	$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\sigma} C_0$			$C_2^* \xleftarrow{\partial^*} C_1^* \xleftarrow{\sigma^*} C_0^*$		
\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}_2	0	\mathbb{Z}
\mathbb{T}	\mathbb{Z}_2	0	\mathbb{T}	0	\mathbb{Z}_2	\mathbb{T}
\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
\mathbb{Z}_3	0	0	\mathbb{Z}_3	0	0	\mathbb{Z}_3
\mathbb{R}	0	0	\mathbb{R}	0	0	\mathbb{R}

Noise Models

Pauli Noise

$$\forall m \in \mathbb{Z}, \mathbb{P}(X(m)) = N_X \exp(-\beta_X V_X(m)),$$

$$\forall \phi \in \mathbb{T}, \mathbb{P}(Z(\phi)) = N_Z \exp(-\beta_Z V_Z(\phi)).$$



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Possible Choice

$$V_Z(\phi) = \sin^2\left(\frac{\phi}{2}\right) \qquad \beta_Z = \frac{1}{\sigma^2}$$

$$V_X(m) = |m| \qquad \beta_X = -\log p$$

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Weight Function

$$W_Z(\phi) = \sum_{j=1}^n V_Z(\phi_j) = \sum_{j=1}^n \sin^2\left(\frac{\phi_j}{2}\right)$$

$$W_X(\mathbf{m}) = \sum_{i=1}^n V_X(m_i) = \|\mathbf{m}\|_1$$

Distances

X Distance

$$d_X = \min_{\mathbf{m} \neq \mathbf{0}} \min_{\mathbf{s} \in \mathbb{Z}^{f_X}} W_X(\mathbf{m}L_X + \mathbf{s}H_X)$$

Distances

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Z Distances

$$d_Z = \min_{\phi} \min_{\nu \in \mathbb{T}^{f_Z}} W_Z(\phi L_Z + \nu H_Z)$$

$$\delta_Z = \min_{\phi} \min_{\nu \in \mathbb{T}^{f_Z}} \frac{W_Z(\phi L_Z + \nu H_Z)}{W_Z(\phi)}$$

X Bound

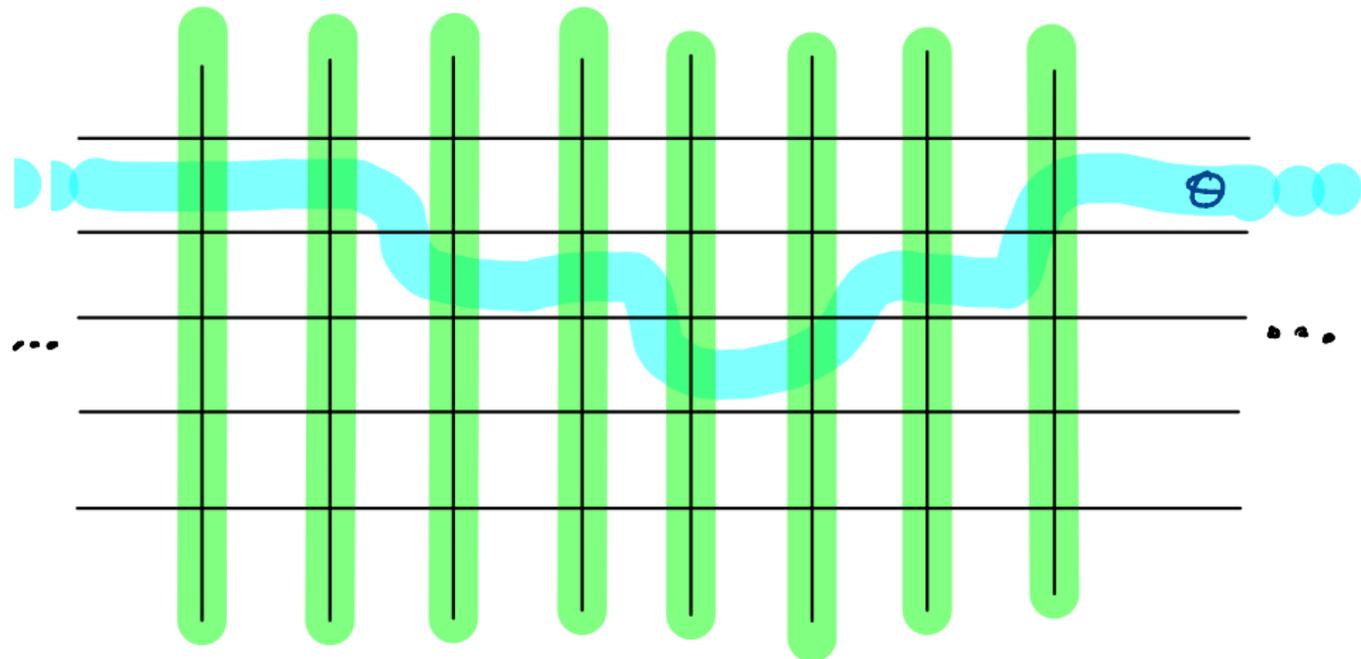
X Distance

Given a quantum rotor code $\mathcal{C}^{\text{rot}}(H_X, H_Z)$, denote as d_X^p the X distance of the corresponding qubit code $\mathcal{C}^p(H_X, H_Z)$, then

