Unimodular Hausdorff and Minkowski Dimensions

François Baccelli[*], Mir-Omid Haji-Mirsadeghi[†], and Ali Khezeli[‡]

February 3, 2020

Abstract

This work introduces two new notions of dimension, namely the *unimodular Minkowski and Hausdorff dimensions*, which are inspired from the classical analogous notions. These dimensions are defined for *unimodular discrete spaces*, introduced this work, which provide a common generalization to stationary point processes under their Palm version and unimodular random rooted graphs. The use of unimodularity in the definitions of dimension is novel. Also, a toolbox of results is presented for the analysis of these dimensions. In particular, analogues of Billingsley's lemma and Frostman's lemma are presented. These lemmas are instrumental in deriving upper bounds on dimensions, whereas lower bounds are obtained from specific coverings. The notions of unimodular Hausdorff measure and unimodular dimension function are also introduced. This toolbox is used to connect the unimodular dimensions to other notions such as growth rate (various further connections will also be considered in future papers of the authors). It is also used to analyze the dimensions of a set of examples pertaining to point processes, branching processes, random graphs, random walks, and self-similar discrete random spaces. This work is structured in two papers, with the present paper being the first.

1 Introduction

Infinite discrete random structures are ubiquitous: random graphs, branching processes, point processes, graphs or zeros of discrete random walks, discrete or Euclidean percolation, to name a few. The main novelty of the present paper is the definition of new notions of dimension for a class of such structures that, heuristically, enjoy a form of statistical homogeneity. The mathematical framework proposed to handle such structures is that of *unimodular (random) discrete spaces*, where unimodularity is defined here by a straightforward version

[*]The University of Texas at Austin, baccelli@math.utexas.edu
[†]Sharif University of Technology, mirsadeghi@sharif.ir
[‡]Tarbiat Modares University, khezeli@modares.ac.ir

of the mass transport principle (the reader can restrict attention to unimodular graphs and skip the definition of unimodular discrete spaces at first reading). This framework unifies unimodular random graphs and networks, stationary point processes (under their Palm version) and point-stationary point processes. It does not require more than a metric; for instance, no edges or no underlying Euclidean space are needed. The statistical homogeneity of such spaces has been used to define localized versions of global notions such as, e.g., intensity. The main novelty of the present paper is the use of this homogeneity to define the notions of *unimodular Minkowski and Hausdorff dimensions*, which are inspired by the analogous classical notions. The definitions are obtained naturally by replacing the infinite sums pertaining to coverings by the expectation of certain random variables at the origin (which is a distinguished point). These definitions are local but capture large scale properties of the space.

The definitions are complemented by a toolbox for the analysis of unimodular dimensions. Several analogues of the important results known about the classical Hausdorff and Minkowski dimensions are established, like for instance the comparison of the unimodular Minkowski and Hausdorff dimensions as well as unimodular versions of Billingsley's lemma and Frostman's lemma. These lemmas allow one to connect the dimension to the (polynomial) growth rate of the space. Analogues of the Hausdorff measure are also defined for unimodular discrete spaces. This can be used to quantify the size of sets with the same dimension. The notion of unimodular dimension function is also defined for a finer quantification of the dimension. While many ideas in this toolbox are imported from the continuum setting, their adaptation is nontrivial and there is no automatic way to import results from the continuum to the discrete setting. For some results, the statements fundamentally differ from their continuum analog; e.g., the statement of Billingsley's lemma and the result that a subspace may have a larger Minkowski dimension in this setting.

Several notions of dimension are already defined in the literature for discrete spaces. For instance, one can mention the growth rate, the notion of *discrete dimension* for discrete subsets of Euclidean spaces [7] and Gromov's notion of *asymptotic dimension* [17]. Various other dimensions and exponents are also defined for graphs (see e.g., Subsection 1.1 below). Connections between unimodular dimensions and some of these earlier notions are clarified. Here are instances of these connections, established under conditions spelled out in the paper: It is shown that the discrete dimension [7] is an upper bound for the unimodular Hausdorff dimension when both notions are defined (i.e., for point-stationary point processes). If the polynomial growth rate exists, then it is equal to the unimodular Hausdorff dimension; If the space admits a scaling limit (such limits are random continuum metric spaces), then the ordinary Hausdorff dimension of the limit is an upper bound for the unimodular Hausdorff dimension. Note that these comparison results imply relations between the growth rate, scaling limits and discrete dimension, which are of independent interest and which are new to the best of the authors' knowledge.

The paper also contains new mathematical results of independent interest. A weak version of Birkhoff's pointwise ergodic theorem is stated for all unimodular discrete spaces. A unimodular version of the max flow min cut theorem is also proved for unimodular one-ended trees, which is used in the proof of the unimodular Frostman lemma. Also, for unimodular one-ended trees, a relation between the growth rate and the height of the root is established as explained below.

Finally, the framework is used to derive concrete results on the dimension of several instances of unimodular random discrete metric spaces. This is done for the zeros and the graph of discrete random walks, sets defined by digit restriction, trees obtained from branching processes and drainage network models, dimension doubling of the simple random walk, etc. Some general results are obtained for all unimodular trees. For instance, a general relation is established between the unimodular dimensions of a unimodular one-ended tree and the tail of the distribution of the height of the root. The dimensions of some unimodular discrete random fractals are also discussed. The latter are defined in this paper as unimodular discrete analogues of self similar sets such as the Koch snowflake, the Sierpinski triangle, etc.

This framework opens several further research directions. There is a large set of open questions on the dimensions of specific instances of such unimodular discrete spaces. In particular, this framework might be useful for the study of some examples which are of interest in mathematical physics (e.g., some unimodular spaces constructed from percolation clusters and self-avoiding random walks). Also, the definitions and many of the results are valid for exponential (or other) gauge functions. They might have applications in group theory (or other areas), where most interesting examples have super-polynomial growth.

1.1 Introduction to the Definitions of Dimension

Recall that the Minkowski dimension of a compact metric space X is defined using the minimum number of balls of radii ϵ needed to cover X. Now, consider a (unimodular) discrete space $\boldsymbol{D}$ (it is useful to have in mind the example $\boldsymbol{D} = \mathbb{Z}^k$ to see how the definitions work). It is convenient to consider coverings of $\boldsymbol{D}$ by balls of equal but large radii. Of course, if $\boldsymbol{D}$ is unbounded, then an infinite number of balls is needed to cover $\boldsymbol{D}$. So one needs another measure to asses how many balls are used in a covering. Let $S \subseteq \boldsymbol{D}$ be the set of centers of the balls in the covering. The idea pursued in this paper is that if $\boldsymbol{D}$ is unimodular, then the *intensity* of S is a measure of *the average number of points of S per points of* $\boldsymbol{D}$ (S should be *equivariant* for the intensity to be defined, as discussed later). This gives rise to the definition of the unimodular Minkowski dimension naturally.

The idea behind the definition of the unimodular Hausdorff dimension is similar. Recall that the α-dimensional *Hausdorff content* of a compact metric space X is defined by considering the infimum of $\sum_i R_i^\alpha$, where the R_i's are the radii of a sequence of balls that cover X. Also, it is convenient to force an upper bound on the radii. Now, consider a unimodular discrete space $\boldsymbol{D}$ and a covering of $\boldsymbol{D}$ by balls which may have different radii. Let $R(v)$ be the radius of the ball centered at v. It is convenient to consider a lower bound on the radii, say $R(\cdot) \geq$

1. Again, if $\boldsymbol{D}$ is unbounded, then $\sum_v R(v)^\alpha$ is always infinite. The idea is to leverage the unimodularity of $\boldsymbol{D}$ and consider *the average of the values* $R(\cdot)^\alpha$ *per point* as a replacement of the sum. Under the unimodularity assumption, this can be defined by $\mathbb{E}\left[R(\boldsymbol{o})^\alpha\right]$, where $\boldsymbol{o}$ stands for the distinguished point of $\boldsymbol{D}$ (called the origin) and by convention, $R(\boldsymbol{o})$ is zero if there is no ball centered at $\boldsymbol{o}$. This is used to define the *unimodular Hausdorff dimension* of $\boldsymbol{D}$ in a natural way.

The literature contains various definitions to study dimension for discrete structures. Here is a brief summary of those relevant in the present context. The connections of the earlier definitions with the proposed ones will be discussed in the next subsection. One is the *growth rate* of the cardinality of a large ball. Another is the *discrete dimension* [7] which uses the idea behind the definition of the Hausdorff dimension by considering coverings of $\Phi \subseteq \mathbb{R}^k$ by large balls and considering the cost $(\frac{r}{r+|x|})^\alpha$ for each ball in the covering, where r and x are the radius and the center of the ball and α is a constant (in fact, this is a modified version of the definition of [7] mentioned in [11]). Other definitions are the *spectral dimension* of a graph (defined in terms of the return probabilities of the simple random walk), the *typical displacement exponent* of a graph (see [13] for both notions), the *isoperimetric dimension* of a graph [12], the *resistance growth exponent* of a graph, the *stochastic dimension* of a partition of $\mathbb{Z}^k$ [8], etc.

In statistical physics, one also assigns dimension and various exponents to finite models. Famous examples are self-avoiding walks and the boundaries of large percolation clusters. More on the matter is provided in [4].

1.2 Organization of the Material and Summary of Results

The material is organized in two companion papers (the current paper and [5]) that will be referred to as Parts I and II respectively. The aim of this subsection is to give a brief summary of the main results and their localizations in the two parts.

Part I is centered on the framework, the definitions and the basic properties of unimodular dimensions. It also contains a large set of examples which will be continued in Part II. These examples stem from point process theory, random graphs, random walks, self-similarity or from analogues in the continuum. Section 2 defines unimodular discrete spaces and *equivariant processes*, which are needed throughout. Section 3 presents the definitions of the unimodular Minkowski and Hausdorff dimensions and the unimodular Hausdorff measure. It also provides some basic properties of unimodular dimensions as part of the toolbox for the analysis of unimodular dimensions. Various examples are discussed in Section 4.

Part II discusses the connections between the proposed dimensions and the *growth rate* of the space. Analogues of the *mass distribution principle* and Billingsley's lemma are presented, which provide upper bounds for the unimodular Hausdorff dimension in terms of the growth rate (weighted versions of these inequalities are also given). General lower bounds are presented as well. These results are very useful for calculating the unimodular dimensions in many examples. An important result in the opposite direction is an analogue of Frostman's lemma. Roughly speaking, it states that the mass distribution principle is sharp if the weights are chosen appropriately. This lemma is a powerful tool to study the unimodular Hausdorff dimension.

In the Euclidean case, another proof of Frostman's lemma is provided using a version of the max-flow min-cut theorem for unimodular one-ended trees, which is of independent interest. Part II also contains a section about examples that completes the examples discussed in Part I.

Some further topics are discussed in a preprint [4] which contains extensions of the notions of dimension and also their connections to classical notions.

2 Unimodular Discrete Spaces

The main objective of this section is the definition of unimodular discrete spaces as a common generalization of unimodular graphs, Palm probabilities and point-stationary point processes. If the reader is familiar with unimodular random graphs, he or she can restrict attention to the case of unimodular graphs and jump to Subsection 2.5 at first reading.

2.1 Notation

The following notation will be used throughout. The set of nonnegative real (resp. integer) numbers is denoted by $\mathbb{R}^{\geq 0}$ (resp. $\mathbb{Z}^{\geq 0}$). The minimum and maximum binary operators are denoted by $\wedge$ and $\vee$ respectively. The number of elements in a set A is denoted by $\#A$, which is a number in $[0,\infty]$. If $P(x)$ is a property about x, the indicator $1_{\{P(x)\}}$ is equal to 1 if $P(x)$ is true and 0 otherwise.

Discrete metric spaces (discussed in details in Subsection 2.2) are denoted by D, D', etc. Graphs are an important class of discrete metric spaces. So the symbols and notations are mostly borrowed from graph theory.

For $r > 0$, $N_r(v) := N_r(D,v)$ refers to the closed r-neighborhood of $v \in D$; i.e., the set of points of D with distance less than or equal to r from v. An exception is made for $r = 0$ (Subsection 3.3), where $N_0(v) := \emptyset$. The diameter of a subset A is denoted by $\mathrm{diam}(A)$. For a function $f : [1,\infty) \to \mathbb{R}^{\geq 0}$, the *polynomial growth rates* and *polynomial decay rates* are defined by the following formulas:

$$\begin{aligned}
\underline{\mathrm{growth}}\,(f) &:= -\overline{\mathrm{decay}}\,(f) := \liminf_{r\to\infty} \log f(r)/\log r,\\
\overline{\mathrm{growth}}\,(f) &:= -\underline{\mathrm{decay}}\,(f) := \limsup_{r\to\infty} \log f(r)/\log r,\\
\mathrm{growth}\,(f) &:= -\mathrm{decay}\,(f) := \lim_{r\to\infty} \log f(r)/\log r.
\end{aligned}$$

2.2 The Space of Pointed Discrete Spaces

Throughout the paper, the metric on any metric space is denoted by d, except when explicitly mentioned. In this paper, it is always assumed that the discrete metric space is **boundedly finite**; i.e., every set included in a ball of finite radius in D is finite. The term **discrete space** will always refer to boundedly finite discrete metric space. A **pointed set** (or a *rooted set*) is a pair (D, o), where D is a set and o a distinguished point of D called the **origin** (or the *root*) of D. Similarly, a **doubly-pointed** set is a triple (D, o_1, o_2), where o_1 and o_2 are two distinguished points of D.

Let Ξ be a complete separable metric space called the **mark space**. A **marked pointed discrete space** is a tuple $(D, o; m)$, where (D, o) is a pointed discrete space and m is a function $m : D \times D \to \Xi$. The mark of a single point x may also be defined by $m(x) := m(x, x)$, where the same symbol m is used for simplicity. An **isomorphism** (or *rooted isomorphism*) between two such spaces $(D, o; m)$ and $(D', o'; m')$ is an isometry $\rho : D \to D'$ such that $\rho(o) = o'$ and $m'(\rho(u), \rho(v)) = m(u, v)$ for all $u, v \in D$. An isomorphism between doubly-pointed marked discrete spaces is defined similarly. An isomorphism from a space to itself is called an **automorphism**.

Most of the examples of discrete spaces in this work are graphs and discrete subsets of the Euclidean space. More precisely, connected and locally-finite simple graphs equipped with the graph-distance metric [2] are instances of discrete spaces. Similarly, *networks*; i.e., graphs equipped with marks on the edges [2], are instances of marked discrete spaces.

Let $\mathcal{D}_*$ (resp. $\mathcal{D}_{**}$) be the set of equivalence classes of pointed (resp. doubly-pointed) discrete spaces under isomorphism. Let $\mathcal{D}'_*$ and $\mathcal{D}'_{**}$ be defined similarly for marked discrete spaces with mark space Ξ (which is usually given). The equivalence class containing (D, o), $(D, o; m)$ etc., is denoted by brackets $[D, o]$, $[D, o; m]$, etc.

Every element of $\mathcal{D}_*$ can be regarded as a *boundedly-compact* measured metric space (where the measure is the counting measure). Therefore, the generalization of the *Gromov-Hausdorff-Prokhorov* metric in [23] defines a metric on $\mathcal{D}_*$. By using the results of [23], one can show that $\mathcal{D}_*$ is a Borel subset of some complete separable metric space, where the proof is skipped for brevity (the arXiv version of this paper also defines a similar metric and proves this result). Similarly, one can show that $\mathcal{D}_{**}, \mathcal{D}'_*$ and $\mathcal{D}'_{**}$ are Borel subsets of some complete separable metric spaces (see [21]). This enables one to define *random pointed discrete spaces*, etc., which are discussed in the next subsection.

2.3 Random Pointed Discrete Spaces

Definition 2.1. A **random pointed discrete space** is a random element in $\mathcal{D}_*$ and is denoted by bold symbols $[\boldsymbol{D}, \boldsymbol{o}]$. Here, $\boldsymbol{D}$ and $\boldsymbol{o}$ represent the discrete space and the origin respectively.

The last paragraph of Subsection 2.2 ensures that a standard probability space can be used in the above definition, and hence, the classical tools of probability theory are available. The probability space is not referred to explicitly in this paper. Note that the whole symbol $[\boldsymbol{D},\boldsymbol{o}]$ represents one random object, which is a random equivalence class of pointed discrete spaces. Therefore, any formula using $\boldsymbol{D}$ and $\boldsymbol{o}$ should be well defined for equivalence classes; i.e., should be invariant under pointed isomorphisms.

The following convention is helpful throughout.

Convention 2.2. In this paper, bold symbols are usually used in the random case or when extra randomness is used. For example, $[D,o]$ refers to a deterministic element of $\mathcal{D}_*$ and $[\boldsymbol{D},\boldsymbol{o}]$ refers to a random pointed discrete space.

Note that the distribution of a random pointed network $[\boldsymbol{D},\boldsymbol{o}]$ is a probability measure on $\mathcal{D}_*$ defined by $\mu(A) := \mathbb{P}\left[[\boldsymbol{D},\boldsymbol{o}]\in A\right]$ for events $A\subseteq\mathcal{D}_*$.

Definition 2.3. A **random pointed marked discrete space** is a random element in $\mathcal{D}'_*$ and is denoted by bold symbols $[\boldsymbol{D},\boldsymbol{o};\boldsymbol{m}]$. Here, $\boldsymbol{D}$, $\boldsymbol{o}$ and $\boldsymbol{m}$ represent the discrete space, the origin and the mark function respectively.

Most examples in this work are either random rooted graphs (or networks) [2] or point processes (i.e., random discrete subset of $\mathbb{R}^k$) and marked point processes that contain 0, where 0 is considered as the origin. Other examples are also studied by considering different metrics on such objects.

2.4 Unimodular Discrete Spaces

Once the notion of random pointed discrete space is defined, the definition of unimodularity is a straight variant of [2]. In what follows, the notation is similarly to [6]. Here, the symbol $g[D,o,v]$ is used as a short form of $g([D,o,v])$. Similarly, brackets $[\cdot]$ are used as a short form of $([\cdot])$.

