Unimodular Billingsley and Frostman Lemmas

François Baccelli,[*] Mir-Omid Haji-Mirsadeghi,[†] and Ali Khezeli[‡]

February 3, 2020

Abstract

The notions of unimodular Minkowski and Hausdorff dimensions are defined in [5] for unimodular random discrete metric spaces. The present paper is focused on the connections between these notions and the polynomial growth rate of the underlying space. It is shown that bounding the dimension is closely related to finding suitable equivariant weight functions (i.e., measures) on the underlying discrete space. The main results are unimodular versions of the mass distribution principle, Billingsley's lemma and Frostman's lemma, which allow one to derive upper bounds on the unimodular Hausdorff dimension from the growth rate of suitable equivariant weight functions. These results allow one to compute or bound both types of unimodular dimensions in a large set of examples in the theory of point processes, unimodular random graphs, and self-similarity. Further results of independent interest are also presented, like a version of the max-flow min-cut theorem for unimodular one-ended trees and a weak form of pointwise ergodic theorems for all unimodular discrete spaces.

1 Introduction

This paper is a companion to [5]. For short, the latter will referred to as Part I. The present paper uses the definitions and symbols of Part I. For cross referencing the definitions and results of Part I, the prefix 'I.' is used. For example, Definition I.3.17 refers to Definition 3.17 in [5].

Part I introduced the notion of unimodular random discrete metric space and two notions of dimension for such spaces, namely the unimodular Minkowski dimension (Section I.3.1) and the unimodular Hausdorff dimension (Section I.3.3).

The present paper is centered on the connections between these dimensions and the *growth rate* of the space, which is the polynomial growth rate of $\#N_r(\boldsymbol{o})$, where $N_r(\boldsymbol{o})$ represents the closed ball of radius r centered at the origin and

---

[*]The University of Texas at Austin, baccelli@math.utexas.edu

[†]Sharif University of Technology, mirsadeghi@sharif.ir

[‡]Tarbiat Modares University, khezeli@modares.ac.ir

$\#N_r(\boldsymbol{o})$ is the number of points in this ball. Section 2 is focused on the basic properties of these connections. It is first shown that the upper and lower polynomial growth rates of $\#N_r(\boldsymbol{o})$ (i.e., limsup and liminf of $\log(\#N_r(\boldsymbol{o}))/\log r$ as $r \to \infty$) provide upper and lower bound for the unimodular Hausdorff dimension, respectively. This is a discrete analogue of Billingsley's lemma (see e.g., [8]). A discrete analogue of the *mass distribution principle* is also provided, which is useful to derive upper bounds on the unimodular Hausdorff dimension. In the Euclidean case (i.e., for point-stationary point processes equipped with the Euclidean metric), it is shown that the unimodular Minkowski dimension is bounded from above by the polynomial decay rate of $\mathbb{E}\left[1/\#N_n(\boldsymbol{o})\right]$. Weighted versions of these inequalities, where a weight is assigned to each point, are also presented. As a corollary, a weak form of Birkhoff's pointwise ergodic theorem is established for all unimodular discrete spaces.

The bounds derived in Section 2 are fundamental for calculating the unimodular dimensions. These bounds are used in Section 3 to complete the examples of Part I. Some new examples are also presented in Section 3 for further illustration of the results.

Section 4 gives a unimodular analogue of Frostman's lemma. Roughly speaking, this result states that there exists a weight function on the points such that the upper bound in the mass distribution principle is sharp. This result is a powerful tool to study the unimodular Hausdorff dimension and is the basis of many of the results of [4]; e.g., connections to scaling limits, discrete dimension and capacity dimension. In the Euclidean case, another proof of the unimodular Frostman lemma is provided using a unimodular version of the max-flow min-cut theorem, which is of independent interest.

It should be noted that the results regarding upper bounds on the dimension (e.g., in the unimodular mass distribution principle, Billingsley lemma and Frostman lemma) are valid for arbitrary gauge functions as well (see Subsection I.3.8.2). The lower bounds are also valid for gauge functions satisfying the doubling condition. However, these more general cases are skipped for simplicity of reading.

2 Connections to Growth Rate

Let D be a discrete space and $o \in D$. The **upper and lower (polynomial) growth rates** of D are

$$\overline{\mathrm{growth}}\left(\#N_r(o)\right) \;=\; \limsup_{r \to \infty} \log \#N_r(o)/\log r,$$
$$\underline{\mathrm{growth}}\left(\#N_r(o)\right) \;=\; \liminf_{r \to \infty} \log \#N_r(o)/\log r.$$

D has **polynomial growth** if $\overline{\mathrm{growth}}\left(\#N_r(o)\right) < \infty$. If the upper and lower growth rates are equal, the common value is called the **growth rate** of D. Note that for $v \in D$, one has $N_r(o) \subseteq N_{r+c}(v)$ and $N_r(v) \subseteq N_{r+c}(o)$, where $c := d(o,v)$. This implies that $\overline{\mathrm{growth}}\left(\#N_r(o)\right)$ and $\underline{\mathrm{growth}}\left(\#N_r(o)\right)$ do not depend on the choice of the point o.

In various situations in this paper, some *weight* in $\mathbb{R}^{\geq 0}$ can be assigned to each point of D. In these cases, it is natural to redefine the growth rate by considering the weights; i.e., by replacing $\#N_r(o)$ with the sum of the weights of the points in $N_r(o)$. This will be formalized below using the notion of *equivariant processes* of Subsection I.2.5. Recall that an equivariant process should be defined for all discrete spaces D. However, if a random pointed discrete space $[\boldsymbol{D}, \boldsymbol{o}]$ is considered, it is enough to define weights in almost every realization (see Subsection I.2.5 for more on the matter). Also, given D, the weights are allowed to be random.

Definition 2.1. An **equivariant weight function** $\boldsymbol{w}$ is an equivariant process (Definition I.2.5) with values in $\mathbb{R}^{\geq 0}$. For all discrete spaces D and $v \in D$, the (random) value $\boldsymbol{w}(v) := \boldsymbol{w}_D(v)$ is called the **weight** of v. Also, for $S \subseteq D$, let $\boldsymbol{w}(S) := \boldsymbol{w}_D(S) := \sum_{v \in S} \boldsymbol{w}(v)$.

The last equation shows that one could also call $\boldsymbol{w}$ an *equivariant measure*.

Assume $[\boldsymbol{D}, \boldsymbol{o}]$ is a unimodular discrete space (Subsection I.2.4). Lemma I.2.11 shows that $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{w}_{\boldsymbol{D}}]$ is a random pointed marked discrete space and is unimodular.

In the following, '$\boldsymbol{w}_{\boldsymbol{D}}(\cdot)$ *is non-degenerate (i.e., not identical to zero) with positive probability*' means that $\mathbb{P}\left[\exists v \in \boldsymbol{D} : \boldsymbol{w}_{\boldsymbol{D}}(v) \neq 0\right] > 0$. If $[\boldsymbol{D}, \boldsymbol{o}]$ is unimodular, Lemma I.2.14 implies that the above condition is equivalent to $\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right] > 0$. Also, '$\boldsymbol{w}_{\boldsymbol{D}}(\cdot)$ *is non-degenerate a.s.*' means $\mathbb{P}\left[\exists v \in \boldsymbol{D} : \boldsymbol{w}_{\boldsymbol{D}}(v) \neq 0\right] = 1$.

2.1 Unimodular Mass Distribution Principle

Theorem 2.2 (Mass Distribution Principle)**.** *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space.*

(i) *Let $\alpha, c, M > 0$ and assume there exists an equivariant weight function $\boldsymbol{w}$ such that $\forall r \geq M : \boldsymbol{w}(N_r(\boldsymbol{o})) \leq cr^\alpha$, a.s. Then, $\mathcal{H}_M^\alpha(\boldsymbol{D})$ defined in (I.3.3) satisfies $\mathcal{H}_M^\alpha(\boldsymbol{D}) \geq \frac{1}{c}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right]$.*

(ii) *If in addition, $\boldsymbol{w}_{\boldsymbol{D}}(\cdot)$ is non-degenerate with positive probability, then $\operatorname{udim}_H(\boldsymbol{D}) \leq \alpha$.*

Proof. Let $\boldsymbol{R}$ be an arbitrary equivariant covering such that $\boldsymbol{R}(\cdot) \in \{0\} \cup [M, \infty)$ a.s. By the assumption on $\boldsymbol{w}$, $\boldsymbol{R}(\boldsymbol{o})^\alpha \geq \frac{1}{c}\boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o}))$ a.s. Therefore,

$$\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \frac{1}{c}\mathbb{E}\left[\boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o}))\right]. \tag{2.1}$$

Consider the *independent coupling* of $\boldsymbol{w}$ and $\boldsymbol{R}$; i.e., for each deterministic discrete space G, choose $\boldsymbol{w}_G$ and $\boldsymbol{R}_G$ independently (see Definition I.2.8). Then, it can be seen that the pair $(\boldsymbol{w}, \boldsymbol{R})$ is an equivariant process. So by Lemma I.2.11, $[\boldsymbol{G}, \boldsymbol{o}; (\boldsymbol{w}, \boldsymbol{R})]$ is unimodular. Now, the mass transport principle (I.2.2) can be used for $[\boldsymbol{G}, \boldsymbol{o}; (\boldsymbol{w}, \boldsymbol{R})]$. By letting $g(u, v) := \boldsymbol{w}(v)1_{\{v \in N_{\boldsymbol{R}}(u)\}}$, one gets $g^+(\boldsymbol{o}) = \boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o}))$. Also, $g^-(\boldsymbol{o}) = \boldsymbol{w}(\boldsymbol{o})\sum_{u \in \boldsymbol{D}} 1_{\{\boldsymbol{o} \in N_{\boldsymbol{R}}(u)\}} \geq \boldsymbol{w}(\boldsymbol{o})$ a.s.,

3

where the last inequality follows from the fact that $\boldsymbol{R}$ is a covering. Therefore, the mass transport principle implies that $\mathbb{E}\left[w(N_{\boldsymbol{R}}(\boldsymbol{o}))\right] \geq \mathbb{E}\left[w(\boldsymbol{o})\right]$ (recall that by convention, $N_{\boldsymbol{R}}(\boldsymbol{o})$ is the empty set when $\boldsymbol{R}(\boldsymbol{o}) = 0$). So by (2.1), one gets $\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \frac{1}{c}\mathbb{E}\left[w(\boldsymbol{o})\right]$. Since this holds for any $\boldsymbol{R}$, one gets that $\mathcal{H}_M^\alpha(\boldsymbol{D}) \geq \frac{1}{c}\mathbb{E}\left[w(\boldsymbol{o})\right]$ and the first claim is proved.

If, with positive probability, $\boldsymbol{w}_{\boldsymbol{D}}(\cdot)$ is non-degenerate, then Lemma I.2.14 implies that $w(\boldsymbol{o}) > 0$ with positive probability. So $\mathbb{E}\left[w(\boldsymbol{o})\right] > 0$. Therefore, $\mathcal{H}_1^\alpha(\boldsymbol{D}) > 0$ and the second claim is proved. $\qquad\square$

2.2   Unimodular Billingsley Lemma

The main result of this subsection is Theorem 2.6. It is based on Lemmas 2.3 and 2.4 below. Lemma 2.3 is a stronger version of the mass distribution principle (Theorem 2.2).

Lemma 2.3 (An Upper Bound)**.** *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space and $\alpha \geq 0$.*

  (i) If there exist $c \geq 0$ and $\boldsymbol{w}$ is an equivariant weight function such that $\limsup_{r\to\infty} \boldsymbol{w}(N_r(\boldsymbol{o}))/r^\alpha \leq c$,   a.s., then $\mathcal{H}_\infty^\alpha(\boldsymbol{D}) \geq \frac{1}{2^\alpha c}\mathbb{E}\left[w(\boldsymbol{o})\right]$.

  (ii) In addition, if $\boldsymbol{w}_{\boldsymbol{D}}(\cdot)$ is non-degenerate with positive probability, then $\mathrm{udim}_H(\boldsymbol{D}) \leq \alpha$.

Proof. Let $c' > c$ be arbitrary. The assumption implies that $\sup\{r \geq 0 : \boldsymbol{w}(N_r(\boldsymbol{o})) > c'r^\alpha\} < \infty$ a.s. For $m \geq 1$, let $A_m := \{v \in \boldsymbol{D} : \forall r \geq m : \boldsymbol{w}(N_r(v)) \leq c'r^\alpha\}$, which is an increasing sequence of equivariant subsets. So

$$\lim_{m\to\infty} \mathbb{P}\left[\boldsymbol{o} \in A_m\right] = 1. \tag{2.2}$$

Let $\boldsymbol{R}$ be an equivariant covering such that $\boldsymbol{R}(\cdot) \in \{0\} \cup [m, \infty)$ a.s. One has

$$\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha 1_{\{N_{\boldsymbol{R}}(\boldsymbol{o}) \cap A_m \neq \emptyset\}}\right]. \tag{2.3}$$

If $N_{\boldsymbol{R}}(\boldsymbol{o}) \cap A_m \neq \emptyset$, then $\boldsymbol{R}(\boldsymbol{o}) \neq 0$ and hence $\boldsymbol{R}(\boldsymbol{o}) \geq m$. In the next step, assume that this is the case. Let v be an arbitrary point in $N_{\boldsymbol{R}}(\boldsymbol{o}) \cap A_m$. By the definition of A_m, one gets that for all $r \geq m$, $\boldsymbol{w}(N_r(v)) \leq c'r^\alpha$. Since $N_{\boldsymbol{R}(\boldsymbol{o})}(\boldsymbol{o}) \subseteq N_{2\boldsymbol{R}(\boldsymbol{o})}(v)$, it follows that $\boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o})) \leq \boldsymbol{w}(N_{2\boldsymbol{R}(\boldsymbol{o})}(v)) \leq 2^\alpha c' \boldsymbol{R}(\boldsymbol{o})^\alpha$. Therefore, (2.3) gives

$$\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \frac{1}{2^\alpha c'}\mathbb{E}\left[\boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o}))1_{\{N_{\boldsymbol{R}}(\boldsymbol{o}) \cap A_m \neq \emptyset\}}\right]. \tag{2.4}$$

By letting $g(u, v) := \boldsymbol{w}(v)1_{\{v \in N_{\boldsymbol{R}}(u)\}}1_{\{N_{\boldsymbol{R}}(u) \cap A_m \neq \emptyset\}}$, one gets that $g^+(\boldsymbol{o}) = \boldsymbol{w}(N_{\boldsymbol{R}}(\boldsymbol{o}))1_{\{N_{\boldsymbol{R}}(\boldsymbol{o}) \cap A_m \neq \emptyset\}}$. Also, since there is a ball $N_{\boldsymbol{R}}(u)$ that covers $\boldsymbol{o}$ a.s., one has $g^-(\boldsymbol{o}) \geq \boldsymbol{w}(\boldsymbol{o})1_{\{\boldsymbol{o} \in A_m\}}$ a.s. Therefore, the mass transport principle (I.2.2) and (2.4) imply that $\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})^\alpha\right] \geq \frac{1}{2^\alpha c'}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})1_{\{\boldsymbol{o} \in A_m\}}\right]$. This implies that $\mathcal{H}_m^\alpha(\boldsymbol{D}) \geq \frac{1}{2^\alpha c'}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})1_{\{\boldsymbol{o} \in A_m\}}\right]$. Using (2.2) and letting m tend to infinity gives $\mathcal{H}_\infty^\alpha(\boldsymbol{D}) \geq \frac{1}{2^\alpha c'}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right]$. Since $c' > c$ is arbitrary, the first claim is proved. Part (ii) is also proved by arguments similar to those in Theorem 2.2. $\qquad\square$

Lemma 2.4 (Lower Bounds). *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space, $\alpha \geq 0$ and $c > 0$. Let $\boldsymbol{w}$ be an arbitrary equivariant weight function such that $\mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] < \infty$.*

(i) If $\exists r_0 : \forall r \geq r_0 : \boldsymbol{w}(N_r(\boldsymbol{o})) \geq cr^\alpha$ a.s., then $\underline{\mathrm{udim}}_M(\boldsymbol{D}) \geq \alpha$.

(ii) If $\underline{\mathrm{growth}}\,(\boldsymbol{w}(N_r(\boldsymbol{o}))) \geq \alpha$ a.s., then $\mathrm{udim}_H(\boldsymbol{D}) \geq \alpha$.

(iii) If $\lim_{\delta \downarrow 0} \liminf_{r \to \infty} \mathbb{P}[\boldsymbol{w}(N_r(\boldsymbol{o})) \leq \delta r^\alpha] = 0$, then $\mathrm{udim}_H(\boldsymbol{D}) \geq \alpha$.

(iv) If $\underline{\mathrm{decay}}\,\left(\mathbb{E}\left[\exp\left(-\frac{\boldsymbol{w}(N_n(\boldsymbol{o}))}{n^\alpha}\right)\right]\right) \geq \alpha$, then $\underline{\mathrm{udim}}_M(\boldsymbol{D}) \geq \alpha$.