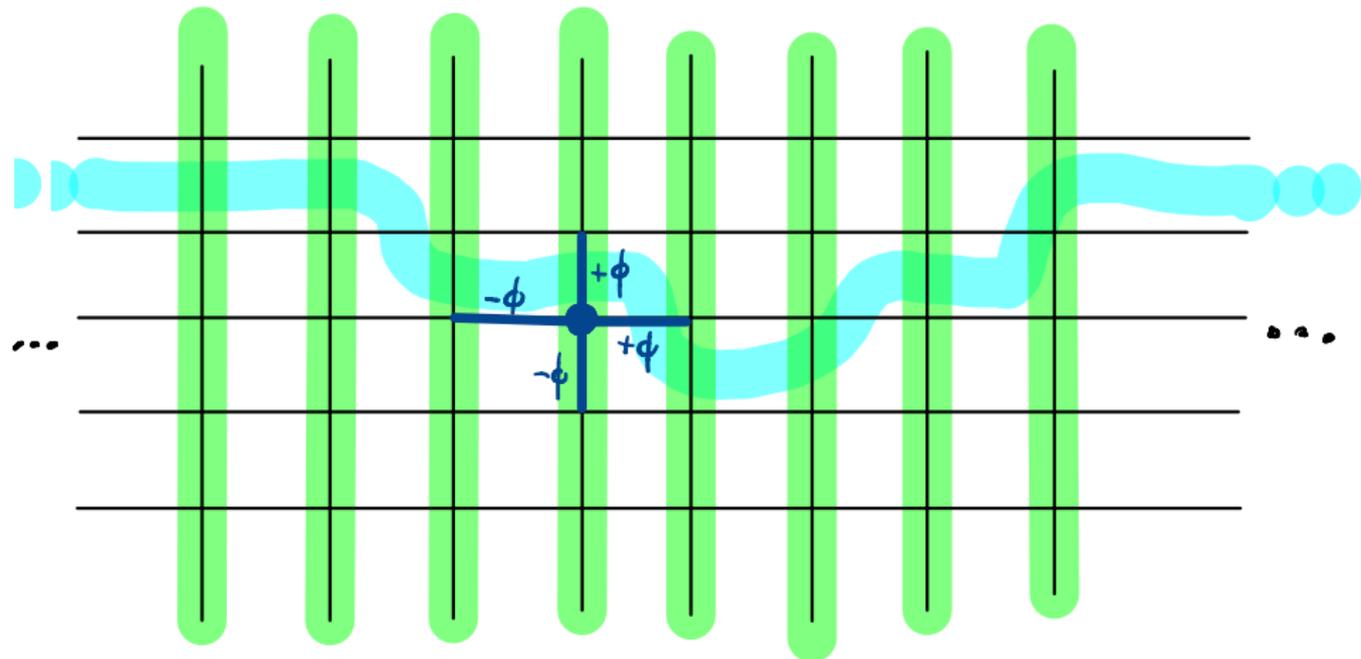
$$d_X \geq \max_{p \in P} d_X^p,$$

where P is the set of qubit dimensions for which there exists a logical X of minimal weight in \mathcal{C}^{rot} non trivial in \mathcal{C}^p .