Definition 2.4. A **unimodular discrete space** is a random pointed discrete space, namely $[\boldsymbol{D},\boldsymbol{o}]$, such that for all measurable functions $g:\mathcal{D}_{**}\to\mathbb{R}^{\geq 0}$,

$$\mathbb{E}\left[\sum_{v\in\boldsymbol{D}} g[\boldsymbol{D},\boldsymbol{o},v]\right] = \mathbb{E}\left[\sum_{v\in\boldsymbol{D}} g[\boldsymbol{D},v,\boldsymbol{o}]\right]. \tag{2.1}$$

Similarly, a **unimodular marked discrete space** is a random pointed marked discrete space $[\boldsymbol{D},\boldsymbol{o};\boldsymbol{m}]$ such that for all measurable functions $g:\mathcal{D}'_{**}\to\mathbb{R}^{\geq 0}$,

$$\mathbb{E}\left[\sum_{v\in\boldsymbol{D}} g[\boldsymbol{D},\boldsymbol{o},v;\boldsymbol{m}]\right] = \mathbb{E}\left[\sum_{v\in\boldsymbol{D}} g[\boldsymbol{D},v,\boldsymbol{o};\boldsymbol{m}]\right]. \tag{2.2}$$

Note that the expectations may be finite or infinite.

When there is no ambiguity, the term $g[D,o,v]$ is also denoted by $g_D(o,v)$ or simply $g(o,v)$. The sum in the left (respectively right) side of (2.1) is called the **outgoing mass from $\boldsymbol{o}$** (respectively **incoming mass into $\boldsymbol{o}$**) and is denoted by $g^+(\boldsymbol{o})$ (respectively $g^-(\boldsymbol{o})$). The same notation can be used for the terms in (2.2). So (2.1) and (2.2) can be summarized by

$$\mathbb{E}\left[g^+(\boldsymbol{o})\right] = \mathbb{E}\left[g^-(\boldsymbol{o})\right].$$

These equations are called the **mass transport principle** in the literature. The reader is encouraged to see [2] and the examples therein for further discussion on the mass transport principle and unimodularity.

As a basic example, every finite metric space D, equipped with a random root $\boldsymbol{o} \in D$ chosen uniformly, is unimodular. Also, the lattices in the Euclidean space rooted at 0; e.g., $[\mathbb{Z}^k, 0]$ and $[\delta\mathbb{Z}^k, 0]$ are unimodular. In addition, unimodularity is preserved under weak convergence, as observed in [10] for unimodular graphs.

The following two examples show that unimodular discrete spaces unify unimodular graphs and point-stationary point processes. Most of the examples in this work are of these types.

Example 2.5 (Unimodular Random Graphs)**.** In the case of random rooted graphs and networks, the concept of unimodularity in Definition 2.4 coincides with that of [2] (see also Remark A.5 of the arXiv version regarding the topologies). Therefore, unimodular random graphs and networks are special cases of unimodular (marked) discrete spaces.

Example 2.6 (Point-Stationary Point Processes)**. Point-stationarity** is defined for point processes Φ in $\mathbb{R}^k$ such that $0 \in \Phi$ a.s. (see e.g., [24]). This definition is equivalent to (2.1) except that g is required to be invariant under translations only (and not under all isometries). This implies that $[\Phi, 0]$ is unimodular. In addition, by considering the mark $\boldsymbol{m}(x, y) := y - x$ on pairs of points of Φ, point-stationarity of Φ will be equivalent to the unimodularity of $[\Phi, 0; \boldsymbol{m}]$ (see also Remark A.5 of the arXiv version regarding the topologies). Note also that Φ can be recovered from $[\Phi, 0; \boldsymbol{m}]$.
For example, if Φ is a stationary point process in $\mathbb{R}^k$ (i.e., its distribution is invariant under all translations) with finite intensity (i.e., finite expected number of points in the unit cube), then the *Palm version* of Φ is a point-stationary point process, where the latter is heuristically obtained by conditioning Φ to contain the origin (see e.g., Section 13 of [14] for the precise definition). Also, if $(X_n)_{n\in\mathbb{Z}}$ is a stochastic process in $\mathbb{R}^k$ with stationary increments such that $X_0 = 0$ and $X_i \neq X_j$ a.s. for every $i \neq j$, then the image of this random walk is a point-stationary point process.

2.5 Equivariant Process on a Unimodular Discrete Space

In many cases in this paper, an unmarked unimodular discrete space $[\boldsymbol{D}, \boldsymbol{o}]$ is given and various ways of assigning marks to $\boldsymbol{D}$ are considered. Intuitively, an *equivariant process* on $\boldsymbol{D}$ is an assignment of (random) marks to $\boldsymbol{D}$ such that the new marked space is unimodular. Formally, it is

> *a unimodular marked discrete space* $[\boldsymbol{D}', \boldsymbol{o}'; \boldsymbol{m}]$ *such that the space* $[\boldsymbol{D}', \boldsymbol{o}']$*, obtained by forgetting the marks, has the same distribution as* $[\boldsymbol{D}, \boldsymbol{o}]$.

In this paper, it is more convenient to work with a disintegrated form of this heuristic, defined below, despite of being more technical. It is not very easy, but can be proved that the two notions are equivalent, but the proof is skipped for brevity (see Lemma 2.11 below and Proposition B.1 of the arXiv version). This claim is similar to *invariant disintegration* for group actions.

In the following, the mark space Ξ is fixed as in Subsection 2.2.

Definition 2.7. Let D be a deterministic discrete space which is boundedly-finite. A **marking** of D is a function from $D \times D$ to Ξ; i.e., an element of $\Xi^{D\times D}$. A **random marking** of D is a random element of $\Xi^{D\times D}$.

Definition 2.8. An **equivariant process $\boldsymbol{Z}$** with values in Ξ is a map that assigns to every deterministic discrete space D a random marking $\boldsymbol{Z}_D$ of D satisfying the following properties:

(i) $\boldsymbol{Z}$ is compatible with isometries in the sense that for every isometry $\rho : D_1 \to D_2$, the random marking $\boldsymbol{Z}_{D_1} \circ \rho^{-1}$ of D_2 has the same distribution as $\boldsymbol{Z}_{D_2}$.

(ii) For every measurable subset $A \subseteq \mathcal{D}'_*$, the following function on $\mathcal{D}_*$ is measurable:

$$[D, o] \mapsto \mathbb{P}\left[[D, o; \boldsymbol{Z}_D] \in A\right].$$

In addition, given a unimodular discrete space $[\boldsymbol{D}, \boldsymbol{o}]$, such a map is also called an **equivariant process on $\boldsymbol{D}$**. In this case, one can also let $\boldsymbol{Z}_{(\cdot)}$ be undefined for a class of discrete spaces, as long as it is defined for almost all realizations of $\boldsymbol{D}$. It is important that extra randomness be allowed here.

Convention 2.9. If D is clear from the context, $\boldsymbol{Z}_D(\cdot)$ is also denoted by $\boldsymbol{Z}(\cdot)$ for simplicity.

Note that in the above definition, D is deterministic and is not an equivalence class of discrete spaces. However, for an equivariant process on $[\boldsymbol{D}, \boldsymbol{o}]$, one can define $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{Z}_{\boldsymbol{D}}]$ as a random pointed marked discrete space with distribution $\mathcal{Q}$ defined by

$$\mathcal{Q}(A) := \int\int 1_A[D, o; m] \mathrm{d}\mathcal{P}_D(m) d\mu([D, o]), \tag{2.3}$$

where $\mathcal{P}_D$ is the distribution of $\boldsymbol{Z}_D$ (for every D) and μ is the distribution of $[\boldsymbol{D}, \boldsymbol{o}]$ (note that only the distribution of $\boldsymbol{Z}_D$ is important and no common probability space is assumed for different D's). It can be seen that $\mathcal{Q}(A)$ is indeed well defined and is a probability measure on $\mathcal{D}'_*$.

The following basic examples help to illustrate the definition.

Example 2.10. By choosing the marks of points (or pairs of points) in an i.i.d. manner, one obtains an equivariant process. Also, the following periodic marking of $\mathbb{Z}$ is an equivariant process on $\mathbb{Z}$: Choose $\boldsymbol{U} \in \{0, 1, \ldots, n-1\}$ uniformly at random and let $\boldsymbol{Z}_{\mathbb{Z}}(x) := 1$ if $x \in n\mathbb{Z} + \boldsymbol{U}$ and $\boldsymbol{Z}_{\mathbb{Z}}(x) := 0$ otherwise. Moreover, given a measurable function $z : \mathcal{D}_{**} \to \Xi$, one can define $\boldsymbol{Z}_D(u, v) := z[D, u, v]$, which is called a *deterministic process* here.

Lemma 2.11. *Let* $[\boldsymbol{D}, \boldsymbol{o}]$ *be a unimodular discrete space. If* $\boldsymbol{Z}$ *is an equivariant process on* $\boldsymbol{D}$*, then* $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{Z}_{\boldsymbol{D}}]$ *is also unimodular.*

The proof is straightforward and skipped for brevity. The converse of this claim also holds (see the arXiv version). It is important here that the distribution of $\boldsymbol{Z}_D$ does not depend on the origin (as in Definition 2.8).

Remark 2.12. One can easily extend the definition of equivariant processes to allow the base space to be marked. Therefore, for point-stationary point processes, one can replace condition (i) by invariance under translations only (see Example 2.6). In particular, every stationary stochastic process on $\mathbb{Z}^k$ defines an equivariant process on $\mathbb{Z}^k$.

Definition 2.13. An **equivariant subset** $\boldsymbol{S}$ is the set of points with mark 1 in some $\{0,1\}$-valued equivariant process. In addition, if $[\boldsymbol{D}, \boldsymbol{o}]$ is a unimodular discrete space, then the **intensity** of $\boldsymbol{S}$ in $\boldsymbol{D}$ is defined by $\rho_{\boldsymbol{D}}(\boldsymbol{S}) := \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}]$.

For example, $\boldsymbol{S}_D := \{v \in D : \#N_1(v) = 4\}$ defines an equivariant subset. Also, let $D = \mathbb{Z}$ and $\boldsymbol{S}_D$ be the set of even numbers with probability p and the set of odd numbers with probability $1-p$. Then, $\boldsymbol{S}$ is an equivariant subset of $\mathbb{Z}$ if and only if $p = \frac{1}{2}$ (notice Condition (i) of Definition 2.8).

Lemma 2.14. *Let* $[\boldsymbol{D}, \boldsymbol{o}]$ *be a unimodular discrete space and* $\boldsymbol{S}$ *an equivariant subset. Then* $\boldsymbol{S}_{\boldsymbol{D}} \neq \emptyset$ *with positive probability if and only if it has positive intensity. Equivalently,* $\boldsymbol{S}_{\boldsymbol{D}} = \boldsymbol{D}$ *a.s. if and only if* $\rho_{\boldsymbol{D}}(\boldsymbol{S}) = 1$.

Proof. The claim is implied by the mass transport principle (2.2) for the function $g[D, u, v; S] := 1_{\{v \in S\}}$. The details are left to the reader. □

The above lemma is a generalization of similar results in [6] and [2].

2.6 Notes and Bibliographical Comments

The mass transport principle is originally introduced in [19]. The concept of unimodular graphs is first defined for deterministic transitive graphs in [9] and developed to random rooted graphs and networks in [2].

Unimodular graphs have many analogies and connections to (Palm versions of) stationary point processes and point-stationary point processes, as discussed in Example 9.5 of [2] and also in [6] and [22]. As already explained in this section, the framework of unimodular discrete spaces, introduced in this section, can be regarded as a common generalization of these concepts.

Special cases of the notion of equivariant processes have been considered in various literature. The first formulation in Subsection 2.5 is considered in [2] for unimodular graphs. *Factors of IID* [25] are special cases of equivariant processes where the marks of the points are obtained from i.i.d. marks (Example 2.10) in an equivariant way. *Covariant subsets* and *covariant partitions* of unimodular graphs are defined similarly in [6], but no extra randomness is allowed therein. In the case of stationary (marked) point processes, the first formulation of Subsection 2.5 is used in the literature. However, the authors believe that the general formulation of Definition 2.8 is new even in those special cases.

3 The Unimodular Minkowski and Hausdorff Dimensions

This section presents the new notions of dimension for unimodular discrete spaces. As mentioned in the introduction, the statistical homogeneity of unimodular discrete spaces is used to define discrete analogous of the Minkowski and Hausdorff dimensions. Also, basic properties of these definitions are discussed.

3.1 The Unimodular Minkowski Dimension

Definition 3.1. Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space and $r \geq 0$. An **equivariant r-covering $\boldsymbol{R}$** of $\boldsymbol{D}$ is an equivariant subset of $\boldsymbol{D}$ (Definition 2.13) such that the set of balls $\{N_r(v) : v \in \boldsymbol{R}_{\boldsymbol{D}}\}$ cover $\boldsymbol{D}$ almost surely. Here, the same symbol $\boldsymbol{R}$ is used for the following equivariant process (Definition 2.8):

$$\boldsymbol{R}(v) := \boldsymbol{R}_{\boldsymbol{D}}(v) := \begin{cases} r, & \text{there is a ball centered at } v \text{ in the covering,} \\ 0, & \text{otherwise,} \end{cases}$$

for $v \in \boldsymbol{D}$. Let $\mathcal{C}_r$ be the set of all equivariant r-coverings. Define

$$\lambda_r := \lambda_r(\boldsymbol{D}) := \inf\{\text{intensity of } \boldsymbol{R} \text{ in } \boldsymbol{D} : \boldsymbol{R} \in \mathcal{C}_r\}, \tag{3.1}$$

where the intensity is defined in Definition 2.13.

Note that λ_r is non-increasing in terms of r. A smaller λ_r heuristically means that *a smaller number of balls per point* is needed to cover $\boldsymbol{D}$. So define

Definition 3.2. The **upper and lower unimodular Minkowski dimensions** of $\boldsymbol{D}$ are defined by

$$\begin{aligned} \overline{\mathrm{udim}}_M(\boldsymbol{D}) &:= \overline{\mathrm{decay}}\,(\lambda_r), \\ \underline{\mathrm{udim}}_M(\boldsymbol{D}) &:= \underline{\mathrm{decay}}\,(\lambda_r), \end{aligned}$$

as $r \to \infty$. If the decay rate of λ_r exists, define the **unimodular Minkowski dimension** of $\boldsymbol{D}$ by

$$\mathrm{udim}_M(\boldsymbol{D}) := \mathrm{decay}\,(\lambda_r)\,.$$

One has

$$0 \leq \underline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \overline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \infty.$$

Remark 3.3. It is essential that extra randomness is allowed in the definition of equivariant r-coverings (based on Definition 2.8). In general, one may have to go beyond i.i.d. marks. See for instance Example 3.4 below.

The following are first illustrations of the definition.

Example 3.4. The randomly shifted lattice $\boldsymbol{S}_n := (2n+1)\mathbb{Z}^k - \boldsymbol{U}_n$, where $\boldsymbol{U}_n \in \{-n, \ldots, n\}^k$ is chosen uniformly, is an equivariant n-covering of $\mathbb{Z}^k$ equipped with the l_∞ metric (other metrics can be treated similarly). This implies that $\lambda_n \leq \mathbb{P}\left[0 \in \boldsymbol{S}_n\right] = (2n+1)^{-k}$, and hence, $\underline{\mathrm{udim}}_M(\mathbb{Z}^k) \geq k$.

Example 3.5. If $\boldsymbol{D}$ is finite with positive probability, then it can be seen that any non-empty equivariant subset has intensity at least $\mathbb{E}[1/\#\boldsymbol{D}]$ (use the mass transport principle when sending mass $1/\#\boldsymbol{D}$ from every point of the subset to every point of $\boldsymbol{D}$). This implies that $\mathrm{udim}_M(\boldsymbol{D}) = 0$.

Remark 3.6 (Bounding the Minkowski Dimension)**.** In all examples in this work, lower bounds on the unimodular Minkowski dimension are obtained by constructing explicit examples of r-coverings. Upper bounds can be obtained by constructing *disjoint* or *bounded* coverings, as discussed in Subsection 3.2 below, or by comparison with the unimodular Hausdorff dimension defined in Subsection 3.3 below (see Theorem 3.22).

3.2 Optimal Coverings for the Minkowski Dimension

Definition 3.7. Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space and $r \geq 0$. If the infimum in the definition of λ_r (3.1) is attained by an equivariant r-covering $\boldsymbol{S}$; i.e., $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}] = \lambda_r$, then $\boldsymbol{S}$ is called an **optimal r-covering** for $\boldsymbol{D}$.

Theorem 3.8. *Every unimodular discrete space has an optimal r-covering for every $r \geq 0$.*

Sketch of the proof. Let $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots$ be a sequence of r-coverings of $\boldsymbol{D}$ such that $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_n] \to \lambda_r$. By a tightness argument and choosing a subsequence if necessary, one may assume that $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{S}_n]$ converges weakly, say to $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{S}]$, where $\boldsymbol{S}$ is an equivariant subset $\boldsymbol{S}$ of $\boldsymbol{D}$ (see the arXiv version for the details). Since each $\boldsymbol{S}_n$ is an r-covering, $\mathbb{P}[\boldsymbol{S}_n \cap N_r(\boldsymbol{o}) = \emptyset] = 0$. It is straightforward to deduce $\mathbb{P}[\boldsymbol{S} \cap N_r(\boldsymbol{o}) = \emptyset] = 0$. So by putting balls of radius r on the points of $\boldsymbol{S}$, the root is covered a.s. So Lemma 2.14 implies that every point is covered a.s.; i.e., $\boldsymbol{S}$ is an r-covering. Also, by weak convergence, $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}] = \lim_n \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_n] = \lambda_r$. This implies that $\boldsymbol{S}$ is an optimal r-covering. □

As a corollary, this implies that $\lambda_r > 0$ for every r since any non-empty equivariant subset has positive intensity. In general, finding an optimal covering is difficult. In some specific examples, the following is easier to study.

Definition 3.9. Let $K < \infty$ and $r \geq 0$. An r-covering of $\boldsymbol{D}$ is **K-bounded** if each point of $\boldsymbol{D}$ is covered at most K times a.s. A sequence $(\boldsymbol{R}_n)_n$ of equivariant coverings of $\boldsymbol{D}$ is called **uniformly bounded** if there is $K < \infty$ such that each $\boldsymbol{R}_n$ is K-bounded.