Proof. The proofs of the first two parts are very similar. The second Part Is proved first.

(ii). Let β, γ and κ be such that $\gamma < \beta < \kappa < \alpha$. Fix $n \in \mathbb{N}$. Let $\boldsymbol{S} = \boldsymbol{S}_{\boldsymbol{D}}$ be the equivariant subset obtained by selecting each point $v \in \boldsymbol{D}$ with probability $1 \wedge (n^{-\beta}\boldsymbol{w}(v))$ (the selection variables are assumed to be conditionally independent given $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{w}]$). Let $\boldsymbol{R}_n(v) = n$ if $v \in \boldsymbol{S}_{\boldsymbol{D}}$, $\boldsymbol{R}_n(v) = 1$ if $N_n(v) \cap \boldsymbol{S}_{\boldsymbol{D}} = \emptyset$, and $\boldsymbol{R}_n(v) = 0$ otherwise. Then $\boldsymbol{R}_n$ is an equivariant covering. It is shown below that $\mathbb{E}[\boldsymbol{R}_n(\boldsymbol{o})^\gamma] \to 0$. Let $M := \sup\{r \geq 0 : \boldsymbol{w}(N_r(\boldsymbol{o})) < r^\kappa\}$. By the assumption, $M < \infty$ a.s. One has

$$
\begin{aligned}
\mathbb{E}[\boldsymbol{R}_n(\boldsymbol{o})^\gamma] &= n^\gamma \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}] + \mathbb{P}[N_n(\boldsymbol{o}) \cap \boldsymbol{S}_{\boldsymbol{D}} = \emptyset] \\
&= n^\gamma \mathbb{E}\left[1 \wedge n^{-\beta}\boldsymbol{w}(\boldsymbol{o})\right] + \mathbb{E}\left[\prod_{v \in N_n(\boldsymbol{o})} \left(1 - (1 \wedge n^{-\beta}\boldsymbol{w}(v))\right)\right] \\
&\leq n^{\gamma-\beta}\mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] + \mathbb{E}\left[\exp\left(-n^{-\beta}\boldsymbol{w}(N_n(\boldsymbol{o}))\right)\right] \\
&= n^{\gamma-\beta}\mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] + \mathbb{E}\left[\exp\left(-n^{-\beta}\boldsymbol{w}(N_n(\boldsymbol{o}))\right)|M < n\right]\mathbb{P}[M < n] \\
&\qquad\qquad + \mathbb{E}\left[\exp\left(-n^{-\beta}\boldsymbol{w}(N_n(\boldsymbol{o}))\right)|M \geq n\right]\mathbb{P}[M \geq n] \\
&\leq n^{\gamma-\beta}\mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] + \exp\left(-n^{\kappa-\beta}\right) + \mathbb{P}[M \geq n],
\end{aligned}
$$

where the first inequality holds because $1 - (1 \wedge x) \leq e^{-x}$ for all $x \geq 0$. Therefore, $\mathbb{E}[\boldsymbol{R}_n(\boldsymbol{o})^\gamma] \to 0$ when $n \to \infty$. It follows that $\mathrm{udim}_H(\boldsymbol{D}) \geq \gamma$. Since γ is arbitrary, this implies $\mathrm{udim}_H(\boldsymbol{D}) \geq \alpha$.

(i). Only a small change is needed in the above proof. For $n \geq r_0$, let $\boldsymbol{R}_n(v) = n$ if either $v \in \boldsymbol{S}_{\boldsymbol{D}}$ or $N_n(v) \cap \boldsymbol{S}_{\boldsymbol{D}} = \emptyset$, and let $\boldsymbol{R}_n(v) = 0$ otherwise. Note that $\boldsymbol{R}_n$ is a covering by balls of equal radii. By the same computations and the assumption $M \leq r_0$, one gets

$$
\mathbb{P}[\boldsymbol{R}_n(\boldsymbol{o}) \neq 0] \leq n^{-\beta}\mathbb{E}[\boldsymbol{w}(\boldsymbol{o})] + \exp\left(-n^{\kappa-\beta}\right),
$$

which is of order $n^{-\beta}$ for large n. This implies that $\underline{\mathrm{udim}}_M(\boldsymbol{D}) \geq \beta$. Since β is arbitrary, one gets $\underline{\mathrm{udim}}_M(\boldsymbol{D}) \geq \alpha$ and the claim is proved.

(iii). Let $\beta < \alpha$. It will be proved below that under the assumption of (iii), there is a sequence $r_1, r_2, \ldots$ such that $\mathbb{E}\left[\exp\left(-r_n^{-\beta}\boldsymbol{w}(N_{r_n}(\boldsymbol{o}))\right)\right] \to 0$. If so, by a slight modification of the proof of part (ii), one can find a sequence of equivariant coverings $\boldsymbol{R}_n$ such that $\mathbb{E}\left[\boldsymbol{R}_n(\boldsymbol{o})^\beta\right] < \infty$ and (iii) is proved.

Let $\epsilon > 0$ be arbitrary. By the assumption, there is $\delta > 0$ and $r \geq 1$ such that $\mathbb{P}\left[\boldsymbol{w}(N_r(\boldsymbol{o})) \leq \delta r^\alpha\right] < \epsilon$. So

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(-r^{-\beta}\boldsymbol{w}(N_r(\boldsymbol{o}))\right)\right] &\leq \mathbb{E}\left[\exp\left(-r^{-\beta}\boldsymbol{w}(N_r(\boldsymbol{o}))\right)|\boldsymbol{w}(N_r(\boldsymbol{o})) > \delta r^\alpha\right] \\
&\quad + \mathbb{P}\left[\boldsymbol{w}(N_r(\boldsymbol{o})) \leq \delta r^\alpha\right] \\
&\leq \exp(-\delta r^{\alpha-\beta}) + \epsilon.
\end{aligned}
$$

Note that for fixed ϵ and δ as above, r can be arbitrarily large. Now, choose r large enough for the right hand side to be at most 2ϵ. This shows that $\mathbb{E}\left[\exp\left(-r^{-\beta}\boldsymbol{w}(N_r(\boldsymbol{o}))\right)\right]$ can be arbitrarily small and the claim is proved.

(iv). As before, let $\boldsymbol{R}_n(v) = n$ if either $v \in \boldsymbol{S_D}$ or $N_n(v) \cap \boldsymbol{S_D} = \emptyset$, and let $\boldsymbol{R}_n(v) = 0$ otherwise. The calculations in the proof of part (ii) show that

$$
\mathbb{P}\left[\boldsymbol{R}_n(\boldsymbol{o}) \neq 0\right] \leq n^{-\beta}\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right] + \mathbb{E}\left[\exp\left(-n^{-\beta}\boldsymbol{w}(N_n(\boldsymbol{o}))\right)\right].
$$

Now, the assumption implies the claim. $\qquad\square$

Remark 2.5. The assumption in part (iii) of Lemma 2.4 is equivalent to the condition that there exists a sequence $r_n \to \infty$ such that the family of random variables $r_n^\alpha/\boldsymbol{w}(N_{r_n}(\boldsymbol{o}))$ is tight. Also, from the proof of the lemma, one can see that this assumption is equivalent to

$$
\liminf_{n\to\infty} \mathbb{E}\left[\exp\left(-\frac{\boldsymbol{w}(N_n(\boldsymbol{o}))}{n^\alpha}\right)\right] = 0.
$$

Theorem 2.6 (Unimodular Billingsley Lemma). *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete metric space. Then, for all equivariant weight functions $\boldsymbol{w}$ such that $0 < \mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right] < \infty$, one has*

$$
\begin{aligned}
\operatorname{ess\,inf}\left(\underline{\operatorname{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)\right) &\leq \operatorname{udim}_H(\boldsymbol{D}) \\
&\leq \operatorname{ess\,inf}\left(\overline{\operatorname{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)\right) \\
&\leq \overline{\operatorname{growth}}\left(\mathbb{E}\left[\boldsymbol{w}(N_r(\boldsymbol{o}))\right]\right).
\end{aligned}
$$

The proof is given below. In fact, in many examples, it is enough to consider $\boldsymbol{w} \equiv 1$ in Billingsley's lemma; i.e., $\boldsymbol{w}(N_r(\boldsymbol{o})) = \#N_r(\boldsymbol{o})$.

Corollary 2.7. *Under the assumptions of Theorem 2.6, if the upper and lower growth rates of $\boldsymbol{D}$ are almost surely constant (e.g., when $[\boldsymbol{D}, \boldsymbol{o}]$ is ergodic), then,*

$$
\underline{\operatorname{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right) \leq \operatorname{udim}_H(\boldsymbol{D}) \leq \overline{\operatorname{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right) \quad a.s. \quad (2.5)
$$

In particular, if $\operatorname{growth}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)$ exists and is constant a.s., then

$$
\operatorname{udim}_H(\boldsymbol{D}) = \operatorname{growth}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right).
$$

In fact, without the assumption of this corollary, an inequality similar to (2.5) is valid for the *sample Hausdorff dimension* of $\boldsymbol{D}$, which will be studied in [4].

Proof of Theorem 2.6. The first inequality is implied by part (ii) of Lemma 2.4. For the second inequality, assume that $\overline{\text{growth}}\,(\boldsymbol{w}(N_r(\boldsymbol{o}))) < \alpha$ with positive probability. On this event, one has $\boldsymbol{w}(N_r(\boldsymbol{o})) \leq r^\alpha$ for large r; i.e., $\limsup_r \boldsymbol{w}(N_r(\boldsymbol{o}))/r^\alpha \leq 1$. Now, Lemma 2.3 implies that $\text{udim}_H(\boldsymbol{D}) \leq \alpha$. This proves the second inequality. The last claim follows because $\overline{\text{growth}}\,(X_n) \leq \overline{\text{growth}}\,(\mathbb{E}\,[X_n])$ for any monotone sequence of nonnegative random variables X_n (see the arXiv version). $\qquad\square$

Remark 2.8. In fact, the assumption $\mathbb{E}\,[\boldsymbol{w}(\boldsymbol{o})] < \infty$ in Theorem 2.6 is only needed for the lower bound while the assumption $\mathbb{E}\,[\boldsymbol{w}(\boldsymbol{o})] > 0$ is only needed for the upper bound. These assumptions are also necessary as shown below.
For example, assume Φ is a point-stationary point process in $\mathbb{R}$ (see Example I.2.6). For $v \in \Phi$, let $\boldsymbol{w}(v)$ be the sum of the distances of v to its next and previous points in Φ. This equivariant weight function satisfies $\boldsymbol{w}(N_r(v)) \geq 2r$ for all r, and hence $\underline{\text{growth}}\,(\boldsymbol{w}(N_r(\boldsymbol{o}))) \geq 1$. But $\text{udim}_H(\Phi)$ can be strictly less than 1 as shown in Subsection 3.3.1.
Also, the condition that $\boldsymbol{w}_D$ is non-degenerate a.s. is trivially necessary for the upper bound.

Corollary 2.9. *Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular random graph equipped with the graph-distance metric. If $\boldsymbol{G}$ is infinite almost surely, then $\underline{\text{udim}}_M(\boldsymbol{G}) \geq 1$ and else, $\text{udim}_M(\boldsymbol{G}) = \text{udim}_H(\boldsymbol{G}) = 0$.*

Proof. If $\boldsymbol{G}$ is infinite a.s., then for $\boldsymbol{w}_G \equiv 1$, one has $\boldsymbol{w}(N_r(\boldsymbol{o})) \geq r$ for all r. So part (i) of Lemma 2.4 implies the first claim. The second claim is implied by Example I.3.18 (this can be deduced from the unimodular Billingsley lemma as well). $\qquad\square$

Corollary 2.10. *The unimodular Minkowski and Hausdorff dimensions of any unimodular two-ended tree are equal to one.*

This result has already been shown in Theorem I.4.1, but can also be deduced from the unimodular Billingsley lemma directly. For this, let $\boldsymbol{w}(v)$ be 1 if v belongs to the trunk of the tree and 0 otherwise.

Problem 2.11. *In the setting of Corollary 2.7, is it always the case that $\text{udim}_H(\boldsymbol{D}) = \overline{\text{growth}}\,(\boldsymbol{w}(N_r(\boldsymbol{o})))$?*

The claim of this problem holds in all of the examples in which both quantities are computed in this work. This problem is a corollary of Problem 2.19 and the unimodular Frostman lemma (Theorem 4.2) below. Note that there are examples where $\underline{\text{growth}}\,(\cdot) \neq \overline{\text{growth}}\,(\cdot)$ as shown in Subsections 3.2.1 and 3.5.1.

2.3 Bounds for Point Processes

The next results use the following equivariant covering. Let φ be a discrete subset of $\mathbb{R}^k$ equipped with the l_∞ metric and $r \geq 1$. Let $C := C_r := [0, r)^k$, $\boldsymbol{U} := \boldsymbol{U}_r$ be a point chosen uniformly at random in $-C$, and consider the

partition $\{C + \boldsymbol{U} + z : z \in r\mathbb{Z}^k\}$ of $\mathbb{R}^k$ by cubes. Then, for each $z \in r\mathbb{Z}^k$, choose a random element in $(C + \boldsymbol{U} + z) \cap \varphi$ independently (if the intersection is nonempty). The distribution of this random element should depend on the set $(C + \boldsymbol{U} + z) \cap \varphi$ in a translation-invariant way (e.g., choose with the uniform distribution or choose the least point in the lexicographic order). Let $\boldsymbol{R} = \boldsymbol{R}_\varphi$ assign the value r to the selected points and zero to the other points of φ. Then, $\boldsymbol{R}$ is an equivariant covering. Also, each point is covered at most 3^k times. So $\boldsymbol{R}$ is 3^k-bounded (Definition I.3.10).

Theorem 2.12 (Minkowski Dimension in the Euclidean Case). *Let Φ be a point-stationary point process in $\mathbb{R}^k$ and assume the metric in Φ is equivalent to the Euclidean metric. Then, for all equivariant weight functions $\boldsymbol{w}$ such that $\boldsymbol{w}_\Phi(0) > 0$ a.s., one has*

$$
\begin{aligned}
\overline{\mathrm{udim}}_M(\Phi) = \overline{\mathrm{decay}}\left(\mathbb{E}\left[\boldsymbol{w}(0)/\boldsymbol{w}(C_r + \boldsymbol{U}_r)\right]\right) &\leq \overline{\mathrm{decay}}\left(\mathbb{E}\left[\boldsymbol{w}(0)/\boldsymbol{w}(N_r(0))\right]\right) \\
&\leq \overline{\mathrm{growth}}\left(\mathbb{E}\left[\boldsymbol{w}(N_r(0))\right]\right), \\
\underline{\mathrm{udim}}_M(\Phi) = \underline{\mathrm{decay}}\left(\mathbb{E}\left[\boldsymbol{w}(0)/\boldsymbol{w}(C_r + \boldsymbol{U}_r)\right]\right) &\leq \underline{\mathrm{decay}}\left(\mathbb{E}\left[\boldsymbol{w}(0)/\boldsymbol{w}(N_r(0))\right]\right) \\
&\leq \underline{\mathrm{growth}}\left(\mathbb{E}\left[\boldsymbol{w}(N_r(0))\right]\right),
\end{aligned}
$$

where $\boldsymbol{U}_r$ is a uniformly at random point in $-C_r$ independent of Φ and $\boldsymbol{w}$.

Proof. By Theorem I.3.32, one may assume the metric on Φ is the l_∞ metric without loss of generality. Given any $r > 0$, consider the equivariant covering $\boldsymbol{R}$ described above, but when choosing a random element of $(C_r + \boldsymbol{U}_r + z) \cap \varphi$, choose point v with probability $\boldsymbol{w}_\varphi(v)/\boldsymbol{w}_\varphi(C_r + \boldsymbol{U}_r + z)$ (conditioned on $\boldsymbol{w}_\varphi$). One gets $\mathbb{P}\left[0 \in \boldsymbol{R}\right] = \mathbb{E}\left[\boldsymbol{w}(0)/\boldsymbol{w}(C_r + \boldsymbol{U}_r)\right]$. As mentioned above, $\boldsymbol{R}$ is equivariant and uniformly bounded (for all $r > 0$). So Lemma I.3.11 implies both equalities in the claim. The inequalities are implied by the facts that $\boldsymbol{w}(C_r + \boldsymbol{U}_r) \leq \boldsymbol{w}(N_r(0))$ and

$$
\mathbb{E}\left[\frac{\boldsymbol{w}(0)}{\boldsymbol{w}(N_r(0))}\right] \mathbb{E}\left[\boldsymbol{w}(N_r(0))\right] \geq \mathbb{E}\left[\sqrt{\boldsymbol{w}(0)}\right]^2 > 0,
$$

which is implied by the Cauchy-Schwartz inequality. $\square$

An example where the decay rate of $\mathbb{E}\left[1/\#N_n(\boldsymbol{o})\right]$ is strictly smaller than the growth rate of $\mathbb{E}\left[\#N_n(\boldsymbol{o})\right]$ can be found in [4].