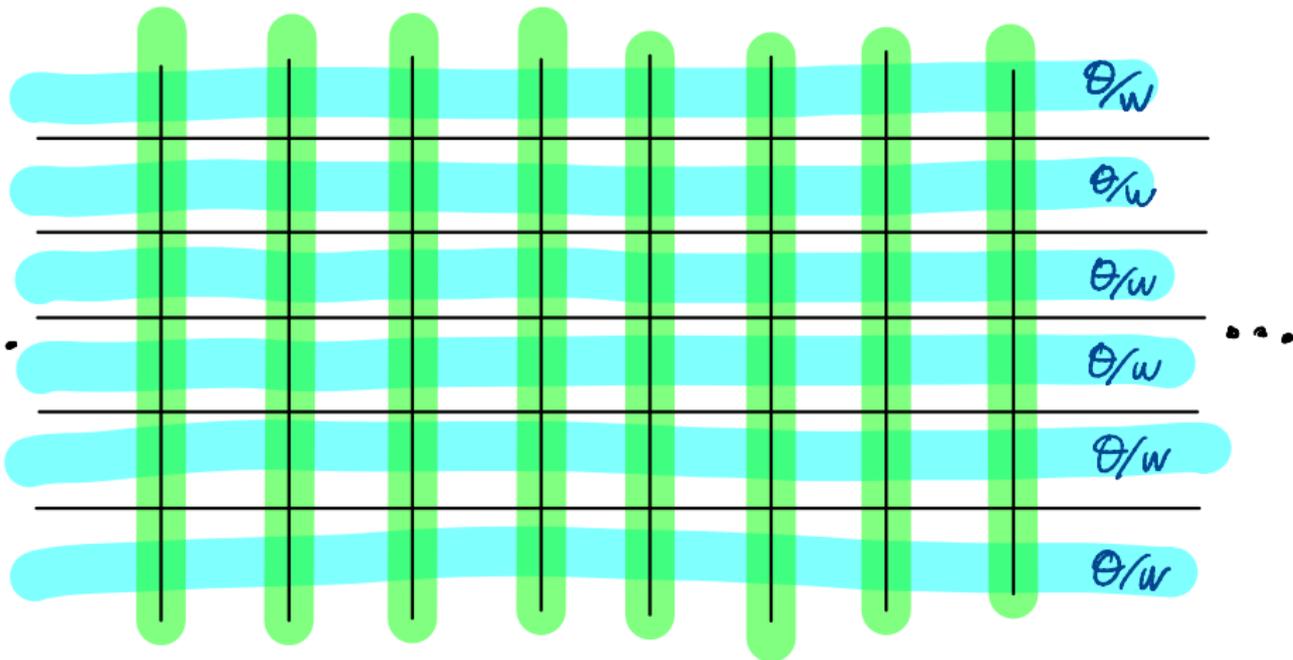
Z Bound and Disjointness



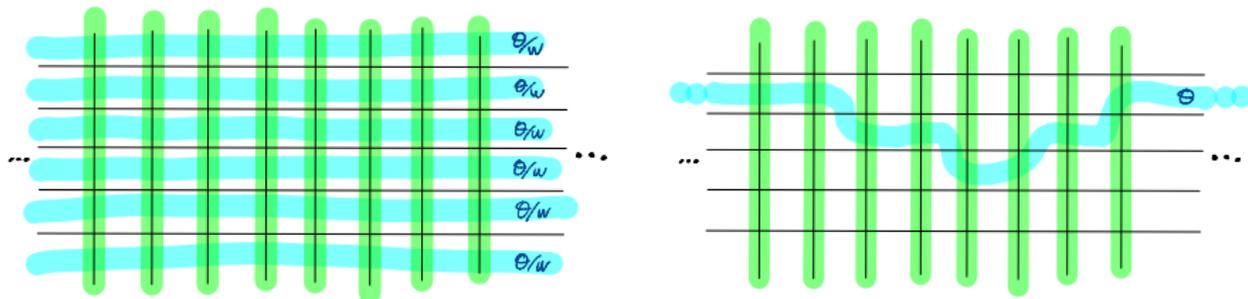
Z Bound and Disjointness



Z Bound and Disjointness



Z Bound and Disjointness

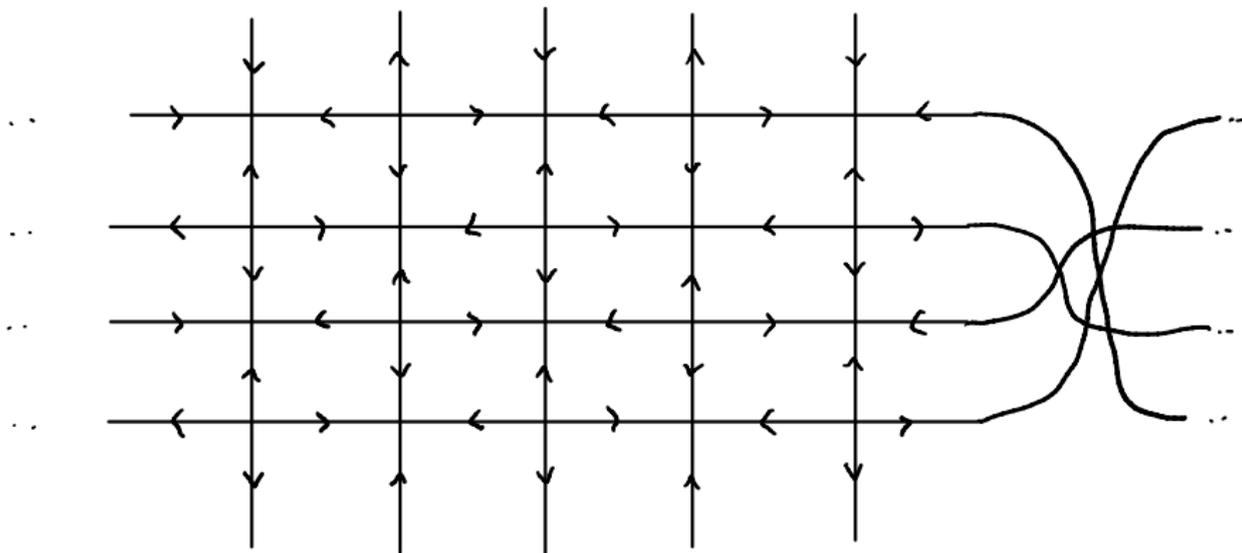