Lemma 3.10. *If $\boldsymbol{R}$ is a K-bounded equivariant r-covering of $\boldsymbol{D}$, then*

$$\frac{1}{K}\mathbb{P}[\boldsymbol{R}(\boldsymbol{o}) \neq 0] \leq \lambda_r \leq \mathbb{P}[\boldsymbol{R}(\boldsymbol{o}) \neq 0]. \tag{3.2}$$

So if $(\boldsymbol{R}_n)_n$ is a sequence of equivariant coverings which is uniformly bounded, with $\boldsymbol{R}_n$ an n-covering for each $n \geq 1$, then

$$\begin{aligned} \overline{\mathrm{udim}}_M(\boldsymbol{D}) &= \overline{\mathrm{decay}}\,(\mathbb{P}[\boldsymbol{R}_n(\boldsymbol{o}) \neq 0]), \\ \underline{\mathrm{udim}}_M(\boldsymbol{D}) &= \underline{\mathrm{decay}}\,(\mathbb{P}[\boldsymbol{R}_n(\boldsymbol{o}) \neq 0]). \end{aligned}$$

Proof. The rightmost inequality in (3.2) is immediate from the definition of λ_r. Let $\boldsymbol{R}'$ be another equivariant r-covering. Let $g(u,v)=1$ if $\boldsymbol{R}'(u)=\boldsymbol{R}(v)=r$ and $d(u,v)\leq r$. Then $g^+(\boldsymbol{o})\leq K1_{\{\boldsymbol{R}'(\boldsymbol{o})\neq 0\}}$ and $g^-(\boldsymbol{o})\geq 1_{\{\boldsymbol{R}(\boldsymbol{o})\neq 0\}}$. Hence by the mass transport principle (2.2), $\frac{1}{K}\mathbb{P}\left[\boldsymbol{R}(\boldsymbol{o})\neq 0\right]\leq \mathbb{P}\left[\boldsymbol{R}'(\boldsymbol{o})\neq 0\right]$ and the leftmost inequality in (3.2) then follows from the definition of λ_r. The last two equalities follow immediately from (3.2). $\square$

Corollary 3.11. *If $\boldsymbol{R}$ is an equivariant **disjoint** r**-covering** of $\boldsymbol{D}$ (i.e., the balls used in the covering are pairwise disjoint a.s.), then it is an optimal r-covering for $\boldsymbol{D}$.*

Example 3.12. The covering of $\mathbb{Z}^k$ (equipped with the l_∞ metric) constructed in Example 3.4 is a disjoint covering. So it is optimal and hence $\mathrm{udim}_M(\mathbb{Z}^k)=k$. For $\mathbb{Z}^k$ equipped with the Euclidean metric, one can construct a 3^k-bounded covering similarly and deduce the same result.

Example 3.13. Let T_k be the k-regular tree. Consider a deterministic covering of T_k by disjoint balls of radius n. By choosing $\boldsymbol{o}$ in one of these balls uniformly at random, it can be seen that an equivariant disjoint n-covering of $[T_k,\boldsymbol{o}]$ is obtained (the proof is left to the reader). So Corollary 3.11 implies that $\lambda_n = 1/\#N_n(\boldsymbol{o})$ which has exponential decay when $k\geq 3$. Hence, $\mathrm{udim}_M(T_k)=\infty$ for $k\geq 3$.

Proposition 3.14. *For any point-stationary point process Φ in $\mathbb{R}$ endowed with the Euclidean metric, by letting $p(r):=\mathbb{P}\left[\Phi\cap(0,r)=\emptyset\right]$, one has*

$$\overline{\mathrm{udim}}_M(\Phi)=\overline{\mathrm{decay}}\left(\frac{1}{r}\int_0^r p(s)\mathrm{d}s\right)\leq 1\wedge\overline{\mathrm{decay}}\left(p(r)\right),$$
$$\underline{\mathrm{udim}}_M(\Phi)=\underline{\mathrm{decay}}\left(\frac{1}{r}\int_0^r p(s)\mathrm{d}s\right)\geq 1\wedge\underline{\mathrm{decay}}\left(p(r)\right).$$

Proof. Let $r>0$ and φ be a discrete subset of $\mathbb{R}$. Let $\boldsymbol{U}_r$ be a random number in $[0,r)$ chosen uniformly. For each $n\in\mathbb{Z}$, put a ball of radius r centered at the largest element of $\varphi\cap[nr+\boldsymbol{U}_r,(n+1)r+\boldsymbol{U}_r)$. Denote this random r-covering of φ by $\boldsymbol{R}_\varphi$. One can see that $\boldsymbol{R}$ is equivariant under translations (see Remark 2.12). This implies that $\boldsymbol{R}$ is an equivariant covering (verifying Condition (ii) of Definition I.2.8 is skipped here). One has

$$\mathbb{P}\left[0\in\boldsymbol{R}_\Phi\right]=\mathbb{P}\left[\Phi\cap(0,\boldsymbol{U}_r)=\emptyset\right]=\frac{1}{r}\int_0^r\mathbb{P}\left[\Phi\cap(0,s)=\emptyset\right]\mathrm{d}s.$$

Now, since $\boldsymbol{R}$ is a 3-bounded covering, Lemma 3.10 implies both equalities in the claim. The rightmost inequalities hold for arbitrary deterministic nonnegative non-decreasing functions and their proof is skipped. $\square$

3.3 The Unimodular Hausdorff Dimension

The definition of the unimodular Hausdorff dimension is based on coverings of the discrete space by balls of possibly different radii. Such a covering can be represented by an assignment of marks to the points, where the mark of a point v represents the radius of the ball centered at v. For reasons explained in the introduction (Subsection 1.1), the radii are assumed to be at least 1. Also, by convention, if there is no ball centered at v, the mark of v is defined to be 0. In relation with this convention, the following notation is used for all discrete spaces D and points $v \in D$:

$$N_r(v) := \begin{cases} \{u \in D : d(v,u) \leq r\}, & r \geq 1, \\ \emptyset, & r = 0. \end{cases}$$

In words, $N_r(v)$ is the *closed ball* of radius r centered at v, except when $r = 0$.

Definition 3.15. Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space. An **equivariant (ball-) covering** $\boldsymbol{R}$ of $\boldsymbol{D}$ is an equivariant process on $\boldsymbol{D}$ (Definition 2.8) with values in $\Xi := \{0\} \cup [1, \infty)$ such that the family of balls $\{N_{\boldsymbol{R}(v)}(v) : v \in \boldsymbol{D}\}$ covers the points of $\boldsymbol{D}$ almost surely. For simplicity, $N_{\boldsymbol{R}(v)}(v)$ will also be denoted by $N_{\boldsymbol{R}}(v)$. Also, for $0 \leq \alpha < \infty$ and $1 \leq M < \infty$, let

$$\mathcal{H}_M^\alpha(\boldsymbol{D}) := \inf \left\{ \mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] : \boldsymbol{R}(v) \in \{0\} \cup [M, \infty),\ \forall v,\ \text{a.s.} \right\}, \qquad (3.3)$$

where the infimum is over all equivariant coverings $\boldsymbol{R}$ such that almost surely, $\forall v \in \boldsymbol{D} : \boldsymbol{R}(v) \in \{0\} \cup [M, \infty)$, and, by convention, $0^0 := 0$. Note that $\mathcal{H}_M^\alpha(\boldsymbol{D})$ is a non-decreasing function of both α and M.

Definition 3.16. Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space. The number $\mathcal{H}_1^\alpha(\boldsymbol{D})$, defined in (3.3), is called the **α-dimensional Hausdorff content** of $\boldsymbol{D}$. The **unimodular Hausdorff dimension** of $\boldsymbol{D}$ is defined by

$$\operatorname{udim}_H(\boldsymbol{D}) := \sup\{\alpha \geq 0 : \mathcal{H}_1^\alpha(\boldsymbol{D}) = 0\}, \qquad (3.4)$$

with the convention that $\sup \emptyset = 0$.

The key point of assuming equivariance in the above definition is that by Lemma 2.11, $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{R}]$ is a unimodular marked discrete space. Note also that extra randomness is allowed in the definition of equivariant coverings. Note also that

$$0 \leq \mathcal{H}_1^\alpha(\boldsymbol{D}) \leq 1,$$

since for the covering by balls of radii 1, one has $\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] = 1$.

Examples 3.17 and 3.20 below provide basic illustrations of the unimodular Hausdorff dimension.

Example 3.17. If $\boldsymbol{D}$ is finite with positive probability, then one can show similarly to Example 3.5 that $\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \mathbb{E}\left[1/\#\boldsymbol{D}\right]$ for every $\boldsymbol{R}$, and hence, $\operatorname{udim}_H(\boldsymbol{D}) = 0$. Also, for the covering $\boldsymbol{S}_n$ of $\mathbb{Z}^k$ constructed in Example 3.4, one has $\mathbb{E}\left[\boldsymbol{S}_n(\boldsymbol{o})^\alpha\right] = (2n+1)^{\alpha-k}$. If $\alpha < k$, this implies that $\mathcal{H}_1^\alpha(\mathbb{Z}^k) = 0$, and hence, $\operatorname{udim}_H(\mathbb{Z}^k) \geq k$. The upper bound $\operatorname{udim}_H(\mathbb{Z}^k) \leq k$ is implied by Lemma 3.18 below. So $\operatorname{udim}_H(\mathbb{Z}^k) = k$.

Lemma 3.18. *Let $[\boldsymbol{D},\boldsymbol{o}]$ be a unimodular discrete space and $\alpha \geq 0$. If there exists $c \geq 0$ such that $\forall r \geq 1 : \#N_r(\boldsymbol{o}) \leq cr^\alpha$ a.s., then* $\mathrm{udim}_H(\boldsymbol{D}) \leq \alpha$.

Proof. Let $\boldsymbol{R}$ be an arbitrary equivariant covering. For all discrete spaces D and $u,v \in D$, let $g_D(u,v)$ be 1 if $d(u,v) \leq \boldsymbol{R}_D(u)$ and 0 otherwise. One has $g^+(u) = \#N_{\boldsymbol{R}}(u)$ and $g^-(u) \geq 1$ a.s. (since $\boldsymbol{R}$ is a covering). By the assumption and the mass transport principle (2.2), one gets

$$\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \frac{1}{c}\mathbb{E}\left[\#N_{\boldsymbol{R}}(\boldsymbol{o})\right] = \frac{1}{c}\mathbb{E}\left[g^+(\boldsymbol{o})\right] = \frac{1}{c}\mathbb{E}\left[g^-(\boldsymbol{o})\right] \geq \frac{1}{c}.$$

Since $\boldsymbol{R}$ is arbitrary, one gets $\mathcal{H}_1^\alpha(\boldsymbol{D}) \geq \frac{1}{c} > 0$, and hence, $\mathrm{udim}_H(\boldsymbol{D}) \leq \alpha$. □

Remark 3.19 (Bounding the Hausdorff Dimension)**.** In most examples in this work, a lower bound on the unimodular Hausdorff dimension is provided, either by comparison with the Minkowski dimension (see Subsection 3.4 below), or by explicit construction of a sequence of equivariant coverings $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots$ such that $\mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^\alpha\right] \to 0$ as $n \to \infty$. Note that this gives $\mathcal{H}_1^\alpha(\boldsymbol{D}) = 0$, which implies that $\mathrm{udim}_H(\boldsymbol{D}) \geq \alpha$. Constructing coverings does not help to find upper bounds for the Hausdorff dimension. The derivation of upper bounds is mainly discussed in Part II. The main tools are *the mass distribution principle* (Theorem II.2.2), which is a stronger form of Lemma 3.18 above, and the *unimodular Billingsley's lemma* (Theorem II.2.6).

Example 3.20. Let $[\boldsymbol{D},\boldsymbol{o}]$ be $[\mathbb{Z},0]$ with probability $\frac{1}{2}$ and $[\mathbb{Z}^2,0]$ with probability $\frac{1}{2}$. It is shown below that $\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D}) = 1$.

For $n \in \mathbb{N}$, the equivariant n-covering of Example 3.4 makes sense for $\boldsymbol{D}$ and is uniformly bounded. One has $\mathbb{P}\left[\boldsymbol{R}(0) > 0\right] = \frac{1}{2}(n^{-1} + n^{-2})$. This implies that $\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{decay}\left(\frac{1}{2}(n^{-1}+n^{-2})\right) = 1$ and also $\mathrm{udim}_H(\boldsymbol{D}) \geq 1$. On the other hand, for any equivariant covering $\boldsymbol{S}$, one has

$$\mathbb{E}\left[\boldsymbol{S}(\boldsymbol{o})\right] \geq \mathbb{E}\left[\boldsymbol{S}(\boldsymbol{o})\,|\boldsymbol{D} = \mathbb{Z}\right]\mathbb{P}\left[\boldsymbol{D} = \mathbb{Z}\right] = \frac{1}{2}\mathbb{E}\left[\boldsymbol{S}(\boldsymbol{o})\,|\boldsymbol{D} = \mathbb{Z}\right].$$

Let $c > 2$. The proof of Lemma 3.18 for $[\mathbb{Z},0]$ implies that $\mathbb{E}\left[\boldsymbol{S}(\boldsymbol{o})\,|\boldsymbol{D} = \mathbb{Z}\right] \geq \frac{1}{c}$. This implies that $\mathcal{H}_1^1(\boldsymbol{D}) \geq \frac{1}{2c} > 0$. So $\mathrm{udim}_H(\boldsymbol{D}) \leq 1$.

Remark 3.21. In Example 3.20 above, different samples of $\boldsymbol{D}$ have different natures heuristically. This is formalized by saying that $[\boldsymbol{D},\boldsymbol{o}]$ is *non-ergodic*; i.e., there is an event $A \subseteq \mathcal{D}_*$ such that the proposition $[D,o] \in A$ does not depend on the origin of D and $0 < \mathbb{P}\left[[\boldsymbol{D},\boldsymbol{o}] \in A\right] < 1$. The concept of ergodicity will be discussed in [4]. In this work, the focus is mainly on the ergodic case. However the definitions and results do not require ergodicity. In the non-ergodic cases, like in Example 3.20, it is desirable to assign a dimension to every sample of $\boldsymbol{D}$. This will be formalized as *sample dimension* in [4].

3.4 Comparison of Hausdorff and Minkowski Dimensions

Theorem 3.22 (Minkowski vs Hausdorff)**.** *One has*

$$\underline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \overline{\mathrm{udim}}_M(\boldsymbol{D}) \leq \mathrm{udim}_H(\boldsymbol{D}).$$

Proof. The first inequality holds by the definition. For the second one, the definition of λ_r (3.1) implies that for every $\alpha \geq 0$ and $r \geq 1$,

$$\inf\{\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] : \boldsymbol{R} \text{ is an equivariant } r\text{-covering}\} = r^\alpha \lambda_r.$$

This readily implies that $\mathcal{H}_1^\alpha(\boldsymbol{D}) \leq r^\alpha \lambda_r$ for every $r \geq 1$. By the definition of the Minkowski dimension, if $\alpha < \overline{\mathrm{udim}}_M(\boldsymbol{D})$, one gets that $\mathcal{H}_1^\alpha(\boldsymbol{D}) = 0$, and hence, $\mathrm{udim}_H(\boldsymbol{D}) \geq \alpha$. This implies the claim. □

Remark 3.23. There exist examples in which the inequalities in Theorem 3.22 are strict (see e.g., Subsections 4.2.2 and 4.4). However, equality holds in most of the examples that arise naturally. So, one can say that the equality $\mathrm{udim}_M(\boldsymbol{D}) = \mathrm{udim}_H(\boldsymbol{D})$ expresses some kind of *regularity* for the unimodular discrete space $\boldsymbol{D}$, regarded as a fractal object.

3.5 The Unimodular Hausdorff Measure

Consider the setting of Subsection 3.3. For $0 \leq \alpha < \infty$, let

$$\mathcal{H}_\infty^\alpha(\boldsymbol{D}) := \lim_{M\to\infty} \mathcal{H}_M^\alpha(\boldsymbol{D}) \in [0, \infty], \tag{3.5}$$

where $\mathcal{H}_M^\alpha(\boldsymbol{D})$ is defined in (3.3). Note that the limit exists because of monotonicity.

Definition 3.24. The α**-dimensional Hausdorff measure** of $\boldsymbol{D}$ is

$$\mathcal{M}^\alpha(\boldsymbol{D}) := \left(\mathcal{H}_\infty^\alpha(\boldsymbol{D})\right)^{-1}. \tag{3.6}$$

Rather than being a measure in the mathematical sense, this is a quantification of the *size* of $\boldsymbol{D}$ and can be used to compare unimodular spaces with equal dimension. The following results gather some elementary properties of the function $\mathcal{H}_M^\alpha$ and the Hausdorff measure.

Lemma 3.25. *One has*

(i) $\mathcal{H}_1^\alpha(\boldsymbol{D}) \leq \mathcal{H}_M^\alpha(\boldsymbol{D}) \leq M^\alpha \mathcal{H}_1^\alpha(\boldsymbol{D})$.

(ii) If $\mathcal{H}_1^\alpha(\boldsymbol{D}) = 0$, *then* $\mathcal{H}_\infty^\alpha(\boldsymbol{D}) = 0$; *i.e.,* $\mathcal{M}^\alpha(\boldsymbol{D}) = \infty$.

(iii) If $\alpha \geq \beta$, *then* $\mathcal{H}_M^\alpha(\boldsymbol{D}) \geq M^{\alpha-\beta}\mathcal{H}_M^\beta(\boldsymbol{D})$.

Proof. (i). If $\boldsymbol{R}$ is an equivariant covering, them $M\boldsymbol{R}$ is also an equivariant covering and satisfies $\forall v \in \boldsymbol{D} : MR(v) \in \{0\} \cup [M, \infty)$ a.s.

(ii). The claim is implied by part (i).