Proposition 2.13. *If Φ is a point-stationary point process in $\mathbb{R}^k$ and the metric on Φ is equivalent to the Euclidean metric, then $\mathrm{udim}_H(\Phi) \leq k$.*

Proof. One may assume the metric on Φ is the l_∞ metric without loss of generality. Let $C := [0,1)^k$ and $\boldsymbol{U}$ be a random point in $-C$ chosen uniformly. For all discrete subsets $\varphi \subseteq \mathbb{R}^k$ and $v \in \varphi$, let $\boldsymbol{C}(v)$ be the cube containing v of the form $C + \boldsymbol{U} + z$ (for $z \in \mathbb{Z}^k$) and $\boldsymbol{w}_\varphi(v) := 1/\#(\varphi \cap \boldsymbol{C}(v))$. Now, $\boldsymbol{w}$ is an equivariant weight function. The construction readily implies that $\boldsymbol{w}(N_r(\boldsymbol{o})) \leq (2r+1)^k$. Moreover, by $\boldsymbol{w} \leq 1$, one has $\mathbb{E}\left[\boldsymbol{w}(0)\right] < \infty$. Therefore, the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\Phi) \leq k$. $\square$

8

Proposition 2.14. *If Ψ is a stationary point process in $\mathbb{R}^k$ with finite intensity and Ψ_0 is its Palm version, then $\mathrm{udim}_M(\Psi_0) = \mathrm{udim}_H(\Psi_0) = k$. Moreover, the modified unimodular Hausdorff measure of Ψ_0 satisfies $\mathcal{M}'_k(\Psi_0) = 2^k \rho(\Psi)$, where $\rho(\Psi)$ is the intensity of Ψ.*

Notice that if $\Psi_0 \subseteq \mathbb{Z}^k$, then the claim is directly implied by Theorem I.3.35. The general case is treated below.

Proof. For the first claim, by Proposition 2.13, it is enough to prove that $\underline{\mathrm{udim}}_M(\Phi) \geq k$. Let Ψ' be a shifted square lattice independent of Ψ (i.e., $\Psi' = \mathbb{Z}^k + \boldsymbol{U}$, where $\boldsymbol{U} \in [0,1)^k$ is chosen uniformly, independently of Ψ). Let $\Psi'' := \Psi \cup \Psi'$. Since Ψ'' is a superposition of two independent stationary point processes, it is a stationary point process itself. By letting $p := \rho(\Psi)/(\rho(\Psi)+1)$, the Palm version Φ'' of Ψ'' is obtained by the superposition of Φ and an independent stationary lattice with probability p (heads), and the superposition of $\mathbb{Z}^k$ and Ψ with probability $1-p$ (tails). So Lemma 2.4 implies that $\underline{\mathrm{udim}}_M(\Phi'') \geq k$. Note that Φ'' has two natural equivariant subsets which, after conditioning to contain the origin, have the same distributions as Φ and $\mathbb{Z}^k$ respectively. Therefore, one can use Theorem I.3.35 to deduce that $\underline{\mathrm{udim}}_M(\Phi) \geq \underline{\mathrm{udim}}_M(\Phi'') = k$. Therefore, Proposition 2.13 implies that $\mathrm{udim}_H(\Phi) = \mathrm{udim}_M(\Phi) = k$.

Also, by using Theorem I.3.35 twice, one gets $\mathcal{M}'_k(\Phi) = p\mathcal{M}'_k(\Phi'')$ and $\mathcal{M}'_k(\mathbb{Z}^k) = (1-p)\mathcal{M}'_k(\Phi'')$. Therefore, $\mathcal{M}'_k(\Phi) = p/(1-p)\mathcal{M}'_k(\mathbb{Z}^k)$. By the definition of $\mathcal{M}'_k$, one can directly show that $\mathcal{M}'_k(\mathbb{Z}^k) = 2^k$ (see also Proposition I.3.30). This implies the claim. $\qquad\square$

The last claim of Proposition 2.14 suggests the following, which is verified when $k = 1$ in the next proposition.

Conjecture 2.15. *If Φ is a point-stationary point process in $\mathbb{R}^k$ which is not the Palm version of any stationary point process, then $\mathcal{M}^k(\Phi) = 0$.*

Proposition 2.16. *Conjecture 2.15 is true when $k = 1$.*

Proof. Denote Φ as $\Phi = \{S_n : n \in \mathbb{Z}\}$ such that $S_0 = 0$ and $S_n < S_{n+1}$ for each n. Then, the sequence $T_n := S_{n+1} - S_n$ is stationary under shifting the indices (see Example I.2.6). The assumption that Φ is not the Palm version of a stationary point process is equivalent to $\mathbb{E}[S_1] = \infty$ (see [10] or Proposition 6 of [16]). Indeed, if $\mathbb{E}[S_1] < \infty$, then one could bias the probability measure by S_1 and then shift the whole process by $-\boldsymbol{U}$, where $\boldsymbol{U} \in [0, S_1]$ is chosen uniformly and independently.

Since $\mathbb{E}[S_1] = \infty$, Birkhoff's pointwise ergodic theorem [19] implies that $\lim_n (T_1 + \cdots + T_n)/n = \infty$. This in turn implies that $\lim_r \#N_r(0)/r = 0$. Therefore, Lemma 2.3 gives that $\mathcal{H}^1_\infty(\Phi) = \infty$; i.e., $\mathcal{M}^1(\Phi) = 0$. $\qquad\square$

2.4 Connections to Birkhoff's Pointwise Ergodic Theorem

The following corollary of the unimodular Billingsley lemma is of independent interest. Note that the statement does not involve dimension.

Theorem 2.17. *Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space. For any two equivariant weight functions $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, if $\mathbb{E}\left[\boldsymbol{w}_1(\boldsymbol{o})\right] < \infty$ and $\boldsymbol{w}_2(\cdot)$ is non-degenerate a.s., then*

$$\underline{\mathrm{growth}}\left(\boldsymbol{w}_1(N_r(\boldsymbol{o}))\right) \leq \overline{\mathrm{growth}}\left(\boldsymbol{w}_2(N_r(\boldsymbol{o}))\right), \quad a.s.$$

In particular, if $\boldsymbol{w}_1(N_r(\boldsymbol{o}))$ and $\boldsymbol{w}_2(N_r(\boldsymbol{o}))$ have well defined growth rates, then their growth rates are equal.

Note that the condition $\mathbb{E}\left[\boldsymbol{w}_1(\boldsymbol{o})\right] < \infty$ is necessary as shown in Remark 2.8.

Proof. Let $\epsilon > 0$ be arbitrary and

$$A := \{[D, o] \in \mathcal{D}_* : \underline{\mathrm{growth}}\left(\boldsymbol{w}_1(N_r(o))\right) > \overline{\mathrm{growth}}\left(\boldsymbol{w}_2(N_r(o))\right) + \epsilon\}.$$

It can be seen that A is a measurable subset of $\mathcal{D}_*$. Assume $\mathbb{P}\left[[\boldsymbol{D}, \boldsymbol{o}] \in A\right] > 0$. Denote by $[\boldsymbol{D}', \boldsymbol{o}']$ the random pointed discrete space obtained by conditioning $[\boldsymbol{D}, \boldsymbol{o}]$ on A. Since A does not depend on the root (i.e., if $[D, o] \in A$, then $\forall v \in D : [D, v] \in A$), by a direct verification of the mass transport principle (I.2.1), one can show that $[\boldsymbol{D}', \boldsymbol{o}']$ is unimodular. So by using the unimodular Billingsley lemma (Theorem 2.6) twice, one gets

$$\mathrm{ess\,inf}\left(\underline{\mathrm{growth}}\left(\boldsymbol{w}_1(N_r(\boldsymbol{o}'))\right)\right) \leq \mathrm{udim}_H(\boldsymbol{D}') \leq \mathrm{ess\,inf}\left(\overline{\mathrm{growth}}\left(\boldsymbol{w}_2(N_r(\boldsymbol{o}'))\right)\right).$$

By the definition of A, this contradicts the fact that $[\boldsymbol{D}', \boldsymbol{o}'] \in A$ a.s. So $\mathbb{P}\left[[\boldsymbol{D}, \boldsymbol{o}] \in A\right] = 0$ and the claim is proved. $\qquad\square$

Remark 2.18. Theorem 2.17 is a generalization of a weaker form of Birkhoff's pointwise ergodic theorem as explained below. In the cases where $\boldsymbol{D}$ is either $\mathbb{Z}$, the Palm version of a stationary point process in $\mathbb{R}^k$ or a point-stationary point process in $\mathbb{R}$, Birkhoff's pointwise ergodic theorem (or its generalizations) implies that $\lim \boldsymbol{w}_1(N_r(\boldsymbol{o}))/\boldsymbol{w}_2(N_r(\boldsymbol{o})) = \mathbb{E}\left[\boldsymbol{w}_1(0)\right]/\mathbb{E}\left[\boldsymbol{w}_2(0)\right]$ a.s. This is stronger than the claim of Theorem 2.17. Note that Theorem 2.17 implies nothing about $\lim \boldsymbol{w}_1(N_r(\boldsymbol{o}))/\boldsymbol{w}_2(N_r(\boldsymbol{o}))$. On the other side, note that *amenability* is not assumed in this Theorem, which is a general requirement in the study of ergodic theorems. However, it will be proved in [5] that, roughly speaking, non-amenability implies $\overline{\mathrm{growth}}\left(\boldsymbol{w}_2(N_r(\boldsymbol{o}))\right) = \infty$, which makes the claim of Theorem 2.17 trivial in this case. In this case, using exponential gauge functions seems more interesting.

Problem 2.19. *Is it true that for every unimodular discrete space $[\boldsymbol{D}, \boldsymbol{o}]$, the growth rates $\overline{\mathrm{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)$ and $\underline{\mathrm{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)$ do not depend on $\boldsymbol{w}$ as long as $0 < \mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})\right] < \infty$?*

2.5 Notes and Bibliographical Comments

As already mentioned, the unimodular mass distribution principle and the unimodular Billingsley lemma have analogues in the continuum setting (see e.g., [8]) and are named accordingly. Note however that there is no direct or systematic reduction to these continuum results. For instance, in the continuum

setting, one should assume that the space under study is a subset of the Euclidean space, or more generally, satisfies the *bounded subcover property* (see e.g., [8]). Theorem 2.6 does not require such assumptions. Note also that the term $\overline{\mathrm{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right)$ in Theorem 2.6 does not depend on the origin in contrast to the analogous term in the continuum version. Similar observations can be made on Theorem 2.2.

3 Examples

This section presents some examples for illustrating the results of the previous section. It also provides further results on the examples introduced in Section I.4.

3.1 General Unimodular Trees

The following is a direct corollary of Theorem I.4.2 and the unimodular Billingsley lemma. Since the statement does not involve dimension, it is of independent interest and believed to be new.

Corollary 3.1. *For every unimodular one-ended tree $[\boldsymbol{T}, \boldsymbol{o}]$ and every equivariant weight function $\boldsymbol{w}$, almost surely,*

$$\overline{\mathrm{decay}}\left(\mathbb{P}\left[h(\boldsymbol{o}) = n\right]\right) \leq \overline{\mathrm{growth}}\left(\boldsymbol{w}(N_r(\boldsymbol{o}))\right) \leq \overline{\mathrm{growth}}\left(\mathbb{E}\left[\boldsymbol{w}(N_r(\boldsymbol{o}))\right]\right).$$

The rest of this subsection is focused on unimodular trees with infinitely many ends.

Proposition 3.2. *Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular tree with infinitely many ends such that $\mathbb{E}\left[\deg(\boldsymbol{o})\right] < \infty$. Then $\boldsymbol{T}$ has exponential growth a.s. and $\mathrm{udim}_H(\boldsymbol{T}) = \infty$.*

In fact, the assumption $\mathbb{E}\left[\deg(\boldsymbol{o})\right] < \infty$ is not necessary. Also, the graph-distance metric on $\boldsymbol{T}$ can be replaced by an arbitrary equivariant metric. These will be proved in [5].

The following proof uses the definitions and results of [1], but they are not recalled for brevity.

Proof of Proposition 3.2. By Corollary 8.10 of [1], $[\boldsymbol{T}, \boldsymbol{o}]$ is *non-amenable* (this will be discussed further in [5]). So Theorem 8.9 of [1] implies that the critical probability p_c of percolation on $\boldsymbol{T}$ is less than one with positive probability. In fact, it can be shown that $p_c < 1$ a.s. (if not, condition on the event $p_c = 1$ to get a contradiction). For any tree, p_c is equal to the inverse of the *branching number*. So the branching number is more than one, which implies that the tree has exponential growth. Finally, the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\boldsymbol{T}) = \infty$. □

The following example shows that the Minkowski dimension can be finite.

Example 3.3. Let T be the 3-regular tree. Split each edge e by adding a random number $\boldsymbol{l}_e$ of new vertices and let $\boldsymbol{T}_0$ be the resulting tree. Let v_e be the middle vertex in this edge (assuming $\boldsymbol{l}_e$ is always odd) and assign marks by $\boldsymbol{m}_0(v_e) := \boldsymbol{l}_e$. Assume that the random variables $\boldsymbol{l}_e$ are i.i.d. If $\mathbb{E}\left[\boldsymbol{l}_e\right] < \infty$, then one can bias the probability measure and choose a new root to obtain a unimodular marked tree, namely $[\boldsymbol{T}, \boldsymbol{o}; \boldsymbol{m}]$ (see Example 9.8 of [1] or [16]). It will be shown below that $\overline{\mathrm{udim}}_M(\boldsymbol{T})$ may be finite.

Let $\boldsymbol{R}$ be an arbitrary equivariant r-covering of $\boldsymbol{T}$. Consider the set of middle vertices $\boldsymbol{A}_r := \{v \in \boldsymbol{T} : \boldsymbol{m}(v) \geq r\}$. Since these vertices have pairwise distance at least r, they belong to different balls in the covering. So, by the mass transport principle, one can show that $\rho(\boldsymbol{R}) \geq \rho(\boldsymbol{A}_r)$, where $\rho(\cdot) = \mathbb{P}\left[\boldsymbol{o} \in \cdot\right]$ denotes the intensity. On the other hand, let $\boldsymbol{S}$ be the equivariant subset of vertices with degree 3. Send unit mass from every point of $\boldsymbol{A}_r$ to its two closest points in $\boldsymbol{S}$. Then the mass transport principle implies that $2\rho(\boldsymbol{A}_r) = 3\rho(\boldsymbol{S})\mathbb{P}\left[\boldsymbol{l}_e \geq r\right]$. Hence, $\rho(\boldsymbol{R}) \geq \frac{3}{2}\rho(\boldsymbol{S})\mathbb{P}\left[\boldsymbol{l}_e \geq r\right]$. This gives that $\overline{\mathrm{udim}}_M(\boldsymbol{T}) \leq \overline{\mathrm{decay}}\left(\mathbb{P}\left[\boldsymbol{l}_e \geq r\right]\right)$, which can be finite. In fact, if $\mathrm{decay}\left(\mathbb{P}\left[\boldsymbol{l}_e \geq r\right]\right)$ exists, Proposition 3.4 below implies that $\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{decay}\left(\mathbb{P}\left[\boldsymbol{l}_e \geq r\right]\right)$.

The following proposition gives a lower bound on the Minkowski dimension.

Proposition 3.4. *Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular tree with infinitely many ends and without leaves. Let $\boldsymbol{S}$ be the equivariant subset of vertices of degree at least 3. For every $v \in \boldsymbol{S}$, let $\boldsymbol{w}(v)$ be the sum of the distances of v to its neighbors in $\boldsymbol{S}$. If $\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})^{\alpha}\right] < \infty$, then $\underline{\mathrm{udim}}_M(\boldsymbol{T}) \geq \alpha$.*

The proof is based on the following simpler result. This will be used in Subsection 3.2.4 as well.