$$LW \sin^2 \frac{\theta}{2w} \leq d_z \leq L \sin^2 \frac{\theta}{2}$$

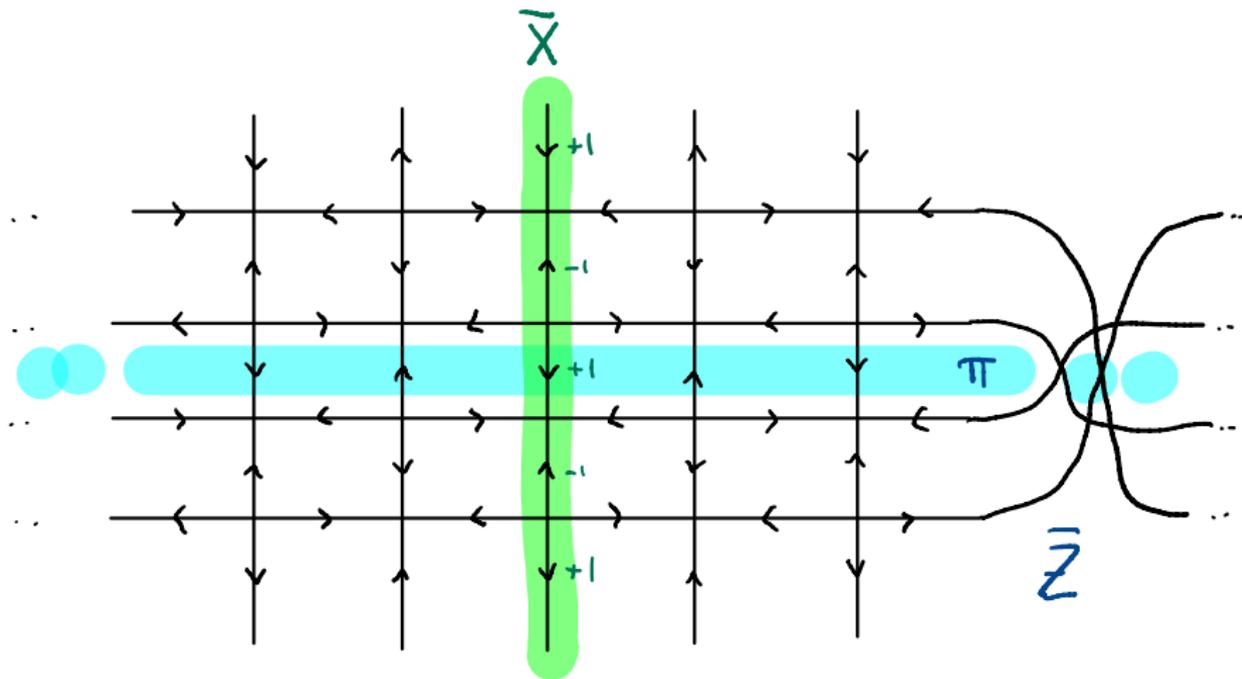
$$L = \# \text{Disjoint set}$$

$$w = d_x$$

Rough Moebius Strip



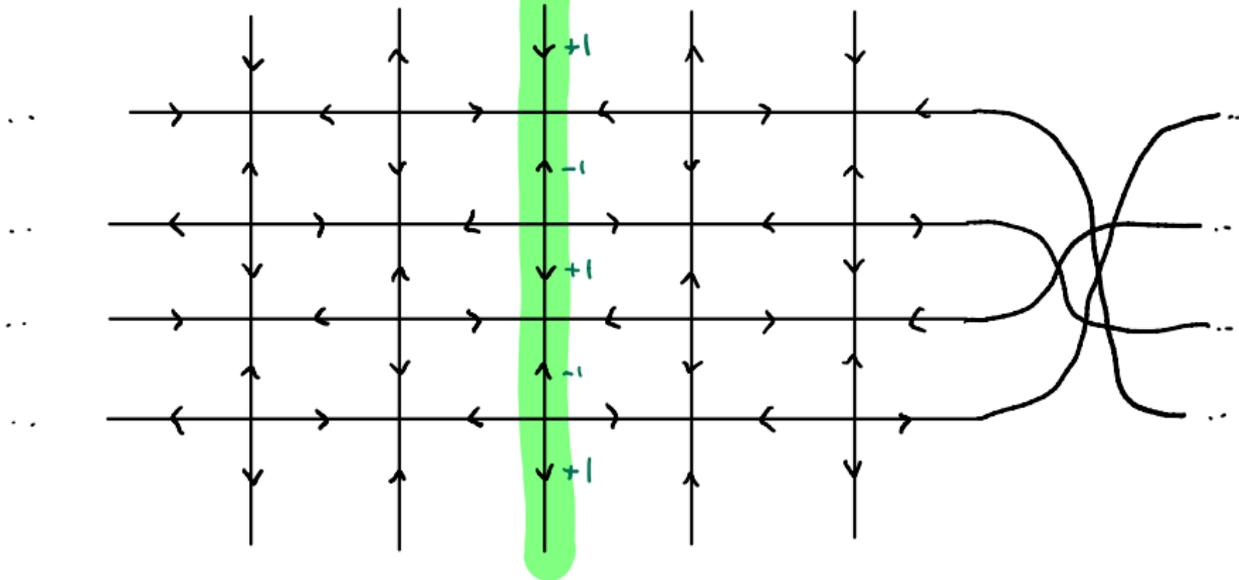
Rough Moebius Strip Distances



Rough Moebius Strip Distances