(iii). If $\boldsymbol{R}$ is an equivariant covering such that $\forall v \in \boldsymbol{D} : \boldsymbol{R}(v) \in \{0\} \cup [M, \infty)$ a.s., then $\boldsymbol{R}(\boldsymbol{o})^\alpha \geq M^{\alpha-\beta}\boldsymbol{R}(\boldsymbol{o})^\beta$ a.s. □

Lemma 3.26. *One has*

$$\begin{aligned} \forall \alpha < \mathrm{udim}_H(\boldsymbol{D}) : \qquad & \mathcal{M}^\alpha(\boldsymbol{D}) = \infty, \\ \forall \alpha > \mathrm{udim}_H(\boldsymbol{D}) : \qquad & \mathcal{M}^\alpha(\boldsymbol{D}) = 0. \end{aligned}$$

Proof. For $\alpha < \mathrm{udim}_H(\boldsymbol{D})$, one has $\mathcal{H}_1^\alpha(\boldsymbol{D}) = 0$. So part (ii) of Lemma 3.25 implies that $\mathcal{M}^\alpha(\boldsymbol{D}) = \infty$. For $\alpha > \mathrm{udim}_H(\boldsymbol{D})$, there exists β such that $\alpha > \beta > \mathrm{udim}_H(\boldsymbol{D})$. For this β, one has $\mathcal{H}_1^\beta(\boldsymbol{D}) > 0$ and part (iii) of the same lemma implies that $\mathcal{H}_M^\alpha(\boldsymbol{D}) \geq M^{\alpha-\beta}\mathcal{H}_M^\beta(\boldsymbol{D}) \geq M^{\alpha-\beta}\mathcal{H}_1^\beta(\boldsymbol{D})$. This implies that $\mathcal{H}_\infty^\alpha(\boldsymbol{D}) = \infty$, which proves the claim. $\square$

Remark 3.27. For $\alpha := \mathrm{udim}_H(\boldsymbol{D})$, the α-dimensional Hausdorff measure of $\boldsymbol{D}$ can be zero, finite or infinite. The lattice $\mathbb{Z}^k$ is a case where $\mathcal{M}^\alpha(\boldsymbol{D})$ is positive and finite (Proposition 3.29 below). Examples II.3.11 and II.3.12 provide examples of the infinite and zero cases respectively.

The exact computation of the Hausdorff measure is generally difficult. The following propositions provide basic examples.

Proposition 3.28 (0-dimensional Hausdorff Measure)**.** *One has*

$$\mathcal{M}^0(\boldsymbol{D}) = \left(\mathbb{E}\left[1/\#\boldsymbol{D}\right]\right)^{-1}.$$

Proof. As in Example 3.17, one gets $\mathcal{H}_M^0(\boldsymbol{D}) \geq \mathbb{E}\left[1/\#\boldsymbol{D}\right]$. It is enough to show that equality holds. If $\boldsymbol{D}$ is finite a.s., this can be proved by putting a single ball of radius $M \vee \mathrm{diam}(\boldsymbol{D})$ centered at a point of $\boldsymbol{D}$ chosen uniformly at random. Second, assume $\boldsymbol{D}$ is infinite a.s. It is enough to construct an equivariant covering $\boldsymbol{R}$ such that $\mathbb{P}\left[\boldsymbol{R}(\boldsymbol{o}) > 0\right]$ is arbitrarily small. Let $p > 0$ be arbitrary and $\boldsymbol{S}$ be the *Bernoulli equivariant subset* obtained by selecting each point with probability p in an i.i.d. manner. For all infinite discrete spaces D and $v \in D$, let $\boldsymbol{\tau}_D(v)$ be the closest point of $\boldsymbol{S}_D$ to v (if there is a tie, choose one of them uniformly at random independently). It can be seen that $\boldsymbol{\tau}_D^{-1}(u)$ is finite almost surely (use the mass transport principle for $g(x,y) := 1_{\{y=\boldsymbol{\tau}_D(x)\}}$). For $u \in \boldsymbol{S}_D$, let $\boldsymbol{R}(u) := 1\vee\mathrm{diam}(\boldsymbol{\tau}^{-1}(u))$ be the diameter of the *Voronoi cell* of u. For $u \in D \setminus \boldsymbol{S}_D$, let $\boldsymbol{R}(u) := 0$. It is clear that $\boldsymbol{R}$ is a covering, and in fact, an equivariant covering. One has $\mathbb{P}\left[\boldsymbol{R}(\boldsymbol{o}) > 0\right] = \mathbb{P}\left[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}\right] = p$, which is arbitrarily small. So the claim is proved in this case.

Finally, assume $\boldsymbol{D}$ is finite with probability q. For all deterministic discrete spaces D, let $\boldsymbol{R}_D$ be one of the above coverings depending on whether D is finite or infinite. It satisfies $\mathbb{P}\left[\boldsymbol{R}(\boldsymbol{o}) > 0\right] = \mathbb{E}\left[1/\#\boldsymbol{D}\right] + p(1-q)$. Since p is arbitrary, the claim is proved. $\square$

Proposition 3.29. *The k-dimensional Hausdorff measure of the scaled lattice $[\delta\mathbb{Z}^k, 0]$, equipped with the l_∞ metric, is given by*

$$\mathcal{M}^k(\delta\mathbb{Z}^k) = (2/\delta)^k.$$

Proof. Let $\boldsymbol{S}_n$ be the covering in Example 3.4 scaled by factor δ. One has $\mathbb{E}\left[\boldsymbol{S}_n(\boldsymbol{o})^k\right] = (n\delta)^k/(2n+1)^k$. This easily implies that $\mathcal{H}_\infty^k(\delta\mathbb{Z}^k) \leq (\delta/2)^k$. On the other hand, the proof of Lemma 3.18 shows that $\mathcal{H}_\infty^k(\delta\mathbb{Z}^k) \geq c\delta^k$, where c is any constant such that $r^k \geq c\#N_r(0)$ for large enough r. It follows that $\mathcal{H}_\infty^k(\delta\mathbb{Z}^k) \geq (\delta/2)^k$ and the claim is proved. $\square$

3.6 The Effect of a Change of Metric

To avoid confusion when considering two metrics, a pointed discrete space is denoted by $((D, d), o)$ here, where d is the metric on D and o is the origin. Note that if d' is another metric on D, then $d' \in \mathbb{R}^{D\times D}$. So d' can be considered as a marking of D in the sense of Definition 2.7 and $((D, d), o; d')$ is a pointed marked discrete space.

Definition 3.30. An **equivariant (boundedly finite) metric** is an $\mathbb{R}$-valued equivariant process $\boldsymbol{d}'$ (Definition 2.8) such that, for all discrete spaces D, $\boldsymbol{d}'_D$ is almost surely (w.r.t. the extra randomness) a metric on D and $(D, \boldsymbol{d}'_D)$ is a boundedly-finite metric space.

If in addition, $[(\boldsymbol{D}, \boldsymbol{d}), \boldsymbol{o}]$ is a unimodular discrete space, then $[(\boldsymbol{D}, \boldsymbol{d}), \boldsymbol{o}; \boldsymbol{d}']$ is a unimodular marked discrete space by Lemma 2.11. It can be seen that $[(\boldsymbol{D}, \boldsymbol{d}'), \boldsymbol{o}; \boldsymbol{d}]$, obtained by swapping the metrics, makes sense as a random pointed marked discrete space (see the arXiv version for the measurability requirement). By verifying the mass transport principle (2.2) directly, it is easy to show that $[(\boldsymbol{D}, \boldsymbol{d}'), \boldsymbol{o}; \boldsymbol{d}]$ is unimodular.

The following result is valid for both the Hausdorff and the (upper and lower) Minkowski dimensions.

Theorem 3.31 (Change of Metric)**.** *Let $[(\boldsymbol{D}, \boldsymbol{d}), \boldsymbol{o}]$ be a unimodular discrete space and $\boldsymbol{d}'$ be an equivariant metric. If $\boldsymbol{d}' \leq c\boldsymbol{d}+a$ a.s., with c and a constants, then the dimension of $(\boldsymbol{D}, \boldsymbol{d}')$ is larger than or equal to that of $(\boldsymbol{D}, \boldsymbol{d})$. Moreover, for every $\alpha \geq 0$, $\mathcal{M}^\alpha(\boldsymbol{D}, \boldsymbol{d}') \geq c^{-\alpha}\mathcal{M}^\alpha(\boldsymbol{D}, \boldsymbol{d})$.*

Proof. The claim is implied by the fact that the ball $N_r((\boldsymbol{D}, \boldsymbol{d}'), v)$ contains the ball $N_{cr+a}((\boldsymbol{D}, \boldsymbol{d}), v)$. □

As a corollary, if $\frac{1}{c}\boldsymbol{d} - a \leq \boldsymbol{d}' \leq c\boldsymbol{d} + a$ a.s., then $(\boldsymbol{D}, \boldsymbol{d}')$ has the same unimodular dimensions as $(\boldsymbol{D}, \boldsymbol{d})$. Also, $c\boldsymbol{D}$ has the same dimension as $\boldsymbol{D}$ and $\mathcal{M}^\alpha(c\boldsymbol{D}) = c^{-\alpha}\mathcal{M}^\alpha(\boldsymbol{D})$.

For instance, this result can be applied to Cayley graphs, which are an important class of unimodular graphs [2]. It follows that the unimodular dimensions of a Cayley graph do not depend on the generating set. In fact, it will be proved in Subsection II.3.6 that these dimensions are equal to the *polynomial growth degree* of H.

Example 3.32. Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular graph. Examples of equivariant metrics on $\boldsymbol{G}$ are the graph-distance metric corresponding to an equivariant spanning subgraph (e.g., the drainage network model of Subsection 4.5 below) and metrics *generated by equivariant edge lengths*. More precisely, if $\boldsymbol{l}$ is an equivariant process which assigns a positive *weight* to the edges of every deterministic graph, then one can let $\boldsymbol{d}'(u, v)$ be the minimum weight of the paths that connect u to v. If $\boldsymbol{d}'$ is a metric for almost every realization of $\boldsymbol{G}$ and is boundedly-finite a.s., then it is an equivariant metric.

3.7 Dimension of Subspaces

Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space and $\boldsymbol{S}$ be an equivariant subset which is almost surely nonempty. Lemma 2.14 implies that $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}] > 0$. So one can consider $[\boldsymbol{S}_{\boldsymbol{D}}, \boldsymbol{o}]$ conditioned on $\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}$. By directly verifying the mass transport principle (2.1), it is easy to see that $[\boldsymbol{S}_{\boldsymbol{D}}, \boldsymbol{o}]$ conditioned on $\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}$ is unimodular (see the similar claim for unimodular graphs in [6]).

Convention 3.33. For an equivariant subset $\boldsymbol{S}$ as above, the unimodular Hausdorff dimension of $[\boldsymbol{S}_{\boldsymbol{D}}, \boldsymbol{o}]$ (conditioned on $\boldsymbol{o} \in S_{\boldsymbol{D}}$) is denoted by $\mathrm{udim}_H(\boldsymbol{S}_{\boldsymbol{D}})$. The same convention is used for the Minkowski dimension, the Hausdorff measure, etc.

Theorem 3.34. *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space and $\boldsymbol{S}$ an equivariant subset such that $\boldsymbol{S}_{\boldsymbol{D}}$ is nonempty a.s. Then,*

(i) One has

$$\begin{aligned} \mathrm{udim}_H(\boldsymbol{S}_{\boldsymbol{D}}) &= \mathrm{udim}_H(\boldsymbol{D}), \\ \overline{\mathrm{udim}}_M(\boldsymbol{S}_{\boldsymbol{D}}) &\geq \overline{\mathrm{udim}}_M(\boldsymbol{D}), \\ \underline{\mathrm{udim}}_M(\boldsymbol{S}_{\boldsymbol{D}}) &\geq \underline{\mathrm{udim}}_M(\boldsymbol{D}). \end{aligned}$$

(ii) If ρ is the intensity of $\boldsymbol{S}$ in $\boldsymbol{D}$, then for every $\alpha \geq 0$, the α-dimensional Hausdorff measure of $\boldsymbol{S}_{\boldsymbol{D}}$ satisfies

$$2^{-\alpha}\rho\, \mathcal{M}^{\alpha}(\boldsymbol{D}) \leq \mathcal{M}^{\alpha}(\boldsymbol{S}_{\boldsymbol{D}}) \leq \rho\, \mathcal{M}^{\alpha}(\boldsymbol{D}).$$

Remark 3.35. Subsection 3.8.1 below defines a modification $\mathcal{M}'_{\alpha}(\boldsymbol{D})$ of the unimodular Hausdorff measure by considering coverings by arbitrary sets. With this definition, one would have $\mathcal{M}'_{\alpha}(\boldsymbol{S}_d) = \rho\, \mathcal{M}'_{\alpha}(\boldsymbol{D})$. This can be proved similarly to Theorem 3.34 with the modification that there is no need to double the radii.

Remark 3.36. In the setting of Theorem 3.34, $\mathrm{udim}_M(\boldsymbol{S}_{\boldsymbol{D}})$ can be strictly larger than $\mathrm{udim}_M(\boldsymbol{D})$ (see e.g., Subsection 4.4). Also, equality is guaranteed if $\boldsymbol{S}_{\boldsymbol{D}}$ is a r-covering of $\boldsymbol{D}$ for some constant r.

Proof of Theorem 3.34. First, part (ii) is proved.

(ii). The definition of $\mathcal{H}^{\alpha}_{\infty}(\boldsymbol{S}_{\boldsymbol{D}})$ implies that there exists a sequence $\boldsymbol{R}_n$ of equivariant coverings of $\boldsymbol{S}_{\boldsymbol{D}}$ such that $\boldsymbol{R}_n(\cdot) \in \{0\} \cup [n, \infty)$ for all $n = 1, 2, \ldots$ and $\mathbb{E}[\boldsymbol{R}_n(\boldsymbol{o})^{\alpha} | \boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}] \to \mathcal{H}^{\alpha}_{\infty}(\boldsymbol{S}_{\boldsymbol{D}})$. One may extend $\boldsymbol{R}_n$ to be defined on $\boldsymbol{D}$ by letting $\boldsymbol{R}_n(v) := 0$ for $v \in \boldsymbol{D} \setminus \boldsymbol{S}_{\boldsymbol{D}}$. Let $\epsilon > 0$ be arbitrary and $\boldsymbol{B}_n \subseteq \boldsymbol{D}$ be the union of $N_{(1+\epsilon)\boldsymbol{R}_n}(v)$ for all $v \in \boldsymbol{D}$. Define $\boldsymbol{R}'_n(u) := (1+\epsilon)\boldsymbol{R}_n(u)$ for $u \in \boldsymbol{B}_n$ and $\boldsymbol{R}'_n(u) := 1/\epsilon$ for $u \notin \boldsymbol{B}_n$. It is clear that $\boldsymbol{R}'_n$ is an equivariant covering of $\boldsymbol{D}$. Also,

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{R}'_n(\boldsymbol{o})^{\alpha}\right] &= (1+\epsilon)^{\alpha}\mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^{\alpha}\right] + \frac{1}{\epsilon^{\alpha}}\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{B}_n\right] \\ &= \rho(1+\epsilon)^{\alpha}\mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^{\alpha} | \boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}\right] + \frac{1}{\epsilon^{\alpha}}\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{B}_n\right]. \quad (3.7) \end{aligned}$$

Since the radii of the balls in $\boldsymbol{R}_n$ are at least n, one gets that $\boldsymbol{B}_n$ includes the ϵn-neighborhood of $\boldsymbol{S}_{\boldsymbol{D}}$. Therefore, $\mathbb{P}[\boldsymbol{o} \notin \boldsymbol{B}_n] \leq \mathbb{P}[N_{\epsilon n}(\boldsymbol{o}) \cap \boldsymbol{S}_{\boldsymbol{D}} = \emptyset]$. Since $\boldsymbol{S}_{\boldsymbol{D}}$ is nonempty a.s., this in turn implies that $\mathbb{P}[\boldsymbol{o} \notin \boldsymbol{B}_n] \to 0$ as $n \to \infty$ (note that the events $N_{\epsilon n}(\boldsymbol{o}) \cap \boldsymbol{S}_{\boldsymbol{D}} = \emptyset$ are nested and converge to the event $\boldsymbol{S}_{\boldsymbol{D}} = \emptyset$). So (3.7) implies that

$$\liminf_{n\to\infty} \mathbb{E}\left[\boldsymbol{R}'_n(\boldsymbol{o})^\alpha\right] = \rho(1+\epsilon)^\alpha \liminf_{n\to\infty} \mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^\alpha \,|\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}\right] = \rho(1+\epsilon)^\alpha \mathcal{H}^\alpha_\infty(\boldsymbol{S}_{\boldsymbol{D}}).$$

Note that the radii of the balls in $\boldsymbol{R}'_n$ are at least $n \wedge (1/\epsilon)$. Therefore, one obtains $\mathcal{H}^\alpha_{1/\epsilon}(\boldsymbol{D}) \leq \rho(1+\epsilon)^\alpha \mathcal{H}^\alpha_\infty(\boldsymbol{S}_{\boldsymbol{D}})$. By letting $\epsilon \to 0$, one gets $\mathcal{H}^\alpha_\infty(\boldsymbol{D}) \leq \rho \mathcal{H}^\alpha_\infty(\boldsymbol{S}_{\boldsymbol{D}})$; i.e., $\mathcal{M}^\alpha(\boldsymbol{S}_{\boldsymbol{D}}) \leq \rho \mathcal{M}^\alpha(\boldsymbol{D})$.

Conversely, let $\boldsymbol{R}_n$ be a sequence of equivariant coverings of $\boldsymbol{D}$ for $n = 1, 2, \ldots$ such that $\boldsymbol{R}_n(\cdot) \in \{0\} \cup [n, \infty)$ a.s. and $\mathbb{E}[\boldsymbol{R}_n(\boldsymbol{o})^\alpha] \to \mathcal{H}^\alpha_\infty(\boldsymbol{D})$. Fix n in the following. Let $\boldsymbol{B} := \boldsymbol{B}_{\boldsymbol{D}} := \{v : N_{\boldsymbol{R}_n}(v) \cap \boldsymbol{S}_{\boldsymbol{D}} \neq \emptyset\}$. For each $v \in \boldsymbol{B}$, let $\boldsymbol{\tau}_n(v)$ be an element chosen uniformly at random in $N_{\boldsymbol{R}_n}(v) \cap \boldsymbol{S}_{\boldsymbol{D}}$. For $v \notin \boldsymbol{B}$, let $\boldsymbol{\tau}_n(v)$ be undefined. For $w \in \boldsymbol{S}_{\boldsymbol{D}}$, let $\boldsymbol{R}'_n(w) := 2\max\{\boldsymbol{R}_n(v) : v \in \boldsymbol{\tau}_n^{-1}(w)\}$. It can be seen that $\boldsymbol{R}'_n$ is an equivariant covering of $\boldsymbol{S}_{\boldsymbol{D}}$. One has

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{R}'_n(\boldsymbol{o})^\alpha\right] &\leq 2^\alpha \mathbb{E}\left[\sum_v \boldsymbol{R}_n(v)^\alpha 1_{\{v \in \boldsymbol{\tau}_n^{-1}(\boldsymbol{o})\}}\right] \\
&= 2^\alpha \mathbb{E}\left[\sum_v \boldsymbol{R}_n(\boldsymbol{o})^\alpha 1_{\{\boldsymbol{o} \in \boldsymbol{\tau}_n^{-1}(v)\}}\right] \\
&\leq 2^\alpha \mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^\alpha\right], \qquad (3.8)
\end{aligned}$$

where the equality is by the mass transport principle. It follows that

$$\rho \liminf_{n\to\infty} \mathbb{E}\left[\boldsymbol{R}'_n(\boldsymbol{o})^\alpha \,|\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}\right] \leq 2^\alpha \mathcal{H}^\alpha_\infty(\boldsymbol{D}).$$

So $\rho \mathcal{H}^\alpha_\infty(\boldsymbol{S}_{\boldsymbol{D}}) \leq 2^\alpha \mathcal{H}^\alpha_\infty(\boldsymbol{D})$. Hence, $\mathcal{M}^\alpha(\boldsymbol{S}_{\boldsymbol{D}}) \geq 2^{-\alpha} \rho\, \mathcal{M}^\alpha(\boldsymbol{D})$ and the claim is proved.