Proposition 3.5. *Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular tree such that the degree of every vertex is at least 3. Let $\boldsymbol{d}'$ be an equivariant metric on $\boldsymbol{T}$. Let $\boldsymbol{w}(v) := \sum_u \boldsymbol{d}'(v, u)$, where the sum is over the 3 neighbors of v which are closest to v under the metric $\boldsymbol{d}'$. If $\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o})^{\alpha}\right] < \infty$, then $\underline{\mathrm{udim}}_M(\boldsymbol{T}, \boldsymbol{d}') \geq \alpha$.*

Proof. Define $\boldsymbol{w}'(v) := \sum_u \boldsymbol{d}'(u, v)^{\alpha}$, where the sum is over the three closest neighbors of v. It is enough to assume that $\boldsymbol{d}'$ is generated by equivariant edge lengths since increasing the edge lengths does not increase the dimension (by Theorem I.3.32). By the same argument, it is enough to assume $\boldsymbol{d}'(u, v) \geq 1$ for all $u \sim v$. Then, it can be seen that there exists a constant c, that depends only on α, such that $\boldsymbol{w}'(N_r(v)) \geq cr^{\alpha}$ for all $v \in \boldsymbol{T}$ and $r \geq 0$ (see Lemma A.3 in the arXiv version). Also, the assumption implies that $\mathbb{E}\left[\boldsymbol{w}'(\boldsymbol{o})\right] < \infty$. So Lemma 2.4 implies that $\underline{\mathrm{udim}}_M(T_3, \boldsymbol{d}') \geq \alpha$ and the claim is proved. $\square$

Proof of Proposition 3.4. For $v \in \boldsymbol{S}$, let $\boldsymbol{w}'(v) := \sum_u d(u, v)^{\alpha}$, where the sum is over the neighbors of v in $\boldsymbol{S}$. For $v \in \boldsymbol{T} \setminus \boldsymbol{S}$, if u_1 and u_2 are the two closest points of $\boldsymbol{S}$ to v, let $g(v, u_i) := d(u_i, v)^{\alpha-1}$ and $\boldsymbol{w}'(v) := g(v, u_1) + g(v, u_2)$. The assumption implies that $\mathbb{E}\left[\boldsymbol{w}'(\boldsymbol{o})\right] < \infty$ (use the mass transport principle for g defined above). Similarly to Proposition 3.5, there exists $c = c(\alpha)$, such that $\boldsymbol{w}'(N_r(v)) \geq cr^{\alpha}$ for all $v \in \boldsymbol{T}$ and $r \geq 0$ (see Lemma A.3 in the arXiv version) and the claim is proved. $\square$

12

3.2 Instances of Unimodular Trees

3.2.1 A Unimodular Tree With No Growth Rate

Recall the generalized canopy tree $[\boldsymbol{T}, \boldsymbol{o}]$ from Subsection I.4.2.2. Here, it is shown that $\underline{\text{growth}}(\boldsymbol{T}) \neq \overline{\text{growth}}(\boldsymbol{T})$ if the parameters are suitably chosen. Similarly, it provides an example where the exponential growth rate does not exist. In the latter, the existence of such trees is proved in [22], but with a more difficult construction.

Choose the sequence $(p_n)_n$ in the definition of $[\boldsymbol{T}, \boldsymbol{o}]$ such that $p_n = c2^{-q_n}$ and $\sum_n p_n = 1$, where c is constant and $q_0 \leq q_1 \leq \cdots$ is a sequence of integers. In this case, $\boldsymbol{T}$ is obtained by splitting the edges of the canopy tree by adding new vertices or concatenating them, depending only on the *level* of the edges. It can be seen that if v is a vertex in the n-th level of $\boldsymbol{T}$, then the number of descendants of v is $(p_0 + \cdots + p_n)/p_n$. It follows that $\underline{\text{growth}}(\boldsymbol{T}) = \underline{\text{decay}}(p_n)$ and $\overline{\text{growth}}(\boldsymbol{T}) = \overline{\text{decay}}(p_n)$. So, by choosing $(p_n)_n$ appropriately, $\boldsymbol{T}$ can have no polynomial (or exponential) growth rate. This proves the claim. Note also that the unimodular Billingsley lemma and Theorem I.4.2 imply that $\text{udim}_H(\boldsymbol{T}) = \overline{\text{growth}}(\boldsymbol{T})$ here.

3.2.2 Unimodular Galton-Watson Trees

Here, it is shown that the unimodular Galton-Watson tree [1] is infinite dimensional. (note that this tree differs from the Eternal Galton-Watson tree of Subsection I.4.2.3 which is a directed tree). Consider an ordinary Galton-Watson tree with offspring distribution $\mu = (p_0, p_1, \ldots)$, where μ is a probability measure on $\mathbb{Z}^{\geq 0}$. The unimodular Galton-Watson tree $[\boldsymbol{T}, \boldsymbol{o}]$ has a similar construction with the difference that the offspring distribution of the origin is different from that of the other vertices: It has for distribution the size-biased version $\hat{\mu} = (\frac{n}{m}p_n)_n$, where m is the mean of μ (assumed to be finite).

In what follows, the trivial case $p_1 = 1$ is excluded. If $m \leq 1$, then $\boldsymbol{T}$ is finite a.s.; i.e., there is extinction a.s. Therefore, $\text{udim}_H(\boldsymbol{T}) = 0$. So assume the *supercritical case*, namely $m > 1$. If $p_0 > 0$, then $\boldsymbol{T}$ is finite with positive probability. So $\text{udim}_H(\boldsymbol{T}) = 0$ for the same reason. Nevertheless, one can condition on non-extinction as follows.

Proposition 3.6. *Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a supercritical unimodular Galton-Watson tree conditioned on non-extinction. Then, $\text{udim}_M(\boldsymbol{T}) = \text{udim}_H(\boldsymbol{T}) = \infty$.*

Proof. The result for the Hausdorff dimension is followed from the unimodular Billingsley lemma (Theorem 2.6) and the Kesten-Stigum theorem [15], which implies that $\lim_n \#N_n(\boldsymbol{o})m^{-n}$ exists and is positive a.s. Computing the Minkowski dimension is more difficult. By part (iv) of Lemma 2.4, it is enough to prove that $\mathbb{E}\left[(1 - n^{-\alpha})^{\#N_n(\boldsymbol{o})}\right]$ has infinite decay rate for every $\alpha \geq 0$. Denote by $[\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{o}}]$ the Galton-Watson tree with the same parameters. Using the fact that $\#N_n(\boldsymbol{o})$ is stochastically larger than $\#N_{n-1}(\widetilde{\boldsymbol{o}})$, one gets that it is enough to prove the last claim for $[\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{o}}]$.

13

For simplicity, the proof is given for the case $p_0 = 0$ only. By this assumption, the probability of extinction is zero. The general case can be proved with similar arguments and by using the decomposition theorem of supercritical Galton-Watson trees (see e.g., Theorem 5.28 of [17]). In fact, the following proof implies the general claim by the fact that the *trunk*, conditioned on non-extinction, is another supercritical unimodular Galton-Watson tree. The latter can be proved similarly to the decomposition theorem.

Let $f(s) := \sum_n p_n s^n$ be the generating function of μ. By classical results of the theory of branching processes, for all $s \le 1$, $\mathbb{E}\left[s^{d_n(\tilde{o})}\right] = f^{(n)}(s)$, where $d_n(\tilde{o}) := \#N_n(\tilde{o}) - \#N_{n-1}(\tilde{o})$ and $f^{(n)}$ is the n-fold composition of f with itself. Let $a > 0$ be fixed and $g(s) := \frac{as}{-s+a+1}$ (such functions are frequently used in the literature on branching processes; see, e.g., [3]). One has $f(0) = g(0) = 0$, $f(1) = g(1) = 1$, $f'(1) = m > 1$, $g'(1) = (1+a)/a$, and f is convex. Therefore, a can be chosen large enough such that $f(s) \le g(s)$ for all $s \in [0,1]$. So

$$f^{(n)}(s) \le g^{(n)}(s) = \frac{a^n s}{a^n + (a+1)^n(1-s)},$$

where the last equality can be checked by induction. Therefore,

$$f^{(n)}(1 - n^{-\alpha}) \le \frac{a^n}{a^n + n^{-\alpha}(a+1)^n}.$$

It follows that decay $\left(f^{(n)}(1 - n^{-\alpha})\right) = \infty$. So the above discussion gives that $\mathbb{E}\left[(1 - n^{-\alpha})^{\#N_n(o)}\right]$ has infinite decay rate and the claim is proved. $\square$

3.2.3 Unimodular Eternal Galton-Watson Trees

Unimodular eternal Galton-Watson (EGW) trees were introduced in Subsection I.4.2.3. The following theorem complements Theorem I.4.9.

Proposition 3.7. *Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular eternal Galton-Watson tree. If the offspring distribution has finite variance, then $\operatorname{udim}_M(\boldsymbol{T}) = \operatorname{udim}_H(\boldsymbol{T}) = 2$.*

Proof. Theorem I.4.9 proves that $\operatorname{udim}_M(\boldsymbol{T}) = 2$. So it remains to prove $\operatorname{udim}_H(\boldsymbol{T}) \le 2$. By the unimodular Billingsley lemma (Theorem 2.6), it is enough to show that $\mathbb{E}\left[\#N_n(\boldsymbol{o})\right] \le cn^2$ for a constant c. Recall from Subsection I.4.1.2 that $F(v)$ represents the parent of vertex v and $D(v)$ denotes the subtree of descendants of v. Write $N_n(\boldsymbol{o}) = Y_0 \cup Y_1 \cup \cdots \cup Y_n$, where $Y_n := N_n(\boldsymbol{o}) \cap D(\boldsymbol{o})$ and $Y_i := N_n(\boldsymbol{o}) \cap D(F^{n-i}(\boldsymbol{o})) \setminus D(F^{n-i-1}(\boldsymbol{o}))$ for $0 \le i < n$. By the explicit construction of EGW trees in [6], Y_n is a critical Galton-Watson tree up to generation n. Also, for $0 \le i < n$, Y_i has the same structure up to generation i, except that the distribution of the first generation is *size-biased minus one* (i.e., $(np_{n+1})_n$ with the notation of Subsection 3.2.2). So the assumption of finite variance implies that the first generation in each Y_i has finite mean, namely m'. Now, one can inductively show that $\mathbb{E}\left[\#Y_n\right] = n$ and $\mathbb{E}\left[\#Y_i\right] = im'$, for $0 \le i < n$. It follows that $\mathbb{E}\left[\#N_n(\boldsymbol{o})\right] \le (1+m')n^2$ and the claim is proved. $\square$

3.2.4 The Poisson Weighted Infinite Tree

The Poisson Weighted Infinite Tree (`PWIT`) is defined as follows (see e.g., [2]). It is a rooted tree $[\boldsymbol{T}, \boldsymbol{o}]$ such that the degree of every vertex is infinite. Regarding $\boldsymbol{T}$ as a family tree with progenitor $\boldsymbol{o}$, the edge lengths are as follows. For every $u \in \boldsymbol{T}$, the set $\{d(u, v) : v \text{ is an offspring of } u\}$ is a Poisson point process on $\mathbb{R}^{\geq 0}$ with intensity function x^k, where $k > 0$ is a given integer. Moreover, for different vertices u, the corresponding Poisson point processes are jointly independent. It is known that the `PWIT` is unimodular (notice that although each vertex has infinite degree, the `PWIT` is boundedly-finite as a metric space). See for example [2] for more details.

Proposition 3.8. *The `PWIT` satisfies* $\mathrm{udim}_M(\texttt{PWIT}) = \mathrm{udim}_H(\texttt{PWIT}) = \infty$.

Proof. Denote the neighbors of $\boldsymbol{o}$ by $v_1, v_2, \ldots$ such that $d(\boldsymbol{o}, v_i)$ is increasing in i. It is straightforward that all moments of $d(\boldsymbol{o}, v_3)$ are finite. Therefore, Proposition 3.5 implies that $\underline{\mathrm{udim}}_M(\boldsymbol{T}) = \infty$ (see also Lemma A.3 in the arXiv version). This proves the claim. $\qquad\square$

3.3 Examples associated with Random Walks

As in Subsection I.4.3, consider the simple random walk $(S_n)_{n \in \mathbb{Z}}$ in $\mathbb{R}^k$, where $S_0 = 0$ and the increments $S_n - S_{n-1}$ are i.i.d.

3.3.1 The Image and The Zeros of the Simple Random Walk

Recall that Theorem I.4.11 studies the unimodular Minkowski dimension of the image of a simple random walk. The following is a complement to this result.

Theorem 3.9. *Let* $\Phi := \{S_n\}_{n \in \mathbb{Z}}$ *be the image of a simple random walk S in $\mathbb{R}$, where $S_0 := 0$. Assume the jumps $S_n - S_{n-1}$ are positive a.s.*

   (i) $\mathrm{udim}_H(\Phi) \leq 1 \wedge \overline{\mathrm{decay}}\left(\mathbb{P}\left[S_1 > r\right]\right)$.

   (ii) If $\beta := \mathrm{decay}\left(\mathbb{P}\left[S_1 > r\right]\right)$ *exists, then* $\mathrm{udim}_M(\Phi) = \mathrm{udim}_H(\Phi) = 1 \wedge \beta$.

Proof. Theorem I.4.11 proves that $\underline{\mathrm{udim}}_M(\Phi) \geq 1 \wedge \underline{\mathrm{decay}}\left(\mathbb{P}\left[S_1 > r\right]\right)$. So it is enough to prove part (i). Since Φ is a point-stationary point process in $\mathbb{R}$ (see Subsection I.4.3.1), Proposition 2.13 implies that $\mathrm{udim}_H(\Phi) \leq 1$. Now, assume $\overline{\mathrm{decay}}\left(\mathbb{P}\left[S_1 > r\right]\right) < \beta$. Then, there exists $c > 0$ such that $\mathbb{P}\left[S_1 > r\right] > cr^{-\beta}$ for all $r \geq 1$. This implies that there exists $C < \infty$ and a random number $r_0 > 0$ such that for all $r \geq r_0$, one has $\#N_r(\boldsymbol{o}) \leq Cr^{\beta} \log\log r$ a.s. (for the proof, see the arXiv version or Theorem 4 of [12]). Therefore, the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\Phi) \leq \beta + \epsilon$ for every $\epsilon > 0$, which in turn implies that $\mathrm{udim}_H(\Phi) \leq \beta$. So the claim is proved. $\qquad\square$

The following proposition complements Theorem I.4.12. It is readily implied by Theorem 3.9 above.

Proposition 3.10. *Let Ψ be the zero set of the symmetric simple random walk on $\mathbb{Z}$ with uniform jumps in $\{\pm 1\}$. Then,* $\mathrm{udim}_M(\Psi) = \mathrm{udim}_H(\Psi) = \frac{1}{2}$.

In this proposition, the guess is that $\mathcal{M}^{1/2}(\Psi) = \infty$. Additionally, by analogy with the zero set of Brownian motion [21], it is natural to guess that $\sqrt{r \log \log r}$ is a dimension function for Ψ (see Subsection I.3.8.2). To prove this, one should strengthen the bound $Cr^\beta \log \log r$ in the proof of Theorem 3.9 and also construct a covering of the set which is better than that of Proposition I.3.15. For the former, one may use Theorem 4 of [12] (it seems that the assumption of [12] on the tail of the jumps is not necessary for having only an upper bound). For the latter, one might try to get ideas from [21] (it is necessary to use intervals with different lengths).

Example 3.11 (Infinite Hausdorff Measure). In Theorem 3.9, assume $\mathbb{P}[S_1 > r] = 1/\log r$ for large enough r. Then, part (ii) of the theorem implies that $\mathrm{udim}_H(\Phi) = 0$. However, since Φ is infinite a.s., it has infinite 0-dimensional Hausdorff measure (Proposition I.3.29).

Example 3.12 (Zero Hausdorff Measure). In Theorem 3.9, assume $\mathbb{P}[S_1 > r] = 1/r$ for large enough r. Then, part (ii) of the theorem implies that $\mathrm{udim}_H(\Phi) = 1$. Since $\mathbb{E}[S_1] = \infty$, Φ is not the Palm version of any stationary point process (see Proposition 2.16). Therefore, Proposition 2.16 implies that $\mathcal{M}^1(\Phi) = 0$.

3.3.2 The Graph of the Simple Random Walk

The graph of the random walk $(S_n)_{n \in \mathbb{Z}}$ is $\Psi := \{(n, S_n) : n \in \mathbb{Z}\} \subseteq \mathbb{R}^{k+1}$. It can be seen that Ψ is a point-stationary point process, and hence, $[\Psi, 0]$ is unimodular (see Subsection I.4.3.1). Since $\#\Psi \cap [-n, n]^{k+1} \leq 2n + 1$, the mass distribution principle (Theorem 2.2) implies that $\mathrm{udim}_H(\Psi) \leq 1$. In addition, if S_1 has finite first moment, then the strong law of large numbers implies that $\lim_n \frac{1}{n} S_n = \mathbb{E}[S_1]$. This implies that $\liminf_n \frac{1}{n} \# \left(\Psi \cap [-n, n]^{k+1} \right) > 0$. Therefore, the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\Psi) \geq 1$. Hence, $\mathrm{udim}_H(\Psi) = 1$.

Below, the focus is on the case $k = 1$ and on the following metric:

$$d((x, y), (x', y')) := \max\{\sqrt{|x - x'|}, |y - y'|\}. \tag{3.1}$$

Theorem I.3.32 implies that, by considering this metric, unimodularity is preserved and dimension is not decreased. Under this metric, the ball $N_n(0)$ is $\Psi \cap [-n^2, n^2] \times [-n, n]$. It is straightforward that $\mathbb{Z}^2$ has growth rate 3 and also Minkowski and Hausdorff dimension 3 under this metric.

Proposition 3.13. *If the jumps are ± 1 uniformly, under the metric* (3.1), *the graph Ψ of the simple random walk satisfies* $\mathrm{udim}_M(\Psi) = \mathrm{udim}_H(\Psi) = 2$.