$$d_x = W$$

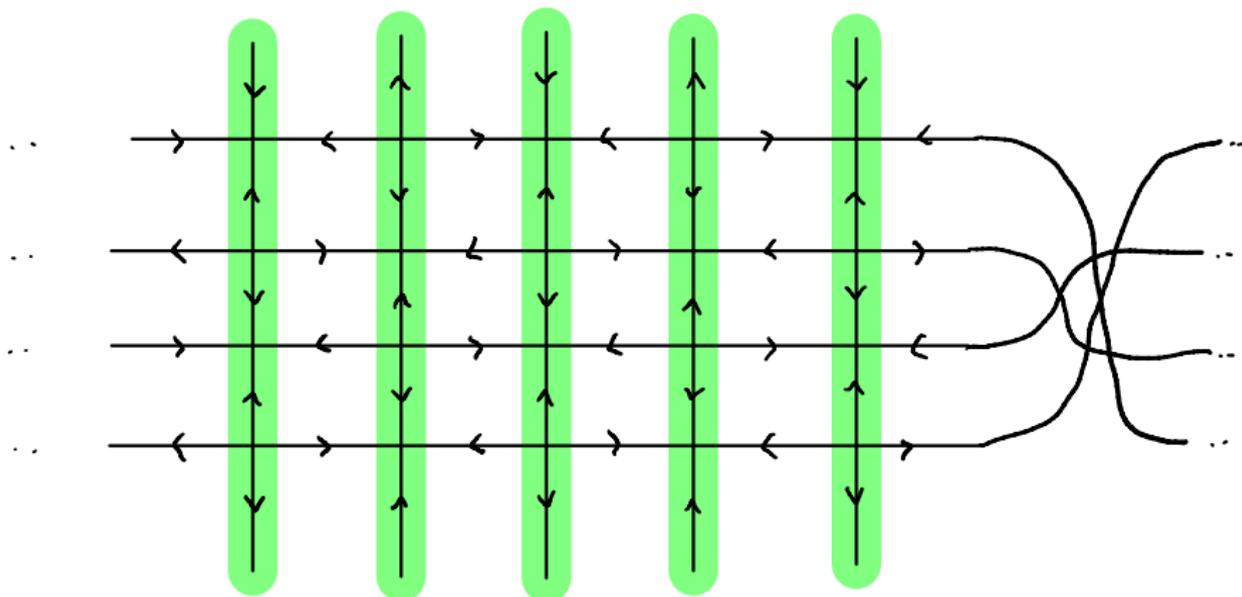
\bar{X}



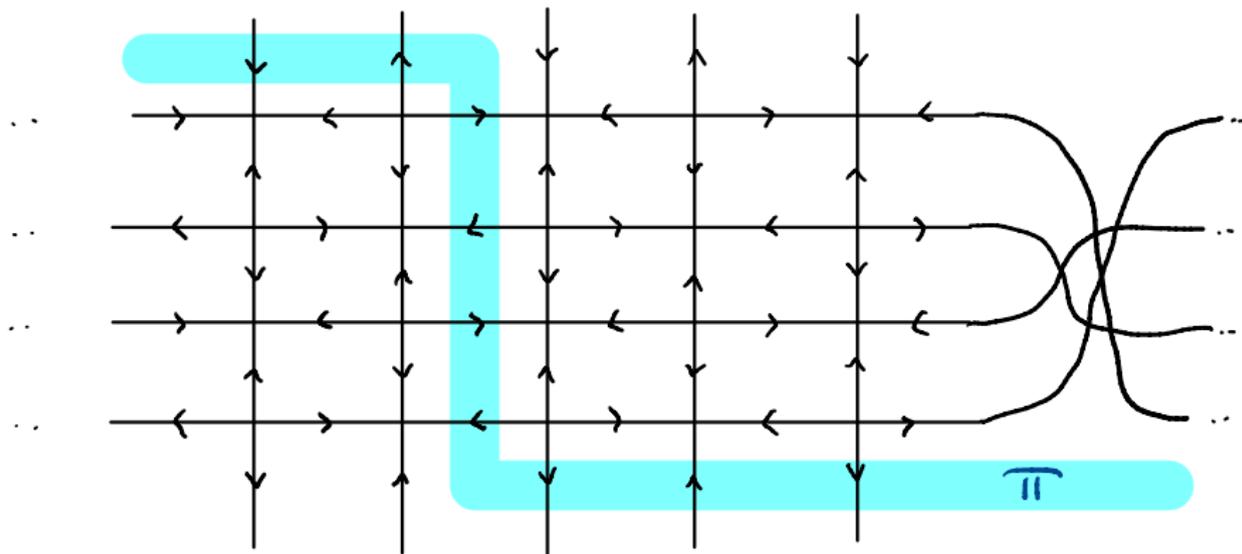
Rough Moebius Strip Distances

Disjoint set of L logical $X \Rightarrow$

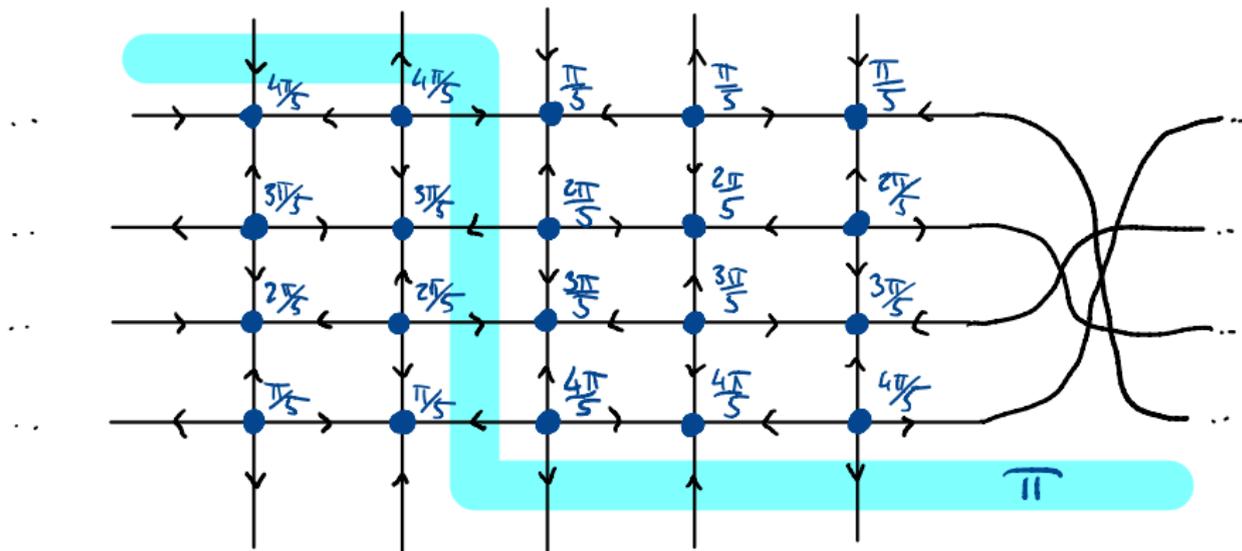
$$d_Z \geq LW \sin \frac{2\pi}{2W}$$