(i). The first claim is implied by part (ii) and Lemma 3.26. For the two other claims, let $\boldsymbol{R}$ be an arbitrary equivariant r-covering of $\boldsymbol{D}$. Similarly to the above arguments, let $\boldsymbol{\tau}(v)$ be an element picked at random in $N_r(v) \cap \boldsymbol{S}_{\boldsymbol{D}}$ and let $\boldsymbol{R}' := \{\tau(v) : v \in \boldsymbol{D}, N_r(v) \cap \boldsymbol{S}_{\boldsymbol{D}} \neq \emptyset\}$. Note that $\boldsymbol{R}'$ is a $2r$-covering of $\boldsymbol{S}_{\boldsymbol{D}}$. So (3.8) implies that for all $\alpha \geq 0$, $\rho \lambda_{2r}(\boldsymbol{S}_{\boldsymbol{D}}) \leq 2^\alpha \lambda_r(\boldsymbol{D})$. The claims follow when taking the log and letting r go to infinity. □

3.8 Other Variants of the Definitions

3.8.1 Covering By Arbitrary Sets

According to Remark 3.35, it is more natural to redefine the Hausdorff measure by considering coverings by subsets which are not necessarily balls (as in the continuum setting). A technical challenge is to define such coverings in an equivariant way. This will be done at the end of this subsection using the notion of equivariant processes of Subsection 2.5. Once an equivariant covering $\boldsymbol{C}$ is defined, one can define the *average diameter of sets $U \in \boldsymbol{C}$ per point* by

$$\mathbb{E}\left[\sum_{U\in\boldsymbol{C}} \frac{1}{\#U} 1_{\{\boldsymbol{o}\in U\}} \mathrm{diam}(U)\right].$$

The same idea is used to redefine $\mathcal{H}_M^\alpha(\boldsymbol{D})$ as follows:

$$\mathcal{H}'_{\alpha,M}(\boldsymbol{D}) := \inf_{\boldsymbol{C}}\left\{\mathbb{E}\left[\sum_{U\in\boldsymbol{C}} \frac{1}{\#U} 1_{\{\boldsymbol{o}\in U\}} \left(M \vee \frac{1}{2}\mathrm{diam}(U)\right)^\alpha\right]\right\},$$

where the infimum is over all equivariant coverings $\boldsymbol{C}$. Here, taking the maximum with M is *similar* to the condition that the subsets have diameter at least $2M$ (note for instance that a ball of radius M might have diameter strictly less than $2M$). Finally, define the **modified unimodular Hausdorff measure** $\mathcal{M}'_\alpha(\boldsymbol{D})$ similarly to (3.6). Remark 3.35 shows an advantage of this definition. Also, the reader can verify that $2^{-\alpha}\mathcal{H}_{2M}^\alpha(\boldsymbol{D}) \leq \mathcal{H}'_{\alpha,M}(\boldsymbol{D}) \leq \mathcal{H}_M^\alpha(\boldsymbol{D})$. Therefore,

$$\mathcal{M}^\alpha(\boldsymbol{D}) \leq \mathcal{M}'_\alpha(\boldsymbol{D}) \leq 2^\alpha \mathcal{M}^\alpha(\boldsymbol{D}).$$

This implies that the notion of unimodular Hausdorff dimension is not changed by this modification. One can also obtain a similar equivalent form of the unimodular Minkowski dimension. This is done by redefining λ_r by considering equivariant coverings by sets of diameter at most $2r$. The details are left to the reader (see also the arXiv version). A similar idea will be used in Subsection 4.1.2 to calculate the Minkowski dimension of one-ended trees.

Finally, here is the promised representation of the above coverings as equivariant processes (it should be noted that such a covering cannot be defined as a numbered sequence of equivariant subsets and the collection should be necessarily unordered). To show the idea, consider a covering $C = \{U_1, U_2, \ldots\}$ of a deterministic discrete space D, where each U_i is bounded. For each U_i, assign the mark $(\boldsymbol{X}_i, \mathrm{diam}(U))$ to every point of U_i, where $\boldsymbol{X}_i \in [0,1]$ is chosen i.i.d. and uniformly. Note that multiple marks are assigned to every point and the covering can be reconstructed from the marks. With this idea, let the mark space Ξ be the set of discrete subsets of $\mathbb{R}^2$ (regard every discrete set as a counting measure and equip Ξ with a metrization of the vague topology). This mark space can be used to represent equivariant coverings by equivariant processes as defined in Subsection 2.5 (for having a complete mark space, one can extend Ξ to the set of discrete multi-sets in $\mathbb{R}^2$).

3.8.2 Gauge Functions and the Unimodular Dimension Function

There exist unimodular discrete spaces $\boldsymbol{D}$ in which the $\mathrm{udim}_H(\boldsymbol{D})$-dimensional Hausdorff measure is either zero or infinity. For such spaces, it is convenient to generalize the unimodular Hausdorff measure as follows. Consider an increasing function $\varphi : \{0\} \cup [1,\infty) \to [0,\infty)$; e.g., $\varphi(r) = r^\alpha$, called a *gauge function*.

Define $\mathcal{H}_M^{\varphi}(\boldsymbol{D})$ by $\inf_{\boldsymbol{R}}\{\mathbb{E}[\varphi(\boldsymbol{R}(\boldsymbol{o}))]\}$ similarly to (3.3). Then, define $\mathcal{M}^{\varphi}(\boldsymbol{D})$ similarly to (3.6). If $0 < \mathcal{M}^{\varphi}(\boldsymbol{D}) < \infty$, then φ is called a **unimodular dimension function** for $\boldsymbol{D}$. This generalization will be used in [4] for having stronger form of the results about connections with scaling limits and some other results.

As an example, it is natural to guess that a dimension function for the zeros of the simple random walk is $\varphi(r) := \sqrt{r \log\log r}$ (see Subsections 4.3 and II.3.3.1 for further discussion). Does there exist a unimodular discrete space without any dimension function? The answer is not known yet. The answer to the analogous question in the continuum setting is positive [16], but the proof ideas don't seem to work in the unimodular discrete setting.

In addition, given a family of gauge functions $(\varphi_\alpha)_{\alpha\geq 0}$ that is increasing in α and such that $\forall\alpha > \beta : \lim_{r\to\infty} \varphi_\alpha(r)/\varphi_\beta(r) = \infty$, one can redefine the unimodular Hausdorff dimension by $\sup\{\alpha : \mathcal{M}^{\varphi_\alpha}(\boldsymbol{D}) = 0\}$ (see e.g., the next paragraph). The reader can redefine the unimodular Minkowski dimension similarly. Then, the results of this section can be extended to this setting except that Theorem 3.34 and the results of Subsection 3.8.1 require the *doubling condition* $\sup_{r\geq 1} \varphi(2r)/\varphi(r) < \infty$. The general result of Subsection 4.1.2 can also be extended under the doubling condition. Also, the upper bounds in the unimodular mass distribution principle, the unimodular Billingsley lemma and the unimodular Frostman lemma in Part II hold in this more general setting (some other results of Part II require the doubling condition). However, for the ease of reading, the results are presented in the original setting of this section.

As an example of the above framework, one can define the **exponential dimension** by considering $\varphi_\alpha(r) := e^{\alpha r}$. It might be useful for studying unimodular spaces with super-polynomial growth, which are more interesting in group theory (see Subsection II.3.6). Note that exponential gauge functions do not satisfy the doubling condition, and hence, the reader should be careful about using the results of this work for such gauge functions.

3.9 Notes and Bibliographical Comments

Several definitions and basic results of this section have analogues in the continuum setting. The following is a list of the analogies. Note however that there is no systematic way of translating the results in the continuum setting to unimodular discrete spaces. In particular, inequalities are most often, but not always, in the other direction. The comparison of the unimodular Minkowski and Hausdorff dimensions (Theorem 3.22) is analogous to the similar comparison in the continuum setting (see e.g., (1.2.3) of [11]), but in the reverse direction. Theorem 3.31 regarding changing the metric is analogous to the fact that the ordinary Minkowski and Hausdorff dimensions are not increased by applying a Lipschitz function. Theorem 3.34 regarding the dimension of subsets is analogous to the fact that the ordinary dimensions do not increase by passing to subsets. Note however that equality holds in Theorem 3.34 for the unimodular Hausdorff dimension (and also for the unimodular Minkowski dimension in most usual examples) in contrast to the continuum setting.

4 Examples

This section presents a large set of examples of unimodular discrete spaces together with discussions about their dimensions. Recall that the tools for bounding the dimensions are summarized in Remarks 3.6 and 3.19. As mentioned in Remark 3.19, bounding the Hausdorff dimension from above usually requires the unimodular mass transport principle or the unimodular Billingsley lemma, which will be stated in Part II. So the upper bounds for most of the following examples are completed in Part II.

4.1 General Unimodular Trees

In this subsection, general results are presented regarding the dimension of unimodular trees with the graph-distance metric. Specific instances are presented later in the section. It turns out that the number of *ends* of the tree plays an important role (an **end** in a tree is an equivalence class of simple paths in the tree, where two such paths are equivalent if their symmetric difference is finite).

It is well known that the number of ends in a unimodular tree belongs to $\{0, 1, 2, \infty\}$ [2]. Unimodular trees without end are finite, and hence, are zero dimensional (Example 3.17). The only point to mention is that there exists an algorithm to construct an optimal n-covering for such trees. This algorithm is similar to the algorithm for one-ended trees, discussed below, and is skipped for brevity. In addition, It will be shown in Part II that unimodular trees with infinitely many ends have exponential growth, and hence, have infinite Hausdorff dimension. The remaining two cases are discussed below.

4.1.1 Unimodular Two-Ended Trees

If T is a tree with two ends, then there is a unique bi-infinite path in T called its **trunk**. Moreover, each connected component of the complement of the trunk is finite.

Theorem 4.1. *For all unimodular two-ended trees* $[\boldsymbol{T}, \boldsymbol{o}]$ *endowed with the graph-distance metric, one has* $\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{T}) = 1$. *Moreover, if* ρ *is the intensity of the trunk of* $\boldsymbol{T}$*, then the modified 1-dimensional Hausdorff measure of* $\boldsymbol{T}$ *is* $\mathcal{M}'_1(\boldsymbol{T}) = 2\rho^{-1}$.

Proof. For all two-ended trees T, let $\boldsymbol{S}_T$ be the trunk of T. Then, $\boldsymbol{S}$ is an equivariant subset (Definition 2.13). Therefore, Theorem 3.34 implies that $\mathrm{udim}_H(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{S}_{\boldsymbol{T}})$. Since the trunk is isometric to $\mathbb{Z}$ as a metric space, Example 3.17 implies that $\mathrm{udim}_H(\boldsymbol{T}) = 1$. In addition, Remark 3.35 and Proposition 3.29 imply that $\mathcal{M}'_1(\boldsymbol{T}) = \rho^{-1}\mathcal{M}'_1(\mathbb{Z}) = 2\rho^{-1}$.

The claim concerning the unimodular Minkowski dimension is implied by Corollary II.2.9 of Part II, which shows that any unimodular infinite graph satisfies $\underline{\mathrm{udim}}_M(\boldsymbol{G}) \geq 1$ (this theorem will not be used throughout). □

4.1.2 Unimodular One-Ended Trees

Unimodular one-ended trees arise naturally in many examples (see [2]). In particular, the (local weak) limit of many interesting sequences of finite trees/graphs are one-ended ([3, 2]). In terms of unimodular dimensions, it will be shown that unimodular one-ended trees are the richest class of unimodular trees.

First, the following notation is borrowed from [6]. Every one-ended tree T can be regarded as a family tree as follows. For every vertex $v \in T$, there is a unique infinite simple path starting from v. Denote by $F(v)$ the next vertex in this path and call it the **parent** of v. By deleting $F(v)$, the connected component containing v is finite. This set is denoted by $D(v)$ and its elements are called the **descendants** of v. The maximum distance of v to its descendants is called the **height** of v and is denoted by $h(v)$.

Theorem 4.2. *If $[\boldsymbol{T}, \boldsymbol{o}]$ is a unimodular one-ended tree endowed with the graph-distance metric, then*

$$\begin{aligned} \overline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) \geq n\right]\right), \qquad (4.1)\\ \underline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \underline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) \geq n\right]\right). \qquad (4.2)\end{aligned}$$

In addition,

$$\mathrm{udim}_H(\boldsymbol{T}) \geq \overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) = n\right]\right) \geq \overline{\mathrm{udim}}_M(\boldsymbol{T}). \qquad (4.3)$$

It is not known whether (4.3) is sharp or not. It should also be noted that $\overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) = n\right]\right)$ can be strictly larger than $1+\overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) \geq n\right]\right)$ (see e.g., Subsection 4.2.2), however, they are equal in most *usual* examples.

Lemma 4.3. *For every unimodular one-ended tree $[\boldsymbol{T}, \boldsymbol{o}]$, the output of the following greedy algorithm is an optimal equivariant n-covering of $\boldsymbol{T}$.*

while *true* **do**
  Let $\boldsymbol{A}$ be the set of vertices which are not yet covered;
  Let $\boldsymbol{T}'$ be the subtree spanned by A and the shortest paths connecting the vertices in A;
  Put balls of radii n at all vertices of height n in $\boldsymbol{T}'$;
end

It should be noted that $\boldsymbol{A}$ might be a disconnected subset of $\boldsymbol{T}$ in the above algorithm. Note also that the algorithm does not finish in finite time, but for each vertex v of $\boldsymbol{T}$, it is determined in finite time whether a ball is put at v or not. So the output of the algorithm is well defined.

The proof of Lemma 4.3 is omitted since it is similar to that of the next lemma. The latter studies a variant of the algorithm which considers coverings by *cones* rather than balls. Nevertheless, it will be shown below that the Minkowski dimensions do not change.

The **cone** with height n at $v \in \boldsymbol{T}$ is defined by $C_n(v) := N_n(v) \cap D(v)$; i.e., the first n generations of the descendants of v, including v itself. Let λ'_n be the infimum intensity of equivariant coverings by cones of height n. The claim is that

$$\lambda'_{2n} \leq \lambda_n \leq \lambda'_n. \tag{4.4}$$

This immediately implies that

$$\underline{\mathrm{udim}}_M(\boldsymbol{T}) = \underline{\mathrm{decay}}\left(\lambda'_n\right), \quad \overline{\mathrm{udim}}_M(\boldsymbol{T}) = \overline{\mathrm{decay}}\left(\lambda'_n\right). \tag{4.5}$$

To prove (4.4), note that any covering by cones of height n is also a covering by balls of radii n. This implies that $\lambda_n \leq \lambda'_n$. Also, if $\boldsymbol{S}$ is a covering by balls of radii n, then $\{F^n(v) : v \in \boldsymbol{S}\}$ is a covering by cones of height $2n$. By the mass transport principle (2.2), one can show that the intensity of the latter is not greater than the intensity of $\boldsymbol{S}$. This implies that $\lambda'_{2n} \leq \lambda_n$. So (4.4) is proved.

Lemma 4.4. *For every unimodular one-ended tree $[\boldsymbol{T}, \boldsymbol{o}]$, the output $\boldsymbol{S}$ of the following greedy algorithm is an optimal equivariant covering of $\boldsymbol{T}$ by cones of height n.*

$\boldsymbol{S} := \emptyset$;
while *true* **do**
  Add all vertices of height n in $\boldsymbol{T}$ to $\boldsymbol{S}$;
  $\boldsymbol{T} := \boldsymbol{T} \setminus \bigcup_{v \in \boldsymbol{S}} D(v)$;
end

Proof. Let $\boldsymbol{A}$ be any equivariant covering of $\boldsymbol{T}$ by cones of height n. Consider a realization $(T; A)$ of $[\boldsymbol{T}; \boldsymbol{A}]$. Let v be a vertex such that $h(v) = n$. Since A is a covering by cones of height n, A should have at least one vertex in $D(v)$ (to see this, consider the farthest leaf from v in $D(v)$). Now, for all such vertices v, delete the vertices in $A \cap D(v)$ from A and then add v to A. Let A_1 be the subset of T obtained by doing this operation for all vertices v of height n. So A_1 is also a covering of T by cones of height n. Now, remove all vertices $\{v : h(v) = n\}$ and their descendants from T to obtain a new one-ended tree. Consider the same procedure for the remaining tree and its intersection with A. Inductively, one obtains a sequence of subsets $A = A_0, A_1, \ldots$ of T such that, for each i, A_i is a covering of T by cones of height n and agrees with $\boldsymbol{S}_T$ on the set of vertices that are removed from the tree up to step i.