Proof. Let $n \in \mathbb{N}$. The ball $N_n(0)$ has at most $2n^2 + 1$ elements. So the mass distribution principle (Theorem 2.2) implies that $\mathrm{udim}_H(\Psi) \leq 2$. For the other side, let $\mathcal{C}$ be the equivariant disjoint covering of $\mathbb{Z}^2$ by translations of

16

the rectangle $[-n^2, n^2] \times [-n, n]$ (similar to Example I.3.13). For each rectangle $\sigma \in \mathcal{C}$, select the right-most point in $\sigma \cap \Psi$ and let $\boldsymbol{S} = \boldsymbol{S}_\Psi$ be the set of selected points. By construction, $\boldsymbol{S}$ gives an n-covering of Ψ and it can be seen that it is an equivariant covering. Let σ_0 be the rectangle containing the origin. By construction, $0 \in \boldsymbol{S}$ if and only if it is either on a right-edge of σ_0 or on a horizontal edge of σ_0 and the random walk stays outside σ_0. The first case happens with probability $1/(2n^2 + 1)$. By classical results concerning the hitting time of random walks, one can obtain that the probability of the second case lies between two constant multiples of n^{-2}. It follows that $\mathbb{P}[0 \in \boldsymbol{S}]$ lies between two constant multiples of n^{-2}. Therefore, $\underline{\mathrm{udim}}_M(\Psi) \geq 2$. This proves the claim. $\qquad\square$

3.4 A Drainage Network Model

Let $[\boldsymbol{T}, \boldsymbol{o}]$ be the one-ended tree in Subsection I.4.5 equipped with the graph-distance metric.

Proposition 3.14. *One has* $\mathrm{udim}_M(\boldsymbol{T}) = \mathrm{udim}_H(\boldsymbol{T}) = \frac{3}{2}$.

Proof. Theorem I.4.14 proves that $\mathrm{udim}_M(\boldsymbol{T}) = \frac{3}{2}$. So it is enough to prove $\mathrm{udim}_H(\boldsymbol{T}) \leq \frac{3}{2}$. To use the unimodular Billingsley lemma, an upper bound on $\mathbb{E}[\#N_n(\boldsymbol{o})]$ is derived. Let $e_{k,l} := \# \left(F^{-k}(F^l(\boldsymbol{o})) \setminus F^{-(k-1)}(F^{l-1}(\boldsymbol{o})) \right)$ be the number of descendants of order k of $F^l(\boldsymbol{o})$ which are not a descendant of $F^{l-1}(\boldsymbol{o})$ (for $l = 0$, let it be just $\#F^{-k}(\boldsymbol{o})$). One has $\#N_n(\boldsymbol{o}) = \sum_{k,l} e_{k,l} 1_{\{k+l \leq n\}}$. It can be seen that $\mathbb{E}[e_{k,l}]$ is equal to the probability that two independent paths of length k and l starting both at $\boldsymbol{o}$ do not collide at another point. Therefore, $\mathbb{E}[e_{k,l}] \leq c(k \wedge l)^{-\frac{1}{2}}$ for some c and all k, l. This implies that (in the following, c is updated at each step to a new constant without changing the notation)

$$\mathbb{E}\left[\sum_{k,l \geq 0} e_{k,l} 1_{\{k+l \leq n\}}\right] \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} ck^{-\frac{1}{2}}(n-k) \leq cn \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} k^{-\frac{1}{2}} \leq cn^{\frac{3}{2}}.$$

The above inequalities imply that $\mathbb{E}[\#N_n(\boldsymbol{o})] \leq cn^{\frac{3}{2}}$ for some c and all n. Therefore, the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\boldsymbol{T}) \leq \frac{3}{2}$. So the claim is proved. $\qquad\square$

3.5 Self Similar Unimodular Spaces

In this subsection, two examples are presented which have some kind of self-similarity heuristically, but do not fit into the framework of Subsection I.4.6.

3.5.1 Unimodular Discrete Spaces Defined by Digit Restriction

Let $J \subseteq \mathbb{Z}^{\geq 0}$. For $n \geq 0$, consider the set of natural numbers with expansion $(a_n a_{n-1} \ldots a_0)$ in base 2 such that $a_i = 0$ for every $i \notin J$. Similarly to the examples in Subsection I.4.6, one can shift this set randomly and take a limit

to obtain a unimodular discrete space. This can be constructed in the following way as well: Let $\boldsymbol{T}_0 := \{0\}$. If $n \in J$, let $\boldsymbol{T}_{n+1} := \boldsymbol{T}_n \cup (\boldsymbol{T}_n \pm 2 \times 2^n)$, where the sign is chosen i.i.d., each sign with probability $1/2$. If $n \notin J$, let $\boldsymbol{T}_{n+1} := \boldsymbol{T}_n$. Finally, let $\Psi := \cup_n \boldsymbol{T}_n$.

The upper and lower asymptotic densities of J in $\mathbb{Z}^{\geq 0}$ are defined by $\overline{d}(J) := \limsup_n \frac{1}{n} J_n$ and $\underline{d}(J) := \liminf_n \frac{1}{n} J_n$, where $J_n := \# J \cap \{0, \ldots, n\}$.

Proposition 3.15. *Almost surely,*

$$
\begin{aligned}
\mathrm{udim}_H(\Psi) = \overline{\mathrm{udim}}_M(\Psi) = \overline{\mathrm{growth}}\,(\# N_n(\boldsymbol{o})) &= \overline{d}(J), \\
\underline{\mathrm{udim}}_M(\Psi) = \underline{\mathrm{growth}}\,(\# N_n(\boldsymbol{o})) &= \underline{d}(J).
\end{aligned}
$$

In particular, this provides another example of a unimodular discrete space where the (polynomial) growth rate does not exist.

Proof. Let $n \geq 0$ be given. Cover $\boldsymbol{T}_n$ by a ball of radius 2^n centered at the minimal element of $\boldsymbol{T}_n$. By the same recursive definition, one can cover $\boldsymbol{T}_{n+1}$ by either 1 or 2 balls of the same radius. Continuing the recursion, an equivariant 2^n-covering $\boldsymbol{R}_n$ is obtained. It is straightforward to see that $\mathbb{P}\left[\boldsymbol{R}_n(\boldsymbol{o}) > 0\right] = 2^{-J_n}$. Since these coverings are uniformly bounded (Definition I.3.10), Lemma I.3.11 implies that $\underline{\mathrm{udim}}_M(\Psi) = \underline{d}(J)$ and $\overline{\mathrm{udim}}_M(\Psi) = \overline{d}(J)$. One has

$$
\# \boldsymbol{T}_m = 2^{J_m}. \tag{3.2}
$$

This implies that $\# N_{2^n}(\boldsymbol{o}) \leq 2^{J_n+1}$. One can deduce that $\overline{\mathrm{growth}}\,(\# N_n(\boldsymbol{o})) \leq \overline{d}(J)$. So the unimodular Billingsley lemma (Theorem 2.6) gives $\mathrm{udim}_H(\Psi) \leq \overline{d}(J)$. This proves the claim. $\square$

3.5.2 Randomized Discrete Cantor set

This subsection proposes a unimodular discrete analogue of the *random Cantor set*, recalled below. Let $0 \leq p \leq 1$ and $b > 1$. The random Cantor set in $\mathbb{R}^k$ [14] (see also [8]) is defined by $\Lambda_k(b, p) := \cap_n E_n$, where E_n is defined by the following random algorithm: Let $E_0 := [0,1]^k$. For each $n \geq 0$ and each *b-adic* cube of edge length b^{-n} in E_n, divide it into b^k smaller b-adic cubes of edge length b^{-n-1}. Keep each smaller b-adic cube with probability p and delete it otherwise independently from the other cubes. Let E_{n+1} be the union of the kept cubes. It is shown in Section 3.7 of [8] that $\Lambda_k(b, p)$ is empty for $p \leq b^{-k}$ and otherwise, has dimension $k + \log_b p$ conditioned on being non-empty.

For each $n \geq 0$, let $\boldsymbol{K}_n$ be the set of lower left corners of the b-adic cubes forming E_n. It is easy to show that $\boldsymbol{K}_n$ tends to $\Lambda_k(b, p)$ a.s. under the Hausdorff metric.

Proposition 3.16. *Let $\boldsymbol{K}'_n$ denote the random set obtained by biasing the distribution of $\boldsymbol{K}_n$ by $\# \boldsymbol{K}_n$. Let $\boldsymbol{o}'_n$ be a point chosen uniformly at random in $\boldsymbol{K}'_n$.*

(i) $[b^n \boldsymbol{K}'_n, \boldsymbol{o}'_n]$ converges weakly to some unimodular discrete space $[\hat{\boldsymbol{K}}, \hat{\boldsymbol{o}}]$.

(ii) If $p < b^{-k}$, then $\hat{\boldsymbol{K}}$ is finite a.s., hence, $\mathrm{udim}_H(\hat{\boldsymbol{K}}) = 0$ a.s.

(iii) If $p \geq b^{-k}$, then $\hat{\boldsymbol{K}}$ is infinite a.s. and

$$\mathrm{udim}_H(\hat{\boldsymbol{K}}) = \mathrm{udim}_M(\hat{\boldsymbol{K}}) = k + \log_b p, \quad a.s.$$

Note that in contrast to the continuum analogue [14], for $p = b^{-k}$, the set is non-empty and even infinite, though still zero dimensional. Also, for $p < b^{-k}$ the set is non-empty as well.

To prove the above proposition, the following construction of $\hat{\boldsymbol{K}}$ will be used. First, consider the usual nested sequence of partitions Π_n of $\mathbb{Z}^k$ by translations of the cube $\{0, \ldots, b^n - 1\}^k$, where $n \geq 0$. To make it *stationary*, shift each Π_n randomly as follows. Let $a_0, a_1, \ldots \in \{0, 1, \ldots, b-1\}^k$ be i.i.d. uniform numbers and let $\boldsymbol{U}_n = \sum_{i=0}^n a_i b^i \in \mathbb{Z}^k$. Shift the partition Π_n by the vector $\boldsymbol{U}_n$ to form a partition denoted by Π'_n. It is easy to see that Π'_n is a nested sequence of partitions.

Lemma 3.17. *Let $(\Pi'_n)_n$ be the stationary nested sequence of partitions of $\mathbb{Z}^k$ defined above. For each $n \geq 0$ and each cube $C \in \Pi'_n$ that does not contain the origin, with probability $1-p$ (independently for different choices of C), mark all points in $C \cap \mathbb{Z}^k$ for deletion. Then, the set of the unmarked points of $\mathbb{Z}^k$, pointed at the origin, has the same distribution as $[\hat{\boldsymbol{K}}, \hat{\boldsymbol{o}}]$ defined in Proposition 3.16.*

Proof of Lemma 3.17. Let Φ be the set of unmarked points in the algorithm. For $n \geq 0$, let $\boldsymbol{C}_n$ be the cube in Π'_n that contains the origin. It is proved below that $\boldsymbol{C}_n \cap \Phi$ has the same distribution as $b^n(\boldsymbol{K}'_n - \boldsymbol{o}_n)$. This implies the claim.

Let $A_n \subseteq [0,1]^k$ be the set of possible outcomes of $\boldsymbol{o}'_n$. One has $\#A_n = b^{kn}$. For $v \in A_n$, it is easy to see that the distribution of $b^n(\boldsymbol{K}'_n - \boldsymbol{o}_n)$, conditioned on $\boldsymbol{o}'_n = v$, coincides with the distribution of $\boldsymbol{C}_n \cap \Phi$ conditioned on $\boldsymbol{C}_n = b^n([0,1)^k - v)$. So it remains to prove that $\mathbb{P}[\boldsymbol{o}'_n = v] = \mathbb{P}\left[\boldsymbol{C}_n = b^n([0,1)^k - v)\right]$, which is left to the reader. $\square$

Here is another description of $\hat{\boldsymbol{K}}$. The nested structure of $\bigcup_n \Pi'_n$ defines a tree as follows. The set of vertices is $\bigcup_n \Pi'_n$. For each $n \geq 0$, connect (the vertex corresponding to) every cube in Π'_n to the unique cube in Π'_{n+1} that contains it. This tree is the canopy tree (Subsection I.4.2.1) with offspring cardinality $N := b^k$, except that the root (the cube $\{0\}$) is always a leaf. Now, keep each vertex with probability p and remove it with probability $1-p$ in an i.i.d. manner. Let $\boldsymbol{T}$ be the connected component of the remaining graph that contains the root. Conditioned on the event that $\boldsymbol{T}$ is infinite, $\hat{\boldsymbol{K}}$ corresponds to the set of leaves in the connected component of the root.

Proof of Proposition 3.16. The unimodular Billingsley lemma is used to get an upper bound on the Hausdorff dimension. For this $\mathbb{E}[\#N_{b^n}(\boldsymbol{o})]$ is studied. Consider the tree $[\boldsymbol{T}, \boldsymbol{o}]$ defined above and obtained by the percolation process on the canopy tree with offspring cardinality $N := b^k$. Let C be any cube in Π'_i that does not contain the origin. Note that the subtree of descendants of C

19

in the percolation cluster (conditioned on keeping C) is a Galton-Watson tree with binomial offspring distribution with parameters (N, p). Classical results on branching processes say $\mathbb{E}\left[\#C \cap \hat{\boldsymbol{K}} \,|\Pi'_i\right] = pm^i$, where $m := pb^k$. So the construction implies that

$$\mathbb{E}\left[\#\boldsymbol{C}_n \cap \hat{\boldsymbol{K}}\right] = 1 + p(N-1)\left(m^{n-1} + m^{n-2} + \cdots + 1\right).$$

For $m > 1$, the latter is bounded by lm^n for some constant l not depending on n. Note that $N_{b^n}(\boldsymbol{o})$ is contained in the union of $\boldsymbol{C}_n$ and $3^k - 1$ other cubes in Π'_n. It follows that $\mathbb{E}\left[\#N_{b^n}(\boldsymbol{o})\right] \leq l'm^n$, where $l' = l + (3^k - 1)p$. So the unimodular Billingsley lemma (Theorem 2.6) implies that $\text{udim}_H(\hat{\boldsymbol{K}}) \leq k + \log_b p$. The claim for $m = 1$ and $m < 1$ are similar.

Consider now the Minkowski dimension. As above, we assume $m > 1$ and the proofs for the other cases are similar. Let $n \geq 0$ be given. By considering the partition Π'_n by cubes, one can construct a b^n-covering $\boldsymbol{R}_n$ as in Theorem 2.12. This covering satisfies $\mathbb{P}\left[\boldsymbol{R}_n(\boldsymbol{o}) \geq 0\right] = \mathbb{E}\left[1/\#(\boldsymbol{C}_n \cap \hat{\boldsymbol{K}})\right]$. Let $[\boldsymbol{T}', \boldsymbol{o}']$ be the eternal Galton-Watson tree of Subsection I.4.2.3 with binomial offspring distribution with parameters (N, p). By regarding $\boldsymbol{T}'$ as a family tree, it is straightforward that $[\boldsymbol{T}, \boldsymbol{o}]$ has the same distribution as the part of $[\boldsymbol{T}', \boldsymbol{o}']$, up to the generation of the root (see [6] for more details on eternal family trees). Therefore, Lemma 5.7 of [6] implies that $\mathbb{E}\left[1/\#(\boldsymbol{C}_n \cap \hat{\boldsymbol{K}})\right] = m^{-n}\mathbb{P}\left[h(\boldsymbol{o}') \geq n\right]$. Since $m > 1$, $\mathbb{P}\left[h(\boldsymbol{o}') \geq n\right]$ tends to the non-extinction probability of the descendants of the root, which is positive. By noticing the fact that the radii of the balls are b^n and the covering is uniformly bounded, one gets that $\text{udim}_M(\hat{\boldsymbol{K}}) = \log_b m = k + \log_b p$.

Finally, it remains to prove that $\hat{\boldsymbol{K}}$ is infinite a.s. when $p = b^{-k}$. In this case, consider the eternal Galton-Watson tree $[\boldsymbol{T}', \boldsymbol{o}']$ as above. Proposition 6.8 of [6] implies that the generation of the root is infinite a.s. This proves the claim. □

3.6 Cayley Graphs

As mentioned in Subsection I.3.6, the dimension of a Cayley graph depends only on the group and not on the generating set. The following result connects it to the growth rate of the group. Note that Gromov's theorem [13] implies that the polynomial growth degree exists and is either an integer or infinity.

Theorem 3.18. *For every finitely generated group H with polynomial growth degree $\alpha \in [0, \infty]$, one has $\text{udim}_M(H) = \text{udim}_H(H) = \alpha$ and $\mathcal{M}^\alpha(H) < \infty$.*

Proof. First, assume $\alpha < \infty$. The result of Bass [7] implies that there are constants $c, C > 0$ such that $\forall r \geq 1 : cr^\alpha < \#N_r(o) \leq Cr^\alpha$, where o is an arbitrary element of H. So the mass distribution principle (Theorem 2.2) and part (i) of Lemma 2.4 imply that $\text{udim}_M(H) = \text{udim}_H(H) = \alpha$. In addition,

(2.1) in the proof of Theorem 2.2 implies that $\mathcal{H}_M^\alpha(H) \geq 1/C$ for all $M \geq 1$, which implies that $\mathcal{M}^\alpha(H) \leq C < \infty$.