Rough Moebius Strip Distances

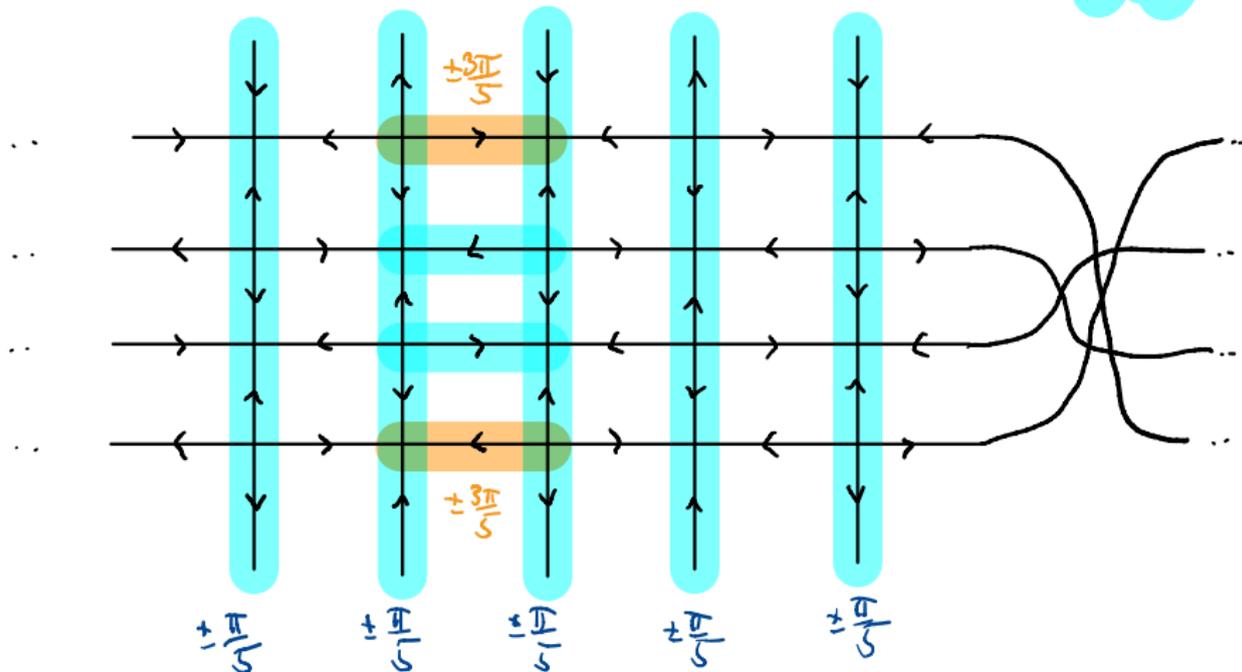


Rough Moebius Strip Distances



Rough Moebius Strip Distances

$$d_z \leq Lw \sin^2 \frac{\pi}{2w} + O(1)$$



Rough Moebius Strip Distances

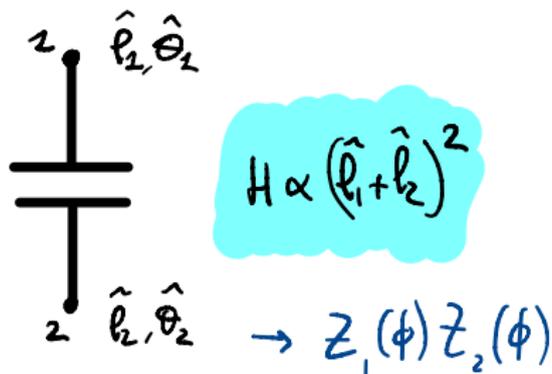