By letting $[\boldsymbol{T}; \boldsymbol{A}]$ be random, the above induction gives a sequence of equivariant subsets $\boldsymbol{A} = \boldsymbol{A}_0, \boldsymbol{A}_1, \ldots$ on $\boldsymbol{T}$. It can be seen that the intensity of $\boldsymbol{A}_1$ is at most that of $\boldsymbol{A}$ (this can be verified by the mass transport principle (2.1)). It is left to the reader to obtain inductively that $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{A}_{i+1}] \leq \mathbb{P}[\boldsymbol{o} \in \boldsymbol{A}_i]$. Also, $\lim_{i\to\infty} \boldsymbol{A}_i = \boldsymbol{S}$ as equivariant subsets of $\boldsymbol{T}$. This implies that $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{A}] \geq \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}]$, hence, $\boldsymbol{S}$ is an optimal covering by cones of height n. □

Lemma 4.5. *Under the above setting, one has*

$$\mathbb{P}\left[h(\boldsymbol{o}) \bmod (n+1) = -1\right] \leq \lambda'_n \leq \mathbb{P}\left[h(\boldsymbol{o}) \bmod \left\lfloor \frac{n}{2} + 1 \right\rfloor = -1\right]. \tag{4.6}$$

Proof. An equivariant covering will be constructed to prove the second inequality in (4.6). Let $A_n := \{v \in \boldsymbol{T} : h(v) \bmod n = -1\}$ and $A'_n := \{F^{n-1}(v) : v \in$

$A_n\}$. The claim is that A'_n is a covering of $\boldsymbol{T}$ by cones of height $2n-2$. Let $v \in \boldsymbol{T}$ be an arbitrary vertex. Let k be such that $(k-1)n-1 < h(v) \le kn-1$. Let j be the first nonnegative integer such that $h(F^j(v)) \ge kn-1$ and let $w := F^j(v)$. One has $0 \le j \le n-1$. By considering the longest path in $D(w)$ from w to the leaves, one finds $z \in D(w)$ such that $h(z) \bmod n = -1$ and $0 \le d(w,z) \le n-1$. Therefore w (and hence v) is a descendant of $F^{n-1}(z)$. Also, $d(w, F^{n-1}(z)) \le n-1$. It follows that $d(v, F^{n-1}(z)) \le 2n-2$. So v is covered by the cone of height $2n-2$ at $F^{n-1}(z)$. Since $F^{n-1}(z) \in A'_n$, it is proved that A'_n gives a $(2n-2)$-covering by cones. It follows that $\lambda'_{2n-2} \le \mathbb{P}[\boldsymbol{o} \in A'_n] \le \mathbb{P}[\boldsymbol{o} \in A_n]$ (where the last inequality can be verified by the mass transport principle (2.1)). This implies the second inequality in (4.6).

To prove the first inequality in (4.6), let $\boldsymbol{S}$ be the optimal covering by cones of height n given by the algorithm in Lemma 4.4. Send unit mass from each vertex $v \in \boldsymbol{S}$ to the first vertex in $v, F(v), \ldots, F^n(v)$ which lies in A_{n+1} (if there is any). So the outgoing mass from v is at most $1_{\{v \in \boldsymbol{S}\}}$. In the next paragraph, it is proved that the incoming mass to each $w \in A_{n+1}$ is at least 1. This in turn (by the mass transport principle) implies that $\mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}] \ge \mathbb{P}[\boldsymbol{o} \in A_{n+1}]$, which proves the first inequality in (4.6).

The final step consists in proving that the incoming mass to each $w \in A_{n+1}$ is at least 1. If $h(w) = n$, then $w \in \boldsymbol{S}$ and the claim is proved. So assume $h(w) > n$. By considering the longest path in $D(w)$ from w, one can find a vertex z such that $w = F^{n+1}(z)$ and $h(z) = h(w) - (n+1)$. This implies that no vertex in $\{F(z), \ldots, F^n(z)\}$ is in A_{n+1}. So to prove the claim, it suffices to show that at least one of these vertices or w itself lies in $\boldsymbol{S}$. Note that in the algorithm in Lemma 4.4, at each step, the height of w decreases by a value at least 1 and at most $n+1$ until w is removed from the tree. So in the last step before w is removed, the height of w is in $\{0, 1, \ldots, n\}$. This is possible only if in the same step of the algorithm, an element of $\{F(z), \ldots, F^n(z), w\}$ is added to $\boldsymbol{S}$. This implies the claim and the lemma is proved. □

Now, the tools needed to prove the main results are available.

Proof of Theorem 4.2. Lemma 4.5 and (4.5) imply that the upper and lower Minkowski dimensions of $\boldsymbol{T}$ are exactly the upper and lower decay rates of $\mathbb{P}[h(\boldsymbol{o}) \bmod n = -1]$ respectively. So one should prove that these rates are equal to the upper and lower decay rates of $\mathbb{P}[h(\boldsymbol{o}) \ge n]$ plus 1.

The first step consists in showing that $\mathbb{P}[h(\boldsymbol{o}) = n]$ is non-increasing in n. To see this, send unit mass from each vertex v to $F(v)$ if $h(v) = n$ and $h(F(v)) = n+1$. Then the outgoing mass is at most $1_{\{h(v)=n\}}$ and the incoming mass is at least $1_{\{h(v)=n+1\}}$. The result is then followed by the mass transport principle. This monotonicity implies that $n \cdot \mathbb{P}[h(\boldsymbol{o}) \bmod n = -1] \ge \mathbb{P}[h(\boldsymbol{o}) \ge n-1]$. Similarly, by monotonicity,

$$\begin{aligned} \frac{n}{2}\mathbb{P}[h(\boldsymbol{o}) \bmod n = -1] &\le \mathbb{P}\left[h(\boldsymbol{o}) \bmod n \in \{-1, -2, \ldots, -\left\lceil \frac{n}{2} \right\rceil\}\right] \\ &\le \mathbb{P}\left[h(\boldsymbol{o}) \ge \left\lfloor \frac{n}{2} \right\rfloor\right]. \end{aligned}$$

These inequalities conclude the proof of (4.1) and (4.2).

It remains to prove (4.3). The second inequality follows from (4.1) and the fact that $\overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o})=n\right]\right) \geq \overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o})\geq n\right]\right)+1$, which is not hard to see. We now prove the first inequality. Fix $0<\epsilon<\alpha<\overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o})=n\right]\right)$. So there is a sequence $n_0<n_2<\cdots$ such that $\mathbb{P}\left[h(\boldsymbol{o})=n_i\right]<n_i^{-\alpha}$ for each i. One may assume the sequence is such that $n_i\geq 2^i$ for each i. Now, for each $k\in\mathbb{N}$, consider the following covering of $\boldsymbol{T}$:

$$\boldsymbol{R}_k(v) := \begin{cases} 2(n_i-n_{i-1}), & \text{if } h(v)=n_i \text{ and } i>k,\\ 2n_k, & \text{if } h(v)=n_k,\\ 0, & \text{otherwise.}\end{cases}.$$

By arguments similar to Lemma 4.5, it can be seen that $\boldsymbol{R}_k$ is indeed a covering. It is claimed that $\mathbb{E}\left[\boldsymbol{R}_k(\boldsymbol{o})^{\alpha-\epsilon}\right]\to 0$ as $k\to\infty$ If the claim is proved, then $\mathrm{udim}_H(\boldsymbol{T})\geq\alpha-\epsilon$ and the proof of (4.3) is concluded. Let $c:=2^{\alpha-\epsilon}$. One has

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{R}_k(\boldsymbol{o})^{\alpha-\epsilon}\right] &= c n_k^{\alpha-\epsilon}\mathbb{P}\left[h(\boldsymbol{o})=n_k\right]+c\sum_{i=k+1}^{\infty}(n_i-n_{i-1})^{\alpha-\epsilon}\mathbb{P}\left[h(\boldsymbol{o})=n_i\right]\\ &\leq c n_k^{-\epsilon}+c\sum_{i=k+1}^{\infty}(n_i-n_{i-1})^{\alpha-\epsilon}n_i^{-\alpha}.\end{aligned}$$

Therefore, it is enough to prove that

$$\sum_{i=1}^{\infty}(n_i-n_{i-1})^{\alpha-\epsilon}n_i^{-\alpha}<\infty. \tag{4.7}$$

It is easy to see that the maximum of the function $(x-n_{i-1})^{\alpha-\epsilon}x^{-\alpha}$ over $x\geq n_{i-1}$ happens at $\frac{\alpha}{\epsilon}n_{i-1}$ and the maximum value is $c'n_{i-1}^{-\epsilon}$, where $c'=(\frac{\alpha}{\epsilon}-1)^{\alpha-\epsilon}$ is a constant. So the left hand side of (4.7) is at most $c'\sum_{i=0}^{\infty}n_i^{-\epsilon}$, which is finite by the assumption $n_i\geq 2^i$. So (4.7) is proved and the proof is completed. □

4.2 Instances of Unimodular Trees

This subsection discusses the dimension of some explicit unimodular trees. More examples are given in Subsection 4.5 below, in Part II (e.g., the unimodular Galton-Watson tree and the Poisson weighted infinite tree), and also in [4] (e.g., uniform spanning forests).

4.2.1 The Canopy Tree

The canopy tree C_k with offspring cardinality k [1] is constructed as follows. Its vertex set is partitioned in levels $L_0, L_1, \ldots$. Each vertex in level n is connected to k vertices in level $n-1$ (if $n\neq 0$) and one vertex (its parent) in level $n+1$. Let $\boldsymbol{o}$ be a random vertex of C_k such that $\mathbb{P}\left[\boldsymbol{o}\in L_n\right]$ is proportional to k^{-n}. Then, $[C_k,\boldsymbol{o}]$ is a unimodular random tree.

Below, three types of metrics are considered on C_k. First, consider the graph-distance metric. Given $n \in \mathbb{N}$, let $S := \{v \in C_k : h(v) \geq n\}$, where $h(v)$ is the height of v defined in Subsection 4.1.2. The set S gives an equivariant n-covering and $\mathbb{P}[\boldsymbol{o} \in S]$ is exponentially small as $n \to \infty$. So $\mathrm{udim}_M(C_k) = \mathrm{udim}_H(C_k) = \infty$.

Second, for each n, let the length of each edge between L_n and L_{n+1} be a^n, where $a > 1$ is constant. Let d_1 be the resulting metric on C_k. Given $r > 0$, let S_1 be the set of vertices having distance at least r/a to L_0 (under d_1). One can show that S_1 is an r-covering of (C_k, d_1) and $\mathrm{decay}(\mathbb{P}[\boldsymbol{o} \in S_1]) = \log k / \log a$. Therefore, $\underline{\mathrm{udim}}_M(C_k, d_1) \geq \log k / \log a$. On the other hand, one can see that the ball of radius a^n centered at $\boldsymbol{o}$ (under d_1) has cardinality of order k^n. One can then use Lemma 3.18 to show that $\mathrm{udim}_H(C_k, d_1) \leq \log k / \log a$. So $\mathrm{udim}_M(C_k, d_1) = \mathrm{udim}_H(C_k, d_1) = \log k / \log a$.

Third, replace a^n by $n!$ in the second case and let d_2 be the resulting metric. Then, the cardinality of the ball of radius r centered at $\boldsymbol{o}$ has order less than r^α for every $\alpha > 0$. One can use Lemma 3.18 again to show that $\mathrm{udim}_H(C_k, d_2) \leq \alpha$. This implies that $\mathrm{udim}_M(C_k, d_2) = \mathrm{udim}_H(C_k, d_2) = 0$.

4.2.2 The Generalized Canopy Tree

This example generalizes the canopy tree of Subsection 4.2.1. The goal is to provide an example where the lower Minkowski dimension, the upper Minkowski dimension and the Hausdorff dimension are all different when suitable parameters are chosen.

Fix $p_0, p_1, \ldots > 0$ such that $\sum p_i = 1$. Let $\boldsymbol{U}_0, \boldsymbol{U}_1, \ldots$ be an i.i.d. sequence of random number in $[0, 1]$ with the uniform distribution. For each $n \geq 0$, let $\Phi_n := \left(\frac{1}{p_n}(\mathbb{Z} + \boldsymbol{U}_n)\right) \times \{n\}$, which is a point process on the horizontal line $y = n$ in the plane. Let $\boldsymbol{o}_n := (\frac{1}{p_n}\boldsymbol{U}_n, n) \in \Phi_n$ and $\Phi := \cup_i \Phi_i$. Then, Φ is a point process in the plane which is stationary under horizontal translations. Choose $\boldsymbol{m}$ independent of the sequence $(\boldsymbol{U}_i)_i$ such that $\mathbb{P}[\boldsymbol{m} = n] = p_n$ for each n. Then, let $\boldsymbol{o} := \boldsymbol{o}_{\boldsymbol{m}}$.

Construct a graph $\boldsymbol{T}$ on Φ as follows: For each n, connect each $x \in \Phi_n$ to its closest point (or closest point on its right) in Φ_{n+1}. Note that $\boldsymbol{T}$ is a forest by definition. However, the next lemma shows that $[\boldsymbol{T}, \boldsymbol{o}]$ is a unimodular tree.

Definition 4.6. The **generalized canopy tree** with parameters $p_0, p_1, \ldots$ is the unimodular tree $[\boldsymbol{T}, \boldsymbol{o}]$ constructed above.

Note that in the case where p_n is proportional to k^{-n} for k fixed, $[\boldsymbol{T}, \boldsymbol{o}]$ is just the ordinary canopy tree C_k of Subsection 4.2.1. Also, one can generalize the above construction by letting Φ_n be a sequence of point processes which are (jointly) stationary under horizontal translations (see e.g., the arXiv version).

Lemma 4.7. *One has*

(i) $[\Phi, \boldsymbol{o}]$*, endowed with the Euclidean metric, is a unimodular discrete space.*

(ii) $\boldsymbol{T}$ *is a tree a.s. and* $[\boldsymbol{T}, \boldsymbol{o}]$ *is unimodular.*

Proof. For part (i), it is enough to show that $\Phi - \boldsymbol{o}$ is a point-stationary point process in the plane (see Example 2.6). This is skipped for brevity (see the arXiv version). The main ingredients are using stationarity of Φ under horizontal translations and the fact that $\Phi_n - \boldsymbol{o}_n$ is point-stationary (the proof is similar to that of the formula for the Palm version of the superposition of stationary point processes, e.g., in [28].)

To prove (ii), note that $\boldsymbol{T}$ can be realized as an equivariant process on Φ (see Definition 2.8 and Remark 2.12). Therefore, by Lemma 2.11 and Theorem 3.31, it is enough to prove that $\boldsymbol{T}$ is connected a.s. Nevertheless, the same lemma implies that the connected component $\boldsymbol{T}'$ of $\boldsymbol{T}$ containing $\boldsymbol{o}$ is a unimodular tree. Since it is one-ended, Theorem 3.9 of [6] implies that the *foils* $\boldsymbol{T}' \cap \Phi_i$ are infinite a.s. By noting that the edges do not cross (as segments in the plane), one obtains that $\boldsymbol{T}' \cap \Phi_i$ should be the whole Φ_i; hence, $\boldsymbol{T}' = \boldsymbol{T}$. Therefore, $\boldsymbol{T}$ is connected a.s. and the claim is proved. □

Proposition 4.8. *The sequence $(p_n)_n$ can be chosen such that*

$$\underline{\mathrm{udim}}_M(\boldsymbol{T}) < \overline{\mathrm{udim}}_M(\boldsymbol{T}) < \mathrm{udim}_H(\boldsymbol{T}),$$

where $\boldsymbol{T}$ is endowed with the graph-distance metric. Moreover, for any $0 \leq \alpha \leq \beta \leq \gamma \leq \infty$, the sequence $(p_n)_n$ can be chosen such that

$$\underline{\mathrm{udim}}_M(\boldsymbol{T}) \leq \alpha, \quad \overline{\mathrm{udim}}_M(\boldsymbol{T}) = \beta, \quad \mathrm{udim}_H(\boldsymbol{T}) \geq \gamma.$$

For example, it is possible to have $\mathrm{udim}_M(\boldsymbol{T}) = 0$ and $\mathrm{udim}_H(\boldsymbol{T}) = \infty$ simultaneously.

Proof. $\boldsymbol{T}$ is a one-ended tree (see Subsection 4.1.2). Assume the sequence $(p_n)_n$ is non-increasing. So the construction implies that there is no leaf of the tree in Φ_n for all $n > 0$. Therefore, for all $n \geq 0$, the height of every vertex in Φ_n is precisely n. So by letting $q_n := \sum_{i \geq n} p_i$, Theorem 4.2 implies that

$$\begin{aligned} \mathrm{udim}_H(\boldsymbol{T}) &\geq \overline{\mathrm{decay}}\,(p_n)\,, \\ \overline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \overline{\mathrm{decay}}\,(q_n)\,, \\ \underline{\mathrm{udim}}_M(\boldsymbol{T}) &= 1 + \underline{\mathrm{decay}}\,(q_n)\,. \end{aligned}$$

For simplicity, assume $0 < \alpha$ and $\gamma < \infty$ (the other cases can be treated similarly). Define $n_0, n_1, \ldots$ recursively as follows. Let $n_0 := 0$. Given that n_i is defined, let n_{i+1} be large enough such that the line connecting points $(n_i, n_i^{-\beta})$ and $(n_{i+1}, n_{i+1}^{-\beta})$ intersects the graph of the function $x^{-\alpha}$ and has slope larger than $-n^{-\gamma}$. Now, let $q_{n_i} := n_i^{-\beta}$ for each i and define q_n linearly in the interval $[n_i, n_{i+1}]$. Let $p_n := q_n - q_{n+1}$. It can be seen that p_n is non-increasing, $\underline{\mathrm{decay}}\,(q_n) \leq \alpha$, $\overline{\mathrm{decay}}\,(q_n) = \beta$ and $\overline{\mathrm{decay}}\,(p_n) \geq \gamma$. □

4.2.3 Unimodular Eternal Galton-Watson Trees

Eternal Galton-Watson (`EGW`) trees are defined in [6]. Unimodular `EGW` trees (in the nontrivial case) can be characterized as unimodular one-ended trees in which the descendants of the root constitute a Galton-Watson tree. Also, the latter Galton-Watson tree is necessarily critical. Here, the trivial case that each vertex has exactly one offspring is excluded (where the corresponding `EGW` tree is a bi-infinite path). In particular, the *Poisson skeleton tree* [3] is an eternal Galton-Watson tree.

Recall that the offspring distribution of a Galton-Watson tree is the probability measure $(p_0, p_1, \dots)$ on $\mathbb{Z}^{\geq 0}$ where p_n is the probability that the root has n offsprings.

Proposition 4.9. *Let* $[\boldsymbol{T}, \boldsymbol{o}]$ *be a unimodular eternal Galton-Watson tree. If the offspring distribution has finite variance, then* $\mathrm{udim}_M(\boldsymbol{T}) = 2$.