Second, assume $\alpha = \infty$. The result of [23] shows that for any $\beta < \infty$, $\#N_r(o) > r^\beta$ for sufficiently large r. Therefore, part (i) of Lemma 2.4 implies that $\underline{\mathrm{udim}}_M(H) \geq \beta$. Hence, $\mathrm{udim}_M(H) = \mathrm{udim}_H(H) = \infty$ and the claim is proved. □

It is natural to expect that $\mathcal{M}^\alpha(H) > 0$ as well, but only a weaker inequality will be proved in Proposition 4.17.

3.7   Notes and Bibliographical Comments

The proof of Proposition 3.2 was suggested by R. Lyons. Bibliographical comments on some of the examples discussed in this section can be found at the end of Section I.4. The example defined by digit restriction (Subsection 3.5.1) is inspired by an example in the continuum setting (see e.g., Examples 1.3.2 of [8]). The randomized discrete Cantor set (Subsection 3.5.2) is inspired by the *random cantor set* (see e.g., Section 3.7 of [8]).

4   Frostman's Theory

This section provides a unimodular version of Frostman's lemma and some of its applications. In a sense to be made precise later, this lemma gives converses to the mass distribution principle. It is a powerful tool in the theoretical analysis of the unimodular Hausdorff dimension. For example, it is used in this section to derive inequalities for the dimension of product spaces and *embedded spaces* (Subsections 4.4.2 and 4.4.3). It is also the basis of many of the results in [5].

4.1   Unimodular Frostman Lemma

The statement of the unimodular Frostman lemma requires the definition of *weighted Hausdorff* content. The latter is based on the notion of *equivariant weighted collections of balls* as follows. For this, the following mark space is needed. Let Ξ be the set of functions $c : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ which are positive in only finitely many points; i.e., $c^{-1}((0, \infty))$ is a finite set. Remark 4.8 below defines a metric on Ξ such that the notion of Ξ-valued equivariant processes (Definition I.2.8) are well defined. Such a process $\boldsymbol{c}$ is called an **equivariant weighted collection of balls**[1]. Consider a unimodular discrete space $[\boldsymbol{D}, \boldsymbol{o}]$ with distribution μ. For $v \in \boldsymbol{D}$, the reader can think of the value $\boldsymbol{c}_r(v) := \boldsymbol{c}(v)(r)$, if positive, to indicate that there is a ball in the collection, with radius r, centered at v, and with **cost** (or weight) $\boldsymbol{c}_r(v)$. Note that extra randomness is allowed in the definition. A ball-covering $\boldsymbol{R}$ can be regarded a special case of this construction by letting $\boldsymbol{c}_r(v)$ be 1 when $r = \boldsymbol{R}(v)$ and 0 otherwise.

---

[1]The term 'weighted' refers to the weighted sums in Definition 4.1 and should not be confused with equivariant weight functions of Definition 2.1

Definition 4.1. Let $f : \mathcal{D}_* \to \mathbb{R}$ be a measurable function and $M \geq 1$. An equivariant weighted collection of balls $\boldsymbol{c}$ is called a (f, M)**-covering** if

$$\forall v \in \boldsymbol{D} : f(v) \leq \sum_{u \in \boldsymbol{D}} \sum_{r \geq M} \boldsymbol{c}_r(u) 1_{\{v \in N_r(u)\}}, \quad a.s., \tag{4.1}$$

where $f(v) := f[\boldsymbol{D}, v]$ for $v \in \boldsymbol{D}$. For $\alpha \geq 0$, define

$$\xi_M^\alpha(f) \quad := \quad \inf \left\{ \mathbb{E}\left[\sum_r \boldsymbol{c}_r(\boldsymbol{o}) r^\alpha \right] : \boldsymbol{c} \text{ is a } (f, M)\text{-covering} \right\},$$

$$\xi_\infty^\alpha(f) \quad := \quad \lim_{M \to \infty} \xi_M^\alpha(f).$$

It is straightforward that every equivariant ball-covering of Definition I.3.16 gives a $(1, 1)$-covering, where 1 is regarded as the constant function $f \equiv 1$ on $\mathcal{D}_*$. This gives (see also Conjecture 4.4 below)

$$\xi_M^\alpha(1) \leq \mathcal{H}_M^\alpha(\boldsymbol{D}). \tag{4.2}$$

Also, by considering the case $\boldsymbol{c}_M(v) := f(v) \vee 0$, one can see that if $f \in L_1(\mathcal{D}_*, \mu)$, then

$$\xi_M^\alpha(f) \leq M^\alpha \mathbb{E}\left[f(\boldsymbol{o}) \vee 0 \right] < \infty.$$

In the following theorem, to be consistent with the setting of the paper, the following notation is used: $w(u) := w([D, u])$ for $u \in D$, and $w(N_r(v)) = \sum_{u \in N_r(v)} w(u)$. Also, recall that a deterministic equivariant weight function is given by a measurable function $w : \mathcal{D}_* \to \mathbb{R}^{\geq 0}$ (see Example I.2.10).

Theorem 4.2 (Unimodular Frostman Lemma)**.** *Let* $[\boldsymbol{D}, \boldsymbol{o}]$ *be a unimodular discrete space,* $\alpha \geq 0$ *and* $M \geq 1$.

(i) *There exists a bounded measurable weight function* $w : \mathcal{D}_* \to \mathbb{R}^{\geq 0}$ *such that* $\mathbb{E}\left[w(\boldsymbol{o}) \right] = \xi_M^\alpha(1)$ *and almost surely,*

$$\forall v \in \boldsymbol{D}, \ \forall r \geq M : w(N_r(v)) \leq r^\alpha. \tag{4.3}$$

(ii) *Given a non-negative function* $h \in L_1(\mathcal{D}_*, \mu)$, *the first condition can be replaced by* $\mathbb{E}\left[w(\boldsymbol{o}) h(\boldsymbol{o}) \right] = \xi_M^\alpha(h)$.

(iii) *In the setting of* (ii), *if* $\boldsymbol{D}$ *has finite* α-*dimensional Hausdorff measure (e.g., when* $\mathrm{udim}_H(\boldsymbol{D}) > \alpha$) *and* $h \not\equiv 0$, *then* $w[\boldsymbol{D}, \boldsymbol{o}] \neq 0$ *with positive probability.*

The proof is given later in this subsection.

Remark 4.3. One can show that (4.3) implies that $\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o}) \right] \leq \xi_M^\alpha(1)$ and $\mathbb{E}\left[\boldsymbol{w}(\boldsymbol{o}) h(\boldsymbol{o}) \right] \leq \xi_M^\alpha(h)$. Therefore, the (deterministic) weight function w given in the unimodular Frostman lemma is a maximal equivariant weight function satisfying (4.3) (it should be noted that the maximal is not unique). The proof is similar to that of the mass distribution principle (Theorem 2.2) and is left to the reader.

Conjecture 4.4. *One has* $\mathcal{H}_M^\alpha(\boldsymbol{D}) = \xi_M^\alpha(1)$.

Lemma 4.7 below proves a weaker inequality. For instance, it can be seen that the conjecture holds for $\mathbb{Z}^k$ (with the l_∞ metric) and for the non-ergodic example of Example I.3.21. In the former, this is obtained by considering the constant weight function $w(\cdot) \equiv \left(\frac{M}{2M+1}\right)^k$, which satisfies the claim of the unimodular Frostman lemma. The latter is similar by letting $w[\mathbb{Z}, 0] := \frac{M}{2M+1}$ and $w[\mathbb{Z}^2, 0] := 0$.

Remark 4.5. The unimodular Frostman lemma implies that, in theory, the mass transport principle (Theorem 2.2) is enough for bounding the Hausdorff dimension from above. However, there are very few examples in which the function w given by the unimodular Frostman lemma can be explicitly computed (in some of the examples, a function w satisfying only (4.3) can be found; e.g., for two-ended trees). Therefore, in practice, the unimodular Billingsley lemma is more useful than the mass transport principle.

The following lemma is needed to prove Theorem 4.2.

Lemma 4.6. *The function* $\xi_M^\alpha : L_1(\mathcal{D}_*, \mu) \to \mathbb{R}$ *is continuous. In fact, it is* M^α-*Lipschitz; i.e.,* $|\xi_M^\alpha(f_1) - \xi_M^\alpha(f_2)| \leq M^\alpha \mathbb{E}\left[|f_1(\boldsymbol{o}) - f_2(\boldsymbol{o})|\right]$.

Proof. Let $\boldsymbol{c}$ be an equivariant weighted collection of balls satisfying (4.1) for f_1. Intuitively, add a ball of radius M at each point v with cost $|f_2(v) - f_1(v)|$. More precisely, let $\boldsymbol{c}'_r(v) := \boldsymbol{c}_r(v)$ for $r \neq M$ and $\boldsymbol{c}'_M(v) := \boldsymbol{c}_M(v) + |f_2(v) - f_1(v)|$. This definition implies that $\boldsymbol{c}'$ satisfies (4.1) for f_2. Also,

$$\xi_M^\alpha(f_2) \leq \mathbb{E}\left[\sum_i \boldsymbol{c}'_i(\boldsymbol{o})i^\alpha\right] = \mathbb{E}\left[\sum_r \boldsymbol{c}_r(\boldsymbol{o})i^\alpha\right] + M^\alpha \mathbb{E}\left[|f_2(\boldsymbol{o}) - f_1(\boldsymbol{o})|\right].$$

Since $\boldsymbol{c}$ was arbitrary, one obtains $\xi_M^\alpha(f_2) \leq \xi_M^\alpha(f_1) + M^\alpha \mathbb{E}\left[|f_2(\boldsymbol{o}) - f_1(\boldsymbol{o})|\right]$, which implies the claim. $\square$

Proof of Theorem 4.2. Part (i) is implied by Part (ii) which is proved below. It is easy to see that $\xi_M^\alpha(tf) = t\xi_M^\alpha(f)$ for all f and $t \geq 0$ and also $\xi_M^\alpha(f_1 + f_2) \leq \xi_M^\alpha(f_1) + \xi_M^\alpha(f_2)$ for all f_1, f_2. Let $h \in L_1(\mathcal{D}_*, \mu)$ be given. By the Hahn-Banach theorem (see Theorem 3.2 of [20]), there is a linear functional $l : L_1(\mathcal{D}_*, \mu) \to \mathbb{R}$ such that $l(h) = \xi_M^\alpha(h)$ and $-\xi_M^\alpha(-f) \leq l(f) \leq \xi_M^\alpha(f)$, for all $f \in L_1$. Since l is sandwiched between two functions which are continuous at 0 and are equal at 0, one gets that l is continuous at 0. Since l is linear, this implies that l is continuous. Since the dual of $L_1(\mathcal{D}_*, \mu)$ is $L_\infty(\mathcal{D}_*, \mu)$, one obtains that there is a function $w \in L_\infty(\mathcal{D}_*, \mu)$ such that $l(f) = \mathbb{E}\left[f(\boldsymbol{o})w(\boldsymbol{o})\right]$, for all $f \in L_1$. Note that if $f \geq 0$, then $\xi_M^\alpha(-f) = 0$ and so $l(f) \geq 0$. This implies that $w(\boldsymbol{o}) \geq 0$ a.s. (otherwise, let $f(\boldsymbol{o}) := 1_{\{w(\boldsymbol{o})<0\}}$ to get a contradiction). Consider a version of w which is nonnegative everywhere. The claim is that w satisfies the requirements.

Let $r \geq M$ be fixed. For all discrete spaces D, let $\boldsymbol{S} := \boldsymbol{S}_D := \{v \in D : w(N_r(v)) > r^\alpha\}$. Define $f_r(v) := \#N_r(v) \cap \boldsymbol{S}$. By the definition of $\boldsymbol{S}_D$, one has

23

$$\mathbb{E}\left[w(N_r(\boldsymbol{o}))1_{\{\boldsymbol{o}\in\boldsymbol{S}\}}\right] \geq r^\alpha \mathbb{P}\left[\boldsymbol{o}\in\boldsymbol{S}\right]. \tag{4.4}$$

Moreover, if $\mathbb{P}\left[\boldsymbol{o}\in\boldsymbol{S}\right] > 0$, then the inequality is strict. On the other hand, by the mass transport principle for the function $(v,u)\mapsto w(u)1_{\{v\in\boldsymbol{S}\}}1_{\{u\in N_r(v)\}}$, one gets

$$
\begin{aligned}
\mathbb{E}\left[w(N_r(\boldsymbol{o}))1_{\{\boldsymbol{o}\in\boldsymbol{S}\}}\right] &= \mathbb{E}\left[w(\boldsymbol{o})\# N_r(\boldsymbol{o})\cap\boldsymbol{S}\right] \\
&= \mathbb{E}\left[w(\boldsymbol{o})f_r(\boldsymbol{o})\right] \\
&= l(f_r) \\
&\leq \xi_M^\alpha(f_r) \\
&\leq r^\alpha\mathbb{P}\left[\boldsymbol{o}\in\boldsymbol{S}\right],
\end{aligned}
$$

where the last inequality is implied by considering the following weighted collection of balls for f_r: put balls of radius r with cost 1 centered at the points in $\boldsymbol{S}$. More precisely, let $\boldsymbol{c}_r(v) := 1_{\{v\in\boldsymbol{S}\}}$ and $\boldsymbol{c}_s(v) := 0$ for $s\neq r$. It is easy to see that this satisfies (4.1) for f_r, which implies the last inequality by the definition of $\xi_M^\alpha(\cdot)$. Thus, equality holds in (4.4). Hence, $\mathbb{P}\left[\boldsymbol{o}\in\boldsymbol{S}\right] = 0$; i.e., $w(N_r(\boldsymbol{o})) \leq r^\alpha$ a.s. Lemma I.2.14 implies that almost surely, $\forall v\in\boldsymbol{D} : w(N_r(v)) \leq r^\alpha$. So the same holds for all rational $r\geq M$ simultaneously. By monotonicity of $w(N_r(v))$ w.r.t. r, one gets that the latter almost surely holds for all $r\geq M$ as desired.

Also, one has $\mathbb{E}\left[w(\boldsymbol{o})h(\boldsymbol{o})\right] = l(h) = \xi_M^\alpha(h)$. Thus, w satisfies the desired requirements.

To prove (iii), assume $h\not\equiv 0$ and $\mathcal{M}^\alpha(\boldsymbol{D}) < \infty$. By Lemma I.3.26, one has $\mathcal{H}_M^\alpha(\boldsymbol{D}) > 0$. So Lemma 4.7 below implies that $\xi_M^\alpha(h) > 0$. Now, the above equation implies that w is not identical to zero. $\qquad\square$

The above proof uses the following lemma.

Lemma 4.7. *Let $[\boldsymbol{D},\boldsymbol{o}]$ be a unimodular discrete metric space.*

(i) By letting $b := \xi_1^\alpha(1)$, one has $b \leq \mathcal{H}_1^\alpha(\boldsymbol{D}) \leq b + b|\log b|$.

(ii) Let $0\neq h\in L_1(\mathcal{D}_*,\mu)$ be a non-negative function. For $M\geq 1$, one has

$$\mathcal{H}_M^\alpha(\boldsymbol{D}) \leq \inf_{a\geq 0}\left\{M^\alpha\mathbb{E}\left[e^{-ah(\boldsymbol{o})}\right] + a\xi_M^\alpha(h)\right\}.$$

(iii) In addition, $\xi_M^\alpha(h) = 0$ if and only if $\mathcal{H}_M^\alpha(\boldsymbol{D}) = 0$.

Proof. (i). By considering the cases where $\boldsymbol{c}(\cdot) \in \{0,1\}$, the first inequality is easily obtained from the definition of $\xi_1^\alpha(1)$. In particular, this implies that $b \leq 1$. The second inequality is implied by part (ii) by letting $h(\cdot) := 1$ and $a := -\log b \geq 0$.