$$d_x = w$$

$$Lw \sin^2 \frac{\pi}{2w} \leq d_z \leq Lw \sin^2 \frac{\pi}{2w} + O(1)$$

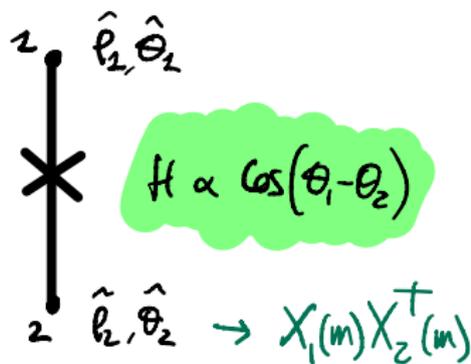
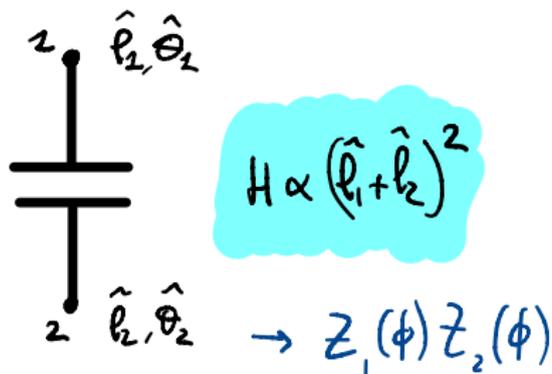
$$d_z \underset{w \rightarrow \infty}{\sim} \frac{\pi^2}{4} \frac{L}{w}$$