Proof. By Kesten's theorem [20] for the Galton-Watson tree formed by the descendants of the root, $\lim_n n\mathbb{P}[h(\boldsymbol{o}) \geq n]$ exists and is positive. It follows that $\mathrm{decay}(\mathbb{P}[h(\boldsymbol{o}) \geq n]) = 1$. So the claim is implied by Theorem 4.2. □

In fact, the same result holds for the Hausdorff dimension of $\boldsymbol{T}$, which will be proved in Theorem II.3.7.

Conjecture 4.10. *If the offspring distribution is in the domain of attraction of an* α*-stable distribution, where* $\alpha \in [1, 2]$*, then*

$$\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{T}) = \frac{\alpha}{\alpha - 1}.$$

4.3 Examples Associated with Random Walks

Let μ be a probability measure on $\mathbb{R}^k$. Consider the simple random walk $(S_n)_{n\in\mathbb{Z}}$, where $S_0 = 0$ and the jumps $S_n - S_{n-1}$ are i.i.d. with distribution μ. In this subsection, unimodular discrete spaces are constructed based on the image and the zero set of this random walk and their dimensions are studied in some special cases. The graph of the simple random walk will be studied in Subsection II.3.3.2.

4.3.1 The Image of the Simple Random Walk

Assume the random walk is transient; i.e., visits every given ball only finitely many times. It follows that the image $\Phi = \{S_n\}_{n\in\mathbb{Z}}$ is a random discrete subset of $\mathbb{R}^k$. If no point is visited more than once a.s. (as in the following theorem), then it can be seen that Φ is a point-stationary point process, hence, $[\Phi, 0]$ is a unimodular discrete space. In the general case, by similar arguments, one should bias the distribution of $[\Phi, 0]$ by the inverse of the *multiplicity* of the origin; i.e., by $1/\#\{n : S_n = 0\}$, to obtain a unimodular discrete space. This claim can be proved by direct verification of the mass transport principle.

Proposition 4.11. *Let $\Phi := \{S_n\}_{n\in\mathbb{Z}}$ be the image of a simple random walk S in $\mathbb{R}$, where $S_0 := 0$. Assume the jumps $S_n - S_{n-1}$ are positive a.s.*

(i) $\underline{\mathrm{udim}}_M(\Phi) = \underline{\mathrm{decay}}\left(\mathbb{E}\left[S_1 1_{\{S_1\leq r\}}\right]\right) \geq 1 \wedge \underline{\mathrm{decay}}\left(\mathbb{P}\left[S_1 > r\right]\right)$.

(ii) $\overline{\mathrm{udim}}_M(\Phi) = \overline{\mathrm{decay}}\left(\mathbb{E}\left[S_1 1_{\{S_1\leq r\}}\right]\right) \leq 1 \wedge \overline{\mathrm{decay}}\left(\mathbb{P}\left[S_1 > r\right]\right)$.

(iii) If $\beta := \mathrm{decay}\left(\mathbb{P}\left[S_1 > r\right]\right)$ *exists, then* $\mathrm{udim}_M(\Phi) = 1 \wedge \beta$.

In fact, the same claims are valid for the Hausdorff dimension as well. This will be shown in Theorem II.3.9.

Proof. For every $r > 0$, one has $\mathbb{P}\left[\Phi \cap (0, r) = \emptyset\right] = \mathbb{P}\left[S_1 \geq r\right]$. So the claims are direct consequences of Proposition 3.14. □

The image of the nearest-neighbor simple random walk in $\mathbb{Z}^k$ will be studied in [4]. It will be shown that it has dimension 2 when $k \geq 2$. Furthermore, a *doubling property* will be proved in this case.

As another example, if $[\boldsymbol{T}, \boldsymbol{o}]$ is any unimodular tree such that the simple random walk on $\boldsymbol{T}$ is transient a.s., then the image of the (two sided) simple random walk on $\boldsymbol{T}$ is another unimodular tree (after biasing by the inverse of the multiplicity of the root). The new tree is two-ended a.s., and hence, is 1-dimensional by Theorem 4.1.

4.3.2 Zeros of the Simple Random Walk

Proposition 4.12. *Let Ψ be the zero set of the symmetric simple random walk on $\mathbb{Z}$ with uniform jumps in $\{\pm 1\}$. Then,* $\mathrm{udim}_M(\Psi) = \frac{1}{2}$.

In fact, the same result holds for the Hausdorff dimension of Ψ, which will be proved in Proposition II.3.10.

Proof. Represent Ψ uniquely as $\Psi := \{S_n : n \in \mathbb{Z}\}$ such that $S_0 := 0$ and $S_n < S_{n+1}$ for each n. Then, $(S_n)_n$ is another simple random walk and Ψ is its image. The distribution of the jump S_1 is explicitly computed in the classical literature on random walks (using the reflection principle). In particular, there exist $c_1, c_2 > 0$ such that $c_1 r^{-\frac{1}{2}} < \mathbb{P}\left[S_1 > r\right] < c_2 r^{-\frac{1}{2}}$ for every $r \geq 1$. Therefore, the claim is implied by Proposition 4.11. □

4.4 A Subspace with Larger Minkowski Dimension

Let $\Phi \subseteq \mathbb{R}$ be an arbitrary point-stationary point process and $0 < \alpha < 1$. Let S_1 be the first point of Φ on the right of the origin. Assume $\beta := \mathrm{decay}\left(\mathbb{P}\left[S_1 > r\right]\right)$ exists (e.g., the case in Theorem 4.11) and $\alpha < \beta < 1$. Then, Proposition 3.14 gives that $\mathrm{udim}_M(\Phi) = \beta$.

Consider the intervals divided by consecutive points of Φ. In each such interval, namely (a, b), add $\lceil (b-a)^\alpha \rceil - 1$ points to split the interval into $\lceil (b-a)^\alpha \rceil$ equal parts. Let Φ' denote the resulting point process. By the assumption

$\alpha < \beta$, one can show that $\mathbb{E}[S_1^\alpha] < \infty$. Now, by biasing the distribution of Φ' by $\lceil S_1^\alpha \rceil$ and changing the origin to a point of $\Phi' \cap [0, S_1)$ chosen uniformly at random, one obtains a point-stationary point process Ψ (see Theorem 5 in [22] and also the examples in [2]). The distribution of Ψ is determined by the following equation (where h is any measurable nonnegative function).

$$\mathbb{E}[h(\Psi)] = \frac{1}{\mathbb{E}[\lceil S_1^\alpha \rceil]} \mathbb{E}\left[\sum_{x \in \Phi' \cap [0, S_1)} h(\Phi' - x)\right]. \tag{4.8}$$

Proposition 4.13. *Let Φ and Ψ be as above. Then, Φ has the same distribution as an equivariant subspace of Ψ (conditioned on having the root) and*

$$\mathrm{udim}_M(\Phi) = \beta > \frac{\beta - \alpha}{1 - \alpha} = \mathrm{udim}_M(\Psi).$$

Before presenting the proof, note that Theorem 3.34 implies that $\mathrm{udim}_H(\Phi) = \mathrm{udim}_H(\Psi)$. Therefore, the proposition implies $\mathrm{udim}_M(\Psi) < \mathrm{udim}_H(\Psi)$.

Proof. Let $\boldsymbol{A}$ be the set of newly-added points in Ψ, which can be defined by adding marks from the beginning and is an equivariant subset of Ψ. By (4.8), one can verify that $\Psi \setminus \boldsymbol{A}$ conditioned on $0 \notin \boldsymbol{A}$ has the same distribution as Φ (see also Proposition 6 in [22]). Also, by letting $c := \mathbb{E}[\lceil S_1^\alpha \rceil]$, (4.8) gives

$$\begin{aligned}
\mathbb{P}[\Psi \cap (0, r) = 0] &= \frac{1}{c}\mathbb{E}\left[\sum_{x \in \Phi' \cap [0, S_1)} 1_{\{(\Phi' - x) \cap (0, r) = \emptyset\}}\right] \\
&= \frac{1}{c}\mathbb{E}\left[\lceil S_1^\alpha \rceil \, 1_{\{\Phi' \cap (0, r) = \emptyset\}}\right] \\
&= \frac{1}{c}\mathbb{E}\left[\lceil S_1^\alpha \rceil \, 1_{\{S_1 / \lceil S_1^\alpha \rceil > r\}}\right].
\end{aligned}$$

One can easily deduce that $\mathrm{decay}\left(\mathbb{P}[\Psi \cap (0, r) = 0]\right) = (\beta - \alpha)/(1 - \alpha)$. Therefore, Proposition 3.14 gives the claim. □

4.5 A Drainage Network Model

Practical observations show that large river basins have a fractal structure. For example, [18] discovered a power law relating the area and the height of river basins. There are various ways to model river basins and their fractal properties in the literature. In particular, [27] formalizes and proves a power law with exponent 3/2 for a specific model called *Howard's model*. Below, the simpler model of [26] is studied. One can ask similar questions for Howard's model or other drainage network models.

Connect each (x, y) in the even lattice $\{(x, y) \in \mathbb{Z}^2 : x + y \bmod 2 = 0\}$ to either $(x - 1, y - 1)$ or $(x + 1, y - 1)$ with equal probability in an i.i.d. manner to obtain a directed graph $\boldsymbol{T}$. Note that the downward path starting at a given vertex is the rotated graph of a simple random walk. It is known that $\boldsymbol{T}$ is connected and is a one-ended tree (see e.g., [27]). Also, by Lemma 2.11, $[\boldsymbol{T}, 0]$ is unimodular.

Note that by considering the Euclidean metric on $\boldsymbol{T}$, the Hausdorff dimension of $\boldsymbol{T}$ is 2. In the following, the graph-distance metric is considered on $\boldsymbol{T}$.

Proposition 4.14. *Under the graph-distance metric, one has* $\mathrm{udim}_M(\boldsymbol{T}) = \frac{3}{2}$.

Before presenting the proof, it is worthwhile mentioning that the same result is valid for the Hausdorff dimension of $\boldsymbol{T}$, which will be proved in Theorem II.3.14.

Proof. The idea is to use Theorem 4.2. Following [27], there are two *backward paths* (going upward) in the odd lattice that surround the descendants $D(\boldsymbol{o})$ of the origin. These two paths have exactly the same distribution as (rotated) graphs of independent simple random walks starting at $(-1,0)$ and $(1,0)$, respectively, until they hit for the first time. In this setting, $h(\boldsymbol{o})$ is exactly the hitting time of these random walks. So classical results on random walks imply that $\mathbb{P}[h(\boldsymbol{o}) \geq n]$ is bounded between two constant multiples of $n^{-\frac{1}{2}}$ for all n. So Theorem 4.2 implies that $\mathrm{udim}_M(\boldsymbol{T}) = \frac{3}{2}$. □

4.6 Self Similar Unimodular Discrete Spaces

This section provides a class of examples of unimodular discrete spaces obtained by discretizing self-similar sets. Let $l \geq 1$ and $f_1, \ldots, f_l$ be similitudes of $\mathbb{R}^k$ with similarity ratios $r_1, \ldots, r_l$ respectively (i.e., $\forall x, y \in \mathbb{R}^k : |f_i(x) - f_i(y)| = r_i |x - y|$). For every $n \geq 0$ and every string $\sigma = (j_1, \ldots, j_n) \in \{1, \ldots, l\}^n$, let $f_\sigma := f_{j_1} \cdots f_{j_n}$. Also let $|\sigma| := n$. Fix a point $o \in \mathbb{R}^k$ (one can similarly start with a finite subset of $\mathbb{R}^k$ instead of a single point). Let $K_0 := \{o\}$ and $K_{n+1} := \bigcup_j f_j(K_n)$ for each $n \geq 0$. Equivalently,

$$K_n = \{f_\sigma(o) : |\sigma| = n\}. \tag{4.9}$$

Recall that if $r_i < 1$ for all i, then by contraction arguments, K_n converges in the Hausdorff metric to *the attractor* of $f_1, \ldots, f_l$ (see e.g., Section 2.1 of [11]). The attractor is the unique compact set $K \subseteq \mathbb{R}^k$ such that $K = \bigcup_i f_i(K)$. In addition, if the f_i's satisfy the *open set condition*; i.e., there is a bounded open set $V \subseteq \mathbb{R}^k$ such that $f_i(V) \subseteq V$ and $f_i(V) \cap f_j(V) = \emptyset$ for each i, j, then the Minkowski and Hausdorff dimensions of K are equal to the *similarity dimension*, which is the unique $\alpha \geq 0$ such that $\sum r_i^\alpha = 1$.

The following is the main result of this section. It introduces a discrete analogue of self-similar sets by scaling the sets K_n and taking local weak limits.

Theorem 4.15. *Let $\boldsymbol{o}_n$ be a point of K_n chosen uniformly at random, where K_n is defined in (4.9). Assume that $r_i = r < 1$ for all i and the open set condition is satisfied. Then,*

(i) $[r^{-n} K_n, \boldsymbol{o}_n]$ *converges weakly to some unimodular discrete space.*

(ii) *The unimodular Minkowski and Hausdorff dimension of the limiting space are equal to* $\alpha := \log l/|\log r|$. *Moreover, it has positive and finite* α*-dimensional Hausdorff measure.*

The proof is given at the end of this subsection. In fact, a point process Ψ in $\mathbb{R}^k$ will be constructed such that $[r^{-n}K_n, \boldsymbol{o}_n]$ converges weakly to $[\Psi, o]$. In addition, $\Psi - o$ is point-stationary. It can also be constructed directly by the algorithm in Remark 4.22 below.

Definition 4.16. The unimodular discrete space in Theorem 4.15 is called a **self similar unimodular discrete space**.

It should be noted that self similar unimodular discrete spaces depend on the choice of the initial point o in general.

The following are examples of unimodular self similar discrete spaces. The reader is also invited to construct a unimodular discrete version of the Sierpinski carpet similarly.

Example 4.17. If $f_1(x) := x/2$ and $f_2(x) := (1+x)/2$, then the limiting space is just $\mathbb{Z}$. Similarly, the lattice $\mathbb{Z}^k$ and the triangular lattice in the plane are self similar unimodular discrete spaces.

Example 4.18 (Unimodular Discrete Cantor Set)**.** Start with two points $K_0 := \{0, 1\}$. Let $f_1(x) := x/3$ and $f_2(x) := (2 + x)/3$. Then, K_n is the set of the interval ends in the n-th step of the definition of the Cantor set. Here, it is easy to see that the random set $\Psi_n := 3^n(K_n - \boldsymbol{o}_n) \subseteq \mathbb{Z}$ converges weakly to the random set $\Psi \subseteq \mathbb{Z}$ defined as follows: $\Psi := \cup_n \boldsymbol{T}_n$, where $\boldsymbol{T}_n$ is defined by letting $\boldsymbol{T}_0 := \{0, \pm 1\}$ and $\boldsymbol{T}_{n+1} := \boldsymbol{T}_n \cup (\boldsymbol{T}_n \pm 2 \times 3^n)$, where the sign is chosen i.i.d., each sign with probability 1/2. Note that each $\boldsymbol{T}_n$ has the same distribution as Ψ_n, but the sequence $\boldsymbol{T}_n$ is nested. In addition, since $\boldsymbol{o}_n$ is chosen uniformly, Ψ_n and Ψ are point-stationary point processes, and hence $[\Psi, 0]$ is unimodular (a deterministic discrete Cantor set exists in the literature which is not unimodular). Theorem 4.15 implies that $\mathrm{udim}_M(\Psi) = \mathrm{udim}_H(\Psi) = \log 2/\log 3$.

Example 4.19 (Unimodular Discrete Koch Snowflake)**.** Let C_n be the set of points in the n-th step of the construction of the Koch snowflake. Let $\boldsymbol{x}_n$ be a random point of C_n chosen uniformly and $\Phi_n := 3^n(C_n - \boldsymbol{x}_n)$. It can be seen that Φ_n tends weakly to a random discrete subset Φ of the triangular lattice which is almost surely a bi-infinite path (note that the cycle disappears in the limit). It can be seen that Φ can be obtained by Theorem 4.15. In this paper, Φ is called the **unimodular discrete Koch snowflake**. Also, Theorem 4.15 implies that $\mathrm{udim}_M(\Phi) = \mathrm{udim}_H(\Phi) = \log 4/\log 3$.
In addition, Φ can be constructed explicitly as $\Phi := \cup_n \boldsymbol{T}_n$, where $\boldsymbol{T}_n$ is a random finite path in the triangular lattice with distinguished end points $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ defined inductively as follows: Let $\boldsymbol{T}_1 := \{\boldsymbol{A}_1, \boldsymbol{B}_1\}$, where $\boldsymbol{A}_1$ is the origin and $\boldsymbol{B}_1$ is a neighbor of the origin in the triangular lattice chosen uniformly at random. For each $n \geq 1$, given $(\boldsymbol{T}_n, \boldsymbol{A}_n, \boldsymbol{B}_n)$, let $(\boldsymbol{T}_{n+1}, \boldsymbol{A}_{n+1}, \boldsymbol{B}_{n+1})$ be

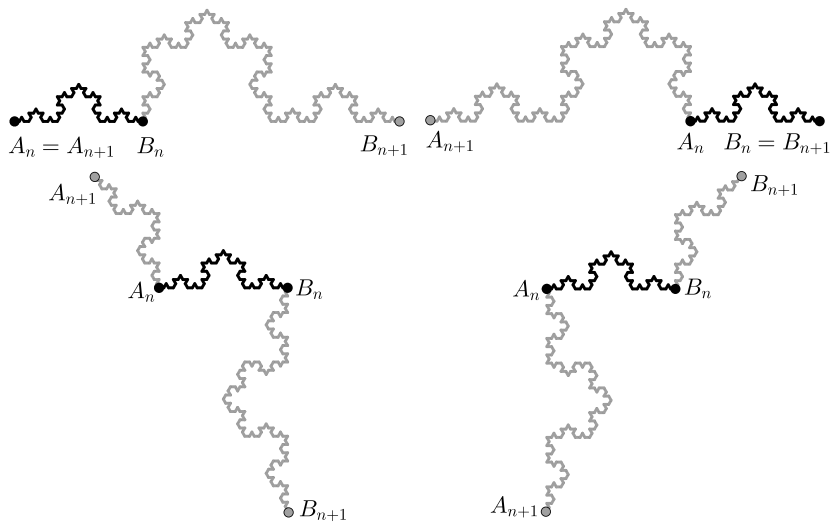

Figure 1: Four ways to attach 3 isometric copies to $\boldsymbol{T}_n$ in the construction of the unimodular discrete Koch snowflake, where each copy is a rotated/translated version of $\boldsymbol{T}_n$ (relative to $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$). Here, $\boldsymbol{T}_n$ is shown in black.

obtained by attaching to $\boldsymbol{T}_n$ three isometric copies of itself as shown in Figure 1. There are 4 ways to attach the copies and one of them should be chosen at random with equal probability (the copies should be attached to $\boldsymbol{T}_n$ relative to the position of $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$). It can be seen that no points overlap.