(ii). Let $b' > \xi_M^\alpha(h)$ be arbitrary. So there exists an equivariant weighted collection of balls $\boldsymbol{c}$ that satisfies (4.1) for h and $\mathbb{E}\left[\sum_{r\geq M}\boldsymbol{c}_r(\boldsymbol{o})r^\alpha\right] \leq b'$. Next, given $a\geq 0$, define an equivariant covering $\boldsymbol{R}$ as follows. For each $v\in\boldsymbol{D}$ and

24

$r \geq M$ such that $\boldsymbol{c}_r(v) > 0$, put a ball of radius r at v with probability $a\boldsymbol{c}_r(v) \wedge 1$. Do this independently for all v and r (one should condition on $\boldsymbol{D}$ first). If more than one ball is put at v, keep only the one with maximum radius. Let $\boldsymbol{S}$ be the union of the chosen balls. For $u \in \boldsymbol{D} \setminus \boldsymbol{S}$, put a ball of radius M at u. This gives an equivariant covering, namely $\boldsymbol{R}$, by balls of radii at least M. Then, one can easily get

$$\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o})\right]^{\alpha} \leq M^{\alpha}\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{S}\right] + \mathbb{E}\left[\sum_{r \geq}(a\boldsymbol{c}_r(\boldsymbol{o}) \wedge 1)r^{\alpha}\right] \leq M^{\alpha}\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{S}\right] + ab'. \ (4.5)$$

To bound $\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{S}\right]$, consider a realization of $[\boldsymbol{D}, \boldsymbol{o}]$. First, if for some $v \in \boldsymbol{D}$ and $r \geq M$, one has $a\boldsymbol{c}_r(v) > 1$ and $\boldsymbol{o} \in N_r(v)$, then $\boldsymbol{o}$ is definitely in $\boldsymbol{S}$. Second, assume this is not the case. By (4.1), one has $\sum_{u \in \boldsymbol{D}} \sum_{r \geq M} \boldsymbol{c}_r(u)1_{\{\boldsymbol{o} \in N_r(u)\}} \geq h(\boldsymbol{o})$. This implies that the probability that $\boldsymbol{o} \notin \boldsymbol{S}$ in this realization is

$$\prod_{(v,r):\boldsymbol{o} \in N_r(v)} (1 - a\boldsymbol{c}_r(v)) \leq \exp\left(-\sum_{(v,r):\boldsymbol{o} \in N_r(v)} a\boldsymbol{c}_r(v)\right) \leq e^{-ah(\boldsymbol{o})}.$$

In both cases, one gets $\mathbb{P}\left[\boldsymbol{o} \notin \boldsymbol{S}\right] \leq \mathbb{E}\left[e^{-ah(\boldsymbol{o})}\right]$. Thus, (4.5) implies that $\mathbb{E}\left[R(\boldsymbol{o})\right]^{\alpha} \leq M^{\alpha}\mathbb{E}\left[e^{-ah(\boldsymbol{o})}\right] + ab'$. Since $a \geq 0$ and $b' > b$ are arbitrary, the claim follows.

(iii). Assume $\xi_M^{\alpha}(h) = 0$. By letting $a \to \infty$ and using the first claim, one obtains that $\mathcal{H}_M^{\alpha}(\boldsymbol{D}) = 0$. Conversely, assume $\mathcal{H}_M^{\alpha}(\boldsymbol{D}) = 0$. The first inequality in part (i) gives that $\xi_M^{\alpha}(a) = 0$ for any constant a. Therefore, $\xi_M^{\alpha}(h) \leq \xi_M^{\alpha}(a) + \xi_M^{\alpha}((h-a) \vee 0)) \leq M^{\alpha}\mathbb{E}\left[(h-a) \vee 0\right]$. By letting a tend to infinity, one gets $\xi_M^{\alpha}(h) = 0$. $\square$

Remark 4.8. In this subsection, the following metric is used on the mark space Ξ. Let Ξ' be the set of finite measures on $\mathbb{R}^2$. By identifying $c \in \Xi$ with the counting measure on the finite set $\{(x, c(x)) : x \in \mathbb{R}^{\geq 0}, c(x) > 0\}$, one can identify Ξ with a Borel subset of Ξ'. It is well known that Ξ' is a complete separable metric space under the Prokhorov metric (see e.g., [9]). So one can define the notion of Ξ'-valued equivariant processes as in Definition I.2.8. Therefore, Ξ-valued equivariant processes also make sense.

4.2 Max-Flow Min-Cut Theorem for Unimodular One-Ended Trees

The result of this subsection is used in the next subsection for a Euclidean version of the unimodular Frostman lemma, but is of independent interest as well.

The max-flow min-cut theorem is a celebrated result in the field of graph theory (see e.g., [11]). In its simple version, it studies the minimum number of edges in a *cut-set* in a finite graph; i.e., a set of edges the deletion of which disconnects two given subsets of the graph. A generalization of the theorem

in the case of trees is obtained by considering cut-sets separating a given finite subset from the set of ends of the tree. This generalization is used to prove a version of Frostman's lemma for compact sets in the Euclidean space (see e.g., [8]).

This subsection presents an analogous result for unimodular one-ended trees. It discusses cut-sets separating the set of leaves from the end of the tree. Since the tree has infinitely many leaves a.s. (see e.g., [6]), infinitely many edges are needed in any such cut-set. Therefore, cardinality cannot be used to study minimum cut-sets. The idea is to use unimodularity for a quantification of the size of a cut-set.

Let $[\boldsymbol{T}, \boldsymbol{o}; \boldsymbol{c}]$ be a unimodular marked one-ended tree with mark space $\mathbb{R}^{\geq 0}$. Assume the mark $\boldsymbol{c}(e)$ of each edge e is well defined and call it the **conductance** of e. Let $\boldsymbol{L}$ be the set of leaves of $\boldsymbol{T}$. As in Subsection I.4.1.2, let $F(v)$ be the parent of vertex v and $D(v)$ be the descendants subtree of v.

Definition 4.9. A **legal equivariant flow** on $[\boldsymbol{T}; \boldsymbol{c}]$ is an equivariant way of assigning extra marks $\boldsymbol{f}(\cdot) \in \mathbb{R}$ to the edges (see Definition I.2.8 and Remark I.2.12), such that almost surely,

(i) for every edge e, one has $0 \leq \boldsymbol{f}(e) \leq \boldsymbol{c}(e)$,

(ii) for every vertex $v \in \boldsymbol{T} \setminus \boldsymbol{L}$, one has

$$\boldsymbol{f}(v, F(v)) = \sum_{w \in F^{-1}(v)} \boldsymbol{f}(w, v). \tag{4.6}$$

Also, an **equivariant cut-set** is an equivariant subset Π of the edges of $[\boldsymbol{T}; \boldsymbol{c}]$ that separates the set of leaves $\boldsymbol{L}$ from the end in $\boldsymbol{T}$.

Note that extra randomness is allowed in the above definition. The reader can think of the value $\boldsymbol{f}(v, F(v))$ as the *flow* from v to $F(v)$. So (4.6) can be interpreted as *conservation of flow* at the vertices except the leaves. Also, the leaves are regarded as the *sources* of the flow.

Since the number of leaves is infinite a.s., the sum of the flows exiting the leaves might be infinite. In fact, it can be seen that unimodularity implies that the sum is always infinite a.s. The idea is to use unimodularity to quantify how *large* is the flow. Similarly, in any equivariant cut-set, the sum of the conductances of the edges is infinite a.s. Unimodularity is also used to quantify the *conductance* of an equivariant cut-set. These are done in Definition 4.11 below.

Below, since each edge of $\boldsymbol{T}$ can be uniquely represented as $(v, F(v))$, the following convention is helpful.

Convention 4.10. For the vertices v of $\boldsymbol{T}$, the symbols $\boldsymbol{f}(v)$ and $\boldsymbol{c}(v)$ are used as abbreviations for $\boldsymbol{f}(v, F(v))$ and $\boldsymbol{c}(v, F(v))$, respectively. Also, by $v \in \Pi$, one means that the edge $(v, F(v))$ is in Π.

Definition 4.11. The **norm** of the legal equivariant flow $\boldsymbol{f}$ is defined as

$$|\boldsymbol{f}| := \mathbb{E}\left[\boldsymbol{f}(\boldsymbol{o})1_{\{\boldsymbol{o}\in\boldsymbol{L}\}}\right].$$

Also, for the equivariant cut-set Π, define

$$\boldsymbol{c}(\Pi) := \mathbb{E}\left[\boldsymbol{c}(\boldsymbol{o})1_{\{\boldsymbol{o}\in\Pi\}}\right] = \mathbb{E}\left[\sum_{w\in F^{-1}(\boldsymbol{o})} \boldsymbol{c}(w)1_{\{w\in\Pi\}}\right],$$

where the last equality follows from the mass transport principle (I.2.2).

An equivariant cut-set Π is called **equivariantly minimal** if there is no other equivariant cut-set which is a subset of Π a.s. If so, it can be seen that it is **almost surely minimal** as well; i.e., in almost every realization, it is a minimal cut set (see the arXiv version).

Lemma 4.12. *If $\boldsymbol{f}$ is a legal equivariant flow and Π is an equivariant cut-set, then $|\boldsymbol{f}| \leq \boldsymbol{c}(\Pi)$. Moreover, if the pair $(\boldsymbol{f},\Pi)$ is equivariant, then*

$$|\boldsymbol{f}| \leq \mathbb{E}\left[\boldsymbol{f}(\boldsymbol{o})1_{\{\boldsymbol{o}\in\Pi\}}\right] \leq \boldsymbol{c}(\Pi).$$

In addition, if Π is minimal, then equality holds in the left inequality.

Proof. One can always consider an *independent coupling* of $\boldsymbol{f}$ and Π (as in the proof of Theorem 2.2). So assume $(\boldsymbol{f},\Pi)$ is equivariant from the beginning. Note that the whole construction (with conductances, the flow and the cut-set) is unimodular (Lemma I.2.11). For every leaf $v \in \boldsymbol{L}$, let $\boldsymbol{\tau}(v)$ be the first ancestor of v such that $(v, F(v)) \in \Pi$. Then, send mass $\boldsymbol{f}(v)$ from each leaf v to $\boldsymbol{\tau}(v)$. By the mass transport principle (I.2.2), one gets

$$
\begin{aligned}
\mathbb{E}\left[\boldsymbol{f}(\boldsymbol{o})1_{\{\boldsymbol{o}\in\boldsymbol{L}\}}\right] &= \mathbb{E}\left[1_{\{\boldsymbol{o}\in\Pi\}} \sum_{v\in\boldsymbol{\tau}^{-1}(\boldsymbol{o})} \boldsymbol{f}(v)\right] \\
&\leq \mathbb{E}\left[1_{\{\boldsymbol{o}\in\Pi\}} \sum_{v\in D(\boldsymbol{o})\cap\boldsymbol{L}} \boldsymbol{f}(v)\right] = \mathbb{E}\left[\boldsymbol{f}(\boldsymbol{o})1_{\{\boldsymbol{o}\in\Pi\}}\right],
\end{aligned}
$$

where the last equality holds because $\boldsymbol{f}$ is a flow. Moreover, if Π is minimal, then the above inequality becomes an equality and the claim follows. $\square$

The main result is the following converse to the above lemma.

Theorem 4.13 (Max-Flow Min-Cut for Unimodular One-Ended Trees)**.** *For every unimodular marked one-ended tree $[\boldsymbol{T}, \boldsymbol{o}; \boldsymbol{c}]$ equipped with conductances $\boldsymbol{c}$ as above, if $\boldsymbol{c}$ is bounded on the set of leaves, then*

$$\max_{f} |\boldsymbol{f}| = \inf_{\Pi} \boldsymbol{c}(\Pi),$$

where the maximum is over all legal equivariant flows $\boldsymbol{f}$ and the infimum is over all equivariant cut-sets Π.

Remark 4.14. The claim of Theorem 4.13 is still valid if the probability measure (the distribution of $[\boldsymbol{T}, \boldsymbol{o}; \boldsymbol{c}]$) is replaced by any (possibly infinite) measure $\mathcal{P}$ on $\mathcal{D}'_*$ supported on one-ended trees, such that $\mathcal{P}(\boldsymbol{o} \in \boldsymbol{L}) < \infty$ and the mass transport principle (I.2.2) holds. The same proof works for this case as well. This will be used in Subsection 4.3.

Proof of Theorem 4.13. For $n \geq 1$, let $\boldsymbol{T}_n$ be the sub-forest of $\boldsymbol{T}$ obtained by keeping only vertices of height at most n in $\boldsymbol{T}$. Each connected component of $\boldsymbol{T}_n$ is a finite tree which contains some leaves of $\boldsymbol{T}$. For each such component, namely T', do the following: if T' has more than one vertex, consider the maximum flow on T' between the leaves and the *top* vertex (i.e., the vertex with maximum height in T'). If there is more than one maximum flow, choose one of them randomly and uniformly. Also, choose a minimum cut-set in T' randomly and uniformly. Similarly, if T' has a single vertex v, do the same for the subgraph with vertex set $\{v, F(v)\}$ and the single edge adjacent to v. By doing this for all components of $\boldsymbol{T}_n$, a (random) function $\boldsymbol{f}_n$ on the edges and a cut-set Π'_n are obtained (by letting $\boldsymbol{f}_n$ be zero on the other edges). Π'_n is always a cut-set, but $\boldsymbol{f}_n$ is not a flow. However, $\boldsymbol{f}_n$ satisfies (4.6) for vertices of $\boldsymbol{T}_n \setminus \boldsymbol{L}$ except the top vertices of the connected components of $\boldsymbol{T}_n$. Also, it can be seen that $\boldsymbol{f}_n$ and Π'_n are equivariant.

For each component T' of $\boldsymbol{T}_n$, the set of leaves of T', excluding the top vertex, is $\boldsymbol{L} \cap T'$. So the max-flow min-cut theorem of Ford-Fulkerson [11] (see e.g., Theorem 3.1.5 of [8]) gives that, for each component T' of $\boldsymbol{T}_n$, one has

$$\sum_{v \in \boldsymbol{L} \cap T'} \boldsymbol{f}_n(v) = \sum_{e \in \Pi'_n \cap T'} \boldsymbol{c}(e).$$

If u is the top vertex of T', let $h(u)$ be the common value in the above equation. By using the mass transport principle (I.2.2) for each of the two representations of $\mathbb{E}\left[h(\boldsymbol{o})\right]$, one can obtain

$$\mathbb{E}\left[\boldsymbol{f}_n(\boldsymbol{o}) 1_{\{\boldsymbol{o} \in \boldsymbol{L}\}}\right] = \mathbb{E}\left[h(\boldsymbol{o})\right] = \mathbb{E}\left[\boldsymbol{c}(\boldsymbol{o}) 1_{\{\boldsymbol{o} \in \Pi'_n\}}\right] = \boldsymbol{c}(\Pi'_n).$$

Since $0 \leq \boldsymbol{f}_n(\cdot) \leq \boldsymbol{c}_n(\cdot)$, one can see that the distributions of $\boldsymbol{f}_n$ are tight (the proof is left to the reader). Therefore, there is a sequence $n_1, n_2, \ldots$ and an equivariant process $\boldsymbol{f}'$ such that $\boldsymbol{f}_{n_i} \to \boldsymbol{f}'$ (weakly). It is not hard to deduce that $\boldsymbol{f}'$ is a legal equivariant flow. Also, since $\boldsymbol{f}'(\boldsymbol{o})$ and $1_{\{\boldsymbol{o} \in \boldsymbol{L}\}}$ are continuous functions of $[\boldsymbol{T}, \boldsymbol{o}; \boldsymbol{f}']$ and their product is bounded (by the assumption on $\boldsymbol{c}$), one gets that

$$\left|\boldsymbol{f}'\right| = \mathbb{E}\left[\boldsymbol{f}'(\boldsymbol{o}) 1_{\{\boldsymbol{o} \in \boldsymbol{L}\}}\right] = \lim_i \mathbb{E}\left[\boldsymbol{f}_{n_i}(\boldsymbol{o}) 1_{\{\boldsymbol{o} \in \boldsymbol{L}\}}\right] = \lim_i \boldsymbol{c}(\Pi'_{n_i}).$$

Therefore, $\max_f |\boldsymbol{f}| \geq \inf_\Pi \boldsymbol{c}(\Pi)$. Note that the maximum of $|\boldsymbol{f}|$ is attained by the same tightness argument as above. So Lemma 4.12 implies that equality holds and the claim is proved. □

4.3 A Unimodular Frostman Lemma for Point Processes

In the Euclidean case, another form of the unimodular Frostman lemma is given below. Its proof is based on the max-flow min-cut theorem of Subsection 4.2. As will be seen, the claim implies that in this case, Conjecture 4.4 holds up to a constant factor (Corollary 4.16). However, the weight function obtained in the theorem needs extra randomness.

Theorem 4.15. *Let Φ be a point-stationary point process in $\mathbb{R}^k$ endowed with the Euclidean metric, and let $\alpha \geq 0$. Then, there exists an equivariant weight function $\boldsymbol{w}$ on Φ such that, almost surely,*

$$\forall v \in \Phi, \ \forall r \geq 1 : \boldsymbol{w}(N_r(v)) \leq r^\alpha \tag{4.7}$$

and

$$\mathbb{E}\left[\boldsymbol{w}(0)\right] \geq 3^{-k} \mathcal{H}_1^\alpha(\Phi). \tag{4.8}$$

In particular, if $\mathcal{H}_1^\alpha(\Phi) > 0$, then $\boldsymbol{w}(0)$ is not identical to zero.

In the following proof, Φ is regarded as a counting measures; i.e., for all $A \subseteq \mathbb{R}^d$, $\Phi(A) := \#(\Phi \cap A)$.