pick $L = w^2$

Superconducting Circuits and Rotors

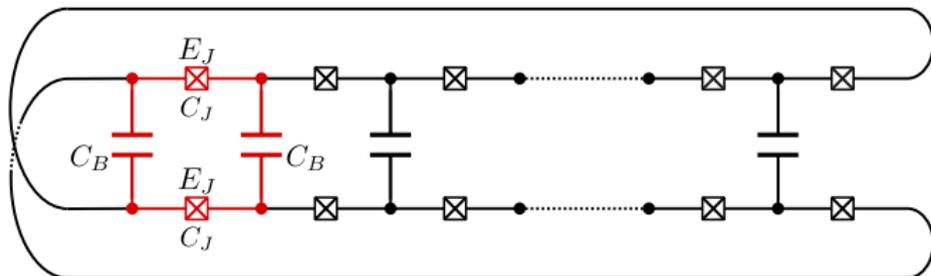


Superconducting Circuits and Rotors

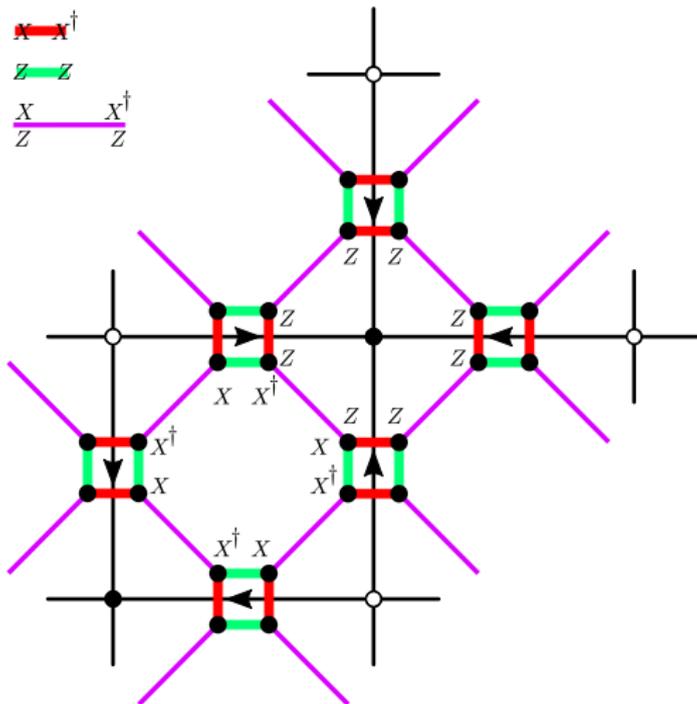


0 - π Qubits

$$\sim \cos(\theta_2 - \theta_1 + \theta_3 - \theta_4)$$

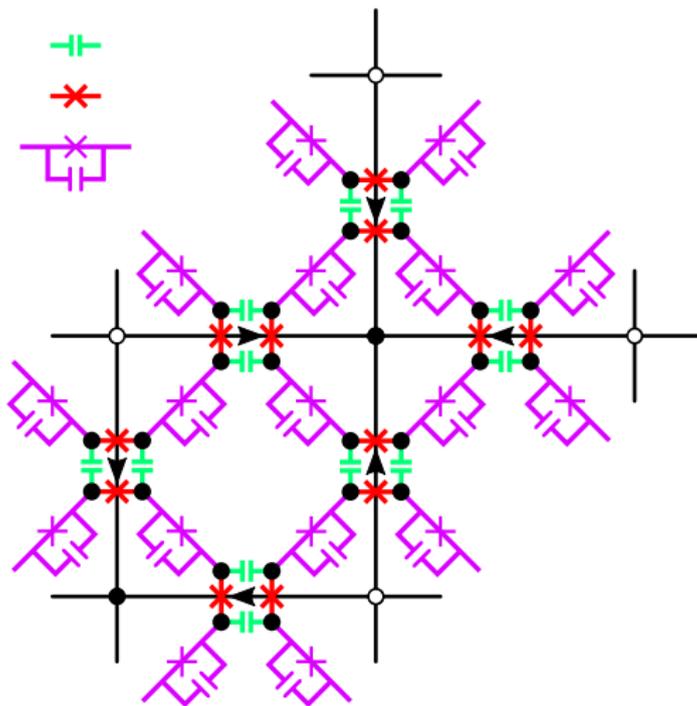


Perturbative Hamiltonian⁴



⁴C. G. Brell, S. T. Flammia, S. D. Bartlett, and A. C. Doherty, New Journal of Physics 13, 053039 (2011)

Superconducting Circuit



Summary and Future Directions

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- Defined Quantum Rotor Codes
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- Potentially generalize protected superconducting qubits

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Future Directions

- Possibility of protected logical quantum rotor in 3D
- Relations between torsion, disjointness, systolic freedom
- Active physical realizations