Remark 4.20. If the r_i's are not all equal, the guess is that there is no scaling of the sequence $[K_n, \boldsymbol{o}_n]$ that converges to a nontrivial unimodular discrete space (which is not a single point). This has been verified by the authors in the case $o \in V$. In this case, by letting $\boldsymbol{a}_n$ be the distance of $\boldsymbol{o}_n$ to its closest point in K_n, it is shown that for any $\epsilon > 0$, $\mathbb{P}\left[\boldsymbol{a}_n/(\bar{r})^n < \epsilon\right] \to \frac{1}{2}$ and $\mathbb{P}\left[\boldsymbol{a}_n/(\bar{r})^n > \frac{1}{\epsilon}\right] \to \frac{1}{2}$, where $\bar{r}$ is the geometric mean of $r_1, \ldots, r_l$. This implies the claim (note that the counting measure matters for convergence; e.g., $\{0, \frac{1}{n}\}$ does not converge to $\{0\}$).

To prove Theorem 4.15, it is useful to consider the following nested version of the sets K_n (note that K_n is not necessarily contained in K_{n+1}, unless o is a fixed point of some f_i). Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots$ be i.i.d. uniform random numbers in $\{1, \ldots, l\}$ and $\boldsymbol{\delta}_n := (\boldsymbol{u}_n, \ldots, \boldsymbol{u}_1)$. Let $\boldsymbol{o}'_n := f_{\boldsymbol{\delta}_n}(o)$. Let $\hat{\boldsymbol{K}}_n := f_{\boldsymbol{\delta}_n}^{-1} K_n = f_{\boldsymbol{u}_1}^{-1} \cdots f_{\boldsymbol{u}_n}^{-1} K_n$. The chosen order of the indices in $\boldsymbol{\delta}_n$ ensures that $\hat{\boldsymbol{K}}_n \subseteq \hat{\boldsymbol{K}}_{n+1}$ for all n. It is easy to see that $[\hat{\boldsymbol{K}}_n, o]$ has the same distribution as $[r^{-n} K_n, \boldsymbol{o}'_n]$. For $v \in \hat{\boldsymbol{K}}_n$, let

$$\boldsymbol{w}_n(v) := \#\{\sigma : |\sigma| = n, f_\sigma(o) = f_{\boldsymbol{\delta}_n}(v)\}.$$

One has $\boldsymbol{w}_n(v) \leq \boldsymbol{w}_{n+1}(v)$. Note that in the case $o \in V$, $\boldsymbol{w}_n(\cdot) = 1$ and the arguments are much simpler. The reader can assume this at first reading.

In the following, for $x \in \mathbb{R}^k$, $B_r(x)$ represents the closed ball of radius r centered at x in $\mathbb{R}^k$.

Lemma 4.21. *Let $\hat{\boldsymbol{K}} := \cup_n \hat{\boldsymbol{K}}_n$ and $\boldsymbol{w}(v) := \lim_n \boldsymbol{w}_n(v)$ for $v \in \hat{\boldsymbol{K}}$.*

(i) $\boldsymbol{w}(\cdot)$ is uniformly bounded.

(ii) Almost surely, $\hat{\boldsymbol{K}}$ is a discrete set.

(iii) The distribution of $[\hat{\boldsymbol{K}}, o]$, biased by $1/\boldsymbol{w}(o)$, is the limiting distribution alluded to in Theorem 4.15.

Proof. (i). Assume $f_{\sigma_1(o)} = \cdots = f_{\sigma_k}(o)$ and $|\sigma_j| = n$ for each $j \leq k$. Let D be a fixed number such that V intersects $B_D(o)$. Now, the sets $f_{\sigma_j}(V)$ for $1 \leq j \leq k$ are disjoint and intersect a common ball of radius Dr^n. Moreover, each of them contains a ball of radius ar^n and each is contained in a ball of radius br^n (for some fixed $a, b > 0$). Therefore, Lemma 2.2.5 of [11] implies that $k \leq (\frac{D+2b}{a})^k =: C$. This implies that $\boldsymbol{w}_n(\cdot) \leq C$ a.s., hence $\boldsymbol{w}(\cdot) \leq C$ a.s.

(ii). Let D be arbitrary as in the previous part. Assume $f^{-1}_{\boldsymbol{\delta}_n} f_{\sigma_j(o)} \in B_D(o)$ for $j = 1, \ldots, k$. Now, for $j = 1, \ldots, k$, the sets $f_{\sigma_j}(V)$ are disjoint and intersect a common ball of radius $2Dr^n$. As in the previous part, one obtains $k \leq (\frac{2D+2b}{a})^k$. Therefore, $\#N_D(o) \leq (\frac{2D+2b}{a})^k$ a.s. Since this holds for all large enough D, one obtains that $\hat{\boldsymbol{K}}$ is a discrete set a.s.

(iii). Note that the distribution of $\boldsymbol{o}'_n$ is just the distribution of $\boldsymbol{o}_n$ biased by the *multiplicities* of the points in K_n. It follows that biasing the distribution of $[\hat{\boldsymbol{K}}_n, o]$ by $1/\boldsymbol{w}_n(o)$ gives just the distribution of $[r^{-n}K_n, \boldsymbol{o}_n]$. The latter is unimodular since $\boldsymbol{o}_n$ is uniform in K_n. So the distribution of $[\hat{\boldsymbol{K}}, o]$ biased by $1/\boldsymbol{w}(o)$ is also unimodular and satisfies the claim of Theorem 4.15. □

Proof of Theorem 4.15. Convergence is proved in Lemma 4.21. The rest of the proof is base on the construction of a sequence of equivariant coverings of $\hat{\boldsymbol{K}}$. In this proof, with an abuse of notation, the dimension of $\hat{\boldsymbol{K}}$ means the dimension of the unimodular space obtained by biasing the distribution of $\hat{\boldsymbol{K}}$ by $1/\boldsymbol{w}(o)$ (see Lemma 4.21). Let $D > \mathrm{diam}(K)$ be given, where K is the attractor of $f_1, \ldots, f_l$. Let $m > 0$ be large enough so that $\mathrm{diam}(K_m) < D$. Note that each element in $\hat{\boldsymbol{K}}$ can be written as $f^{-1}_{\boldsymbol{\delta}_n} f_\sigma(o)$ for some n and some string σ of length n. Let $\boldsymbol{\gamma}_m$ be a string of length m chosen uniformly at random and independently of other variables. For an arbitrary n and a string σ of length n, let

$$\boldsymbol{U}_\sigma := f^{-1}_{\boldsymbol{\delta}_{n+m}} f_\sigma(K_m),$$
$$\boldsymbol{z}_\sigma := f^{-1}_{\boldsymbol{\delta}_{n+m}} f_\sigma f_{\boldsymbol{\gamma}_m}(o).$$

Note that $\boldsymbol{U}_\sigma \subseteq \hat{\boldsymbol{K}}$ is always a scaling of K_m with ratio r^{-m} and $\boldsymbol{z}_\sigma \in \boldsymbol{U}_\sigma$. Now, define the following covering of $\hat{\boldsymbol{K}}$:

$$\boldsymbol{R}_m(v) := \begin{cases} Dr^{-m}, & \text{if } v = \boldsymbol{z}_\sigma \text{ for some } \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

It can be seen that $\boldsymbol{R}_m$ gives an equivariant covering. Also, note that $\boldsymbol{R}_m(o) > 0$ if and only if $f_\sigma f_{\boldsymbol{\gamma}_m}(o) = f_{\boldsymbol{\delta}_{n+m}}(o)$ for some n and some string σ of length n. Let $A_{n,m}(o)$ be the set of possible outcomes for $\boldsymbol{\gamma}_m$ such that there exists a string σ of length n such that the last equation holds. One can see that this set is increasing with n and deduce that $\boldsymbol{w}_m(o) \leq \# A_{n,m}(o) \leq \boldsymbol{w}_{n+m}(o)$. By letting $\boldsymbol{w}'_m(o) := \# \cup_n A_{n,m}(o)$, it follows that $\boldsymbol{w}_m(o) \leq \boldsymbol{w}'_m(o) \leq \boldsymbol{w}(o)$. According to the above discussion, $\boldsymbol{R}_m(o) > 0$ if and only if $\boldsymbol{\gamma}_m \in \cup_n A_{n,m}(o)$. So

$$\mathbb{P}\left[\boldsymbol{R}_m(o) > 0 \,|\, \boldsymbol{u}_0, \boldsymbol{u}_1, \ldots\right] = \boldsymbol{w}'_m(o) r^{m\alpha}.$$

Therefore, by considering the biasing that makes $\hat{\boldsymbol{K}}$ unimodular, one gets

$$\mathbb{E}\left[\frac{1}{\boldsymbol{w}(o)} 1_{\{\boldsymbol{R}_m(o)>0\}}\right] = \mathbb{E}\left[\frac{\boldsymbol{w}'_m(o) r^{m\alpha}}{\boldsymbol{w}(o)}\right] \leq r^{m\alpha}. \tag{4.10}$$

Since the balls in the covering have radius Dr^{-m}, one gets $\underline{\mathrm{udim}}_M(\hat{\boldsymbol{K}}) \geq \alpha$.

On the other hand, by (4.10) and monotone convergence, one finds that

$$\mathbb{E}\left[\frac{1}{\boldsymbol{w}(o)} 1_{\{\boldsymbol{R}_m(o)>0\}}\right] \geq \frac{1}{2} r^{m\alpha},$$

for large enough m. Similar to the proof of part (i) of Lemma 4.21, one can show that the sequence of coverings $\boldsymbol{R}_m$ (for $m = 1, 2, \ldots$) is uniformly bounded. Therefore, Lemma 3.10 implies that $\mathrm{udim}_M(\hat{\boldsymbol{K}}) = \alpha$. Moreover, since $\mathbb{E}\left[\boldsymbol{R}_m(o)^\alpha / \boldsymbol{w}(o)\right]$ is bounded (by D^α), one can get that $\mathcal{M}^\alpha(\hat{\boldsymbol{K}}) > 0$.

Lemma 3.18 will be used to bound the Hausdorff dimension. Let $D > 1$ be arbitrary. Choose m such that $r^{-m} \leq D < r^{-m-1}$. By Lemma 4.21, there are finitely many points in $\hat{\boldsymbol{K}} \cap B_D(o)$. Therefore, one finds n such that $\hat{\boldsymbol{K}} \cap B_D(o) = \hat{\boldsymbol{K}}_{n+m} \cap B_D(o)$. It follows that the sets $\{\boldsymbol{U}_\sigma : |\sigma| = n\}$ cover $\hat{\boldsymbol{K}}_{n+m}$. Now, assume $\sigma_1, \ldots, \sigma_k$ are strings of length n such that $\boldsymbol{U}_{\sigma_i}$ are distinct and intersects $B_D(o)$. One obtains that

$$\# B_D(o) \cap \hat{\boldsymbol{K}} \leq \sum_{j=1}^{k} \# B_D(o) \cap \boldsymbol{U}_{\sigma_j} \leq k l^m = k r^{-\alpha m} \leq k D^\alpha. \tag{4.11}$$

Consider the sets $\boldsymbol{V}_{\sigma_j} := f^{-1}_{\boldsymbol{\delta}_{n+m}} f_{\sigma_j}(V)$ which are disjoint (since σ_j's have the same length). Note that if $\epsilon > \mathrm{diam}(V \cup \{o\})$ is fixed, then the ϵ-neighborhood of V contains K_m. Therefore, all $\boldsymbol{V}_{\sigma_j}$'s intersect a common ball of radius $D + \epsilon r^{-m} \leq (1+\epsilon)D$. Moreover, each of them contains a ball of radius $ar^{-m} \geq arD$ and is contained in a ball of radius $br^{-m} \leq bD$ (for some $a, b > 0$ not depending on D). Therefore, Lemma 2.2.5 of [11] implies that $k \leq (\frac{(1+\epsilon)+2b}{ar})^k$. Therefore, (4.11) implies that

$$\# B_D(o) \cap \hat{\boldsymbol{K}} \leq C D^\alpha, \quad \text{a.s.}$$

Therefore, Lemma 3.18 implies that $\mathrm{udim}_H(\hat{\boldsymbol{K}}) \leq \alpha$. Moreover, the proof of the lemma shows that $\mathcal{M}^\alpha(\hat{\boldsymbol{K}}) < \infty$. This completes the proof. □

Remark 4.22. Motivated by Examples 4.18 and 4.19, it can be seen that every unimodular self similar discrete space can be constructed by successively attaching copies of a set to itself. This is expressed in the following algorithm.

$\hat{\boldsymbol{K}}_0 := \{o\}$;
Let g_0 be the identity map;
Choose i.i.d. random numbers $i_1, i_2, \ldots$ uniformly in $\{1, \ldots, l\}$;
for $n = 1, 2, \ldots$ **do**
let $\hat{\boldsymbol{K}}_n$ consist of l isometric copies of $\hat{\boldsymbol{K}}_{n-1}$ as follows

$$\hat{\boldsymbol{K}}_n := \bigcup_{j=1}^{l} g_{n-1} f_{i_n}^{-1} f_j g_{n-1}^{-1}(\hat{\boldsymbol{K}}_{n-1});$$

Let $g_n := g_{n-1} f_{i_n}^{-1}$;
end

4.7 Notes and Bibliographical Comments

Some of the examples in this section, listed below, are motivated by analogous examples in the continuum setting. In fact, the unimodular dimensions of these examples are equal to the ordinary dimensions of the analogous continuum examples. This connection will be discussed further in [4] via scaling limits.

Theorem 4.9 and Conjecture 4.10 regarding the `EGW` are inspired by the dimension of the Brownian continuum random tree and stable trees respectively (see Theorem 5.5 of [15]), which are scaling limits of Galton-Watson trees conditioned to be large. The zero set of the simple random walk (Theorem 4.12) is analogous to the zero set of Brownian motion. Self-similar unimodular discrete spaces are inspired by continuum self-similar sets (see e.g., Section 2.1 of [11]) as discussed in Subsection 4.6.

Acknowledgements

Supported in part by a grant of the Simons Foundation (#197982) to The University of Texas at Austin.

References

[1] M. Aizenman and S. Warzel. The canopy graph and level statistics for random operators on trees. *Math. Phys. Anal. Geom.*, 9(4):291–333 (2007), 2006.

[2] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12:no. 54, 1454–1508, 2007.

[3] D. Aldous and J. M. Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 1–72. Springer, Berlin, 2004.

[4] F. Baccelli, M.-O. Haji-Mirsadeghi, and A. Khezeli. On the dimension of unimodular discrete spaces. *preprint*,https://www.researchgate.net/publication/328137256_On_the_Dimension_of_Unimodular_Discrete_Spaces.

[5] F. Baccelli, M.-O. Haji-Mirsadeghi, and A. Khezeli. Unimodular Billingsley and Frostman lemmas. *preprint*.

[6] F. Baccelli, M. O. Haji-Mirsadeghi, and A. Khezeli. Eternal Family Trees and dynamics on unimodular random graphs. In *Unimodularity in randomly generated graphs*, volume 719 of *Contemp. Math.*, pages 85–127. Amer. Math. Soc., 2018.

[7] M. T. Barlow and S. J. Taylor. Defining fractal subsets of $\mathbf{Z}^d$. *Proc. London Math. Soc. (3)*, 64(1):125–152, 1992.

[8] I. Benjamini, H. Kesten, Y. Peres, and O. Schramm. Geometry of the uniform spanning forest: transitions in dimensions $4, 8, 12, \ldots$. *Ann. of Math. (2)*, 160(2):465–491, 2004.

[9] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm. Group-invariant percolation on graphs. *Geom. Funct. Anal.*, 9(1):29–66, 1999.

[10] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13, 2001.

[11] C. J. Bishop and Y. Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017.

[12] F. R. K. Chung and S.-T. Yau. Eigenvalues of graphs and Sobolev inequalities. *Combin. Probab. Comput.*, 4(1):11–25, 1995.

[13] N. Curien, T. Hutchcroft, and A. Nachmias. Geometric and spectral properties of causal maps. *arXiv preprint arXiv:1710.03137*, 2017.

[14] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II*. Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[15] T. Duquesne and J. F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields*, 131(4):553–603, 2005.

[16] M. Elekes and T. Keleti. Borel sets which are null or non-σ-finite for every translation invariant measure. *Adv. Math.*, 201(1):102–115, 2006.

[17] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[18] J. T. Hack. *Studies of longitudinal stream profiles in Virginia and Maryland*, volume 294. US Government Printing Office, 1957.

[19] O. Häggström. Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.*, 25(3):1423–1436, 1997.

[20] H. Kesten, P. Ney, and F. Spitzer. The Galton-Watson process with mean one and finite variance. *Teor. Verojatnost. i Primenen.*, 11:579–611, 1966.

[21] A. Khezeli. A unified framework for generalizing the Gromov-Hausdorff metric. *preprint*.

[22] A. Khezeli. Shift-coupling of random rooted graphs and networks. In *Unimodularity in randomly generated graphs*, volume 719 of *Contemp. Math.*, pages 175–211. Amer. Math. Soc., Providence, RI, 2018.

[23] A. Khezeli. Metrization of the Gromov-Hausdorff (-Prokhorov) topology for boundedly-compact metric spaces. *Stochastic Process. Appl.*, 2019.

[24] G. Last and H. Thorisson. Invariant transports of stationary random measures and mass-stationarity. *Ann. Probab.*, 37(2):790–813, 2009.

[25] R. Lyons. Factors of IID on trees. *Combin. Probab. Comput.*, 26(2):285–300, 2017.

[26] B. G. Nguyen. Percolation of coalescing random walks. *J. Appl. Probab.*, 27(2):269–277, 1990.

[27] R. Roy, K. Saha, and A. Sarkar. Hack's law in a drainage network model: a Brownian web approach. *Ann. Appl. Probab.*, 26(3):1807–1836, 2016.

[28] R. Schneider and W. Weil. *Stochastic and integral geometry*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.