Proof. Let $b > 1$ be an arbitrary integer (e.g., $b = 2$). For every integer $n \geq 0$, let $\boldsymbol{Q}_n$ be the stationary partition of $\mathbb{R}^k$ by translations of the cube $[0, b^n)^k$ as in Subsection 2.3. Consider the nested coupling of these partitions for $n \geq 0$ (i.e., every cube of $\boldsymbol{Q}_n$ is contained in some cube of $\boldsymbol{Q}_{n+1}$ for every $n \geq 0$) independent of Φ. Let $\boldsymbol{T}_0$ be the tree whose vertices are the cubes in $\cup_n \boldsymbol{Q}_n$ and the edges are between all pairs of nested cubes in $\boldsymbol{Q}_n$ and $\boldsymbol{Q}_{n+1}$ for all n. Let $\boldsymbol{T} \subseteq \boldsymbol{T}_0$ be the subtree consisting of the cubes $\boldsymbol{q}_n(v)$ for all $v \in \Phi$ and $n \geq 0$. The set $\boldsymbol{L}$ of the leaves of $\boldsymbol{T}$ consists of the cubes $\boldsymbol{q}_0(v)$ for all $v \in \Phi$. Let $\boldsymbol{\sigma} := \boldsymbol{q}_0(0) \in \boldsymbol{L}$. Note that in the correspondence $v \mapsto \boldsymbol{q}_0(v)$, each cube $\sigma \in \boldsymbol{L}$ corresponds to $\Phi(\sigma) \geq 1$ points of Φ. Therefore, by verifying the mass transport principle, it can be seen that the distribution of $[\boldsymbol{L}, \boldsymbol{\sigma}]$, biased by $1/\Phi(\boldsymbol{\sigma})$, is unimodular; i.e.,

$$\mathbb{E}\left[\frac{1}{\Phi(\boldsymbol{\sigma})} \sum_{\sigma' \in \boldsymbol{L}} g(\boldsymbol{L}, \boldsymbol{\sigma}, \sigma')\right] = \mathbb{E}\left[\frac{1}{\Phi(\boldsymbol{\sigma})} \sum_{\sigma' \in \boldsymbol{L}} g(\boldsymbol{L}, \sigma', \boldsymbol{\sigma})\right],$$

for every measurable $g \geq 0$. In addition, g can be allowed to depend on $\boldsymbol{T}$ in this equation (but the sum is still on $\sigma' \in \boldsymbol{L}$). Therefore, one can assume the metric on $\boldsymbol{L}$ is the graph-distance metric induced from $\boldsymbol{T}$ (see Theorem I.3.32). Moreover, Theorem 5 of [16] implies that by a further biasing and choosing a new root for $\boldsymbol{T}$, one can make $\boldsymbol{T}$ unimodular. More precisely, the following (possibly infinite) measure on $\mathcal{D}_*$ is unimodular:

$$\mathcal{P}[A] := \mathbb{E}\left[\sum_{n \geq 0} \frac{1}{e_n} 1_A[\boldsymbol{T}, \boldsymbol{q}_n(0)]\right], \tag{4.9}$$

where $e_n := \Phi(\boldsymbol{q}_n(0))$. Let $\mathcal{E}$ denote the integral operator w.r.t. the measure $\mathcal{P}$. For any equivariant flow $\boldsymbol{f}$ on $\boldsymbol{T}$, the norm of $\boldsymbol{f}$ w.r.t. the measure $\mathcal{P}$ (see Remark 4.14) satisfies

$$|\boldsymbol{f}| = \mathcal{E}[\boldsymbol{f} \cdot \mathbf{1}_{\boldsymbol{L}}] = \mathbb{E}\left[\sum_{n \geq 0} \frac{1}{e_n} \boldsymbol{f}(\boldsymbol{q}_n(0)) \mathbf{1}_{\{\boldsymbol{q}_n(0) \in \boldsymbol{L}\}}\right] = \mathbb{E}\left[\frac{1}{\Phi(\boldsymbol{\sigma})} \boldsymbol{f}(\boldsymbol{\sigma})\right],$$

where the second equality is by (4.9). Consider the conductance function $\boldsymbol{c}(\tau) := b^{n\alpha}$ for all cubes τ of edge length b^n in $\boldsymbol{T}$ and all n. Therefore, Theorem 4.13 and Remark 4.14 imply that the maximum of $\mathbb{E}\left[\boldsymbol{f}(\boldsymbol{\sigma})\right]$ over all equivariant legal flows $\boldsymbol{f}$ on $[\boldsymbol{T}, \boldsymbol{\sigma}]$ is attained (note that $[\boldsymbol{T}, \boldsymbol{\sigma}]$ is not unimodular, but the theorem can be used for $\mathcal{P}$). Denote by $\boldsymbol{f}_0$ the maximum flow. Let $\boldsymbol{w}$ be the weight function on Φ defined by $\boldsymbol{w}(v) = \delta \boldsymbol{f}_0(\boldsymbol{q}_0(v))/\Phi(\boldsymbol{q}_0(v))$, for all $v \in \Phi$, where $\delta := (b+1)^{-k}$. The claim is that $\boldsymbol{w}$ satisfies the requirements (4.7) and (4.8). Since $\boldsymbol{f}_0$ is a legal flow, it follows that for every cube $\sigma \in \boldsymbol{T}$, one has

$$\boldsymbol{w}(\sigma) = \delta \boldsymbol{f}_0(\sigma) \leq \delta \boldsymbol{c}(\sigma) = \delta b^{n\alpha}.$$

Each cube σ of edge length $r \in [b^n, b^{n+1})$ in $\mathbb{R}^k$ can be covered with at most $(b+1)^k$ cubes of edge length b^n in $\boldsymbol{T}_0$. If $n \geq 0$, the latter are either in $\boldsymbol{T}$ or do not intersect Φ. So the above inequality implies that $\boldsymbol{w}(\sigma) \leq r^\alpha$. So (4.7) is proved for $\boldsymbol{w}$.

To prove (4.8), given any equivariant cut-set Π of $\boldsymbol{T}$, a covering of Φ can be constructed as follows: For each cube $\sigma \in \Pi$ of edge length say n, let $\boldsymbol{\tau}(\sigma)$ be one of the points in $\sigma \cap \Phi$ chosen uniformly at random and put a ball of radius b^n centered at $\boldsymbol{\tau}(\sigma)$. Note that this ball contains σ. Do this independently for all cubes in $\boldsymbol{T}$. If a point in Φ is chosen more than once, consider only the largest radius assigned to it. It can be seen that this gives an equivariant covering of Φ, namely $\boldsymbol{R}$. One has

$$
\begin{aligned}
\mathbb{E}\left[\boldsymbol{R}(0)^\alpha\right] &\leq \mathbb{E}\left[\sum_{n \geq 0} b^{n\alpha} \mathbf{1}_{\{\boldsymbol{q}_n(0) \in \Pi\}} \mathbf{1}_{\{0 = \boldsymbol{\tau}(\boldsymbol{q}_n(0))\}}\right] \\
&= \mathbb{E}\left[\sum_{n \geq 0} \frac{b^{n\alpha}}{e_n} \mathbf{1}_{\{\boldsymbol{q}_n(0) \in \Pi\}}\right].
\end{aligned}
$$

On the other hand, by (4.9), one can see that

$$\boldsymbol{c}(\Pi) = \mathbb{E}\left[\sum_{n \geq 0} \frac{1}{e_n} \boldsymbol{c}(\boldsymbol{q}_n(0)) \mathbf{1}_{\{\boldsymbol{q}_n(0) \in \Pi\}}\right] = \mathbb{E}\left[\sum_{n \geq 0} \frac{b^{n\alpha}}{e_n} \mathbf{1}_{\{\boldsymbol{q}_n(0) \in \Pi\}}\right].$$

Therefore, $\mathbb{E}\left[\boldsymbol{R}(0)^\alpha\right] \leq \boldsymbol{c}(\Pi)$. So $\mathcal{H}_1^\alpha(\Phi) \leq \boldsymbol{c}(\Pi)$. Since Π is an arbitrary equivariant cut-set, by the unimodular max-flow min-cut theorem established above (Theorem 4.13) and the maximality of the flow $\boldsymbol{f}_0$, one gets that $\mathcal{H}_1^\alpha(\Phi) \leq |\boldsymbol{f}_0| = \mathbb{E}\left[\boldsymbol{f}_0(\boldsymbol{\sigma})/\Phi(\boldsymbol{\sigma})\right] = \delta^{-1} \mathbb{E}\left[\boldsymbol{w}(0)\right]$. So the claim is proved. □

30

The following corollary shows that in the setting of Theorem 4.15, the claim of Conjecture 4.4 holds up to a constant factor (compare this with Lemma 4.7).

Corollary 4.16. *For all point-stationary point processes Φ in $\mathbb{R}^k$ endowed with the Euclidean metric and all $\alpha \geq 0$, $3^{-k}\mathcal{H}_1^\alpha(\Phi) \leq \xi_1^\alpha(\Phi) \leq \mathcal{H}_1^\alpha(\Phi)$.*

Proof. The claim is directly implied by (4.2), Theorem 4.15 and Remark 4.3. □

4.4 Applications

The following are some basic applications of the unimodular Frostman lemma. This lemma is also the basis of many results of [4].

4.4.1 Cayley Graphs

Proposition 4.17. *For every finitely generated group H with polynomial growth degree $\alpha \in [0, \infty]$, one has $\xi_\infty^\alpha(H) < \infty$.*

Note that if Conjecture 4.4 holds, then this result implies $\mathcal{M}^\alpha(H) > 0$, as conjectured in Subsection 3.6.

Proof. By Theorem 3.18, $\mathcal{M}^\alpha(H) < \infty$. So the unimodular Frostman lemma (Theorem 4.2) implies that for every $M \geq 1$, there exists $w : \mathcal{D}_* \to \mathbb{R}^{\geq 0}$ such that $w(N_M(e)) \leq M^\alpha$ and $\mathbb{E}\left[w(e)\right] = \xi_M^\alpha(H)$, where e is the neutral element of H. Since the Cayley graph of H is transitive and w is defined up to rooted isomorphisms, $w(H, \cdot)$ is constant. Hence, $w(H, v) = \xi_M^\alpha(H)$ for all $v \in H$. Therefore, $\xi_M^\alpha(H)\#N_M(e) \leq M^\alpha$. Thus, $\xi_M^\alpha(H) \leq 1/c$, where c is as in the proof of Theorem 3.18. By letting $M \to \infty$, one gets $\xi_\infty^\alpha(H) \leq 1/c < \infty$. □

4.4.2 Dimension of Product Spaces

Let $[\boldsymbol{D}_1, \boldsymbol{o}_1]$ and $[\boldsymbol{D}_2, \boldsymbol{o}_2]$ be independent unimodular discrete metric spaces. By considering any of the usual product metrics; e.g., the sup metric or the p product metric, the **independent product** $[\boldsymbol{D}_1 \times \boldsymbol{D}_2, (\boldsymbol{o}_1, \boldsymbol{o}_2)]$ makes sense as a random pointed discrete space. It is not hard to see that the latter is also unimodular (see also Proposition 4.11 of [1]).

Proposition 4.18. *Let $[\boldsymbol{D}_1 \times \boldsymbol{D}_2, (\boldsymbol{o}_1, \boldsymbol{o}_2)]$ represent the independent product of $[\boldsymbol{D}_1, \boldsymbol{o}_1]$ and $[\boldsymbol{D}_2, \boldsymbol{o}_2]$ defined above. Then,*

$$\mathrm{udim}_H(\boldsymbol{D}_1) + \underline{\mathrm{udim}}_M(\boldsymbol{D}_2) \leq \mathrm{udim}_H(\boldsymbol{D}_1 \times \boldsymbol{D}_2) \leq \mathrm{udim}_H(\boldsymbol{D}_1) + \mathrm{udim}_H(\boldsymbol{D}_2).$$
$$(4.10)$$

Proof. By Theorem I.3.32, one can assume the metric on $\boldsymbol{D}_1 \times \boldsymbol{D}_2$ is the sup metric without loss of generality. So $N_r(v_1, v_2) = N_r(v_1) \times N_r(v_2)$.

The upper bound is proved first. For $i = 1, 2$, let $\alpha_i > \mathrm{udim}_H(\boldsymbol{D}_i)$ be arbitrary. By the unimodular Frostman lemma (Theorem 4.2), there is a nonnegative measurable functions w_i on $\mathcal{D}_*$ such that $\forall v \in \boldsymbol{D}_i : \forall r \geq 1 : w_i(N_r(v)) \leq$

r^α, a.s. In addition, $w_i(\boldsymbol{o}_i) \neq 0$ with positive probability. Consider the equivariant weight function $\boldsymbol{w}$ on $\boldsymbol{D}_1 \times \boldsymbol{D}_2$ defined by $\boldsymbol{w}(v_1, v_2) := w_1[\boldsymbol{D}_1, v_1] \times w_2[\boldsymbol{D}_2, v_2]$. It is left to the reader to show that $\boldsymbol{w}$ is an equivariant weight function. One has $\boldsymbol{w}(N_r(v_1, v_2)) = w_1(N_r(v_1))w_2(N_r(v_2)) \leq r^{\alpha_1 + \alpha_2}$. Also, by the independence assumption, $\boldsymbol{w}(\boldsymbol{o}_1, \boldsymbol{o}_2) \neq 0$ with positive probability. Therefore, the mass distribution principle (Theorem 2.2) implies that $\mathrm{udim}_H(\boldsymbol{D}_1 \times \boldsymbol{D}_2) \leq \alpha_1 + \alpha_2$. This proves the upper bound.

For the lower bound in the claim, let $\alpha < \mathrm{udim}_H(\boldsymbol{D}_1)$, $\beta < \underline{\mathrm{udim}}_M(\boldsymbol{D}_2)$ and $\epsilon > 0$ be arbitrary. It is enough to find an equivariant covering $\boldsymbol{R}$ of $\boldsymbol{D}_1 \times \boldsymbol{D}_2$ such that $\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o}_1, \boldsymbol{o}_2)^{\alpha+\beta}\right] < \epsilon$. One has $\underline{\mathrm{decay}}(\lambda_r(\boldsymbol{D}_2)) > \beta$, where λ_r is defined in (I.3.1). So there is $M > 0$ such that $\forall r \geq M : \lambda_r(\boldsymbol{D}_2) < r^{-\beta}$. So for every $r \geq M$, there is an equivariant r-covering of $\boldsymbol{D}_2$ with intensity less than $r^{-\beta}$. On the other hand, since $\alpha < \mathrm{udim}_H(\boldsymbol{D}_1)$, one has $\mathcal{H}_M^\alpha(\boldsymbol{D}_1) = 0$ (by Lemma I.3.26). Therefore there is an equivariant covering $\boldsymbol{R}_1$ of $\boldsymbol{D}_1$ such that $\mathbb{E}\left[\boldsymbol{R}_1(\boldsymbol{o}_1)^\beta\right] < \epsilon$ and $\forall v \in \boldsymbol{D}_1 : \boldsymbol{R}_1(v) \in \{0\} \cup [M, \infty)$ a.s. Choose the extra randomness in $\boldsymbol{R}_1$ independently from $[\boldsymbol{D}_2, \boldsymbol{o}_2]$. Given a realization of $[\boldsymbol{D}_1, \boldsymbol{o}_1]$ and $\boldsymbol{R}_1$, do the following: Let $v_1 \in \boldsymbol{D}_1$ such that $\boldsymbol{R}_1(v_1) \neq 0$ (and hence, $\boldsymbol{R}_1(v_1) \geq M$). One can find an equivariant subset $\boldsymbol{S}_{v_1}$ of $\boldsymbol{D}_2$ that gives a covering of $\boldsymbol{D}_2$ by balls of radius $\boldsymbol{R}_1(v_1)$ and has intensity less than $\boldsymbol{R}_1(v_1)^{-\beta}$. Do this independently for all $v_1 \in \boldsymbol{D}_1$. Now, for all $(v_1, v_2) \in \boldsymbol{D}_1 \times \boldsymbol{D}_2$, define

$$\boldsymbol{R}(v_1, v_2) := \begin{cases} \boldsymbol{R}_1(v_1) & \text{if } \boldsymbol{R}_1(v_1) \neq 0 \text{ and } v_2 \in \boldsymbol{S}_{v_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\boldsymbol{R}$ is a covering of $\boldsymbol{D}_1 \times \boldsymbol{D}_2$ and it can be seen that it is an equivariant covering. Also, given $[\boldsymbol{D}_1, \boldsymbol{o}_1]$ and $\boldsymbol{R}_1$, the probability that $\boldsymbol{o}_2 \in \boldsymbol{S}_{\boldsymbol{o}_1}$ is less than $\boldsymbol{R}_1(\boldsymbol{o}_1)^{-\beta}$. So one gets

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o}_1, \boldsymbol{o}_2)^{\alpha+\beta}\right] &= \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{R}(\boldsymbol{o}_1, \boldsymbol{o}_2)^{\alpha+\beta} | [\boldsymbol{D}_1, \boldsymbol{o}_1], \boldsymbol{R}_1\right]\right] \\ &< \mathbb{E}\left[\boldsymbol{R}_1(\boldsymbol{o}_1)^{\alpha+\beta}\boldsymbol{R}_1(\boldsymbol{o}_1)^{-\beta}\right] = \mathbb{E}\left[\boldsymbol{R}_1(\boldsymbol{o}_1)^\alpha\right] < \epsilon. \end{aligned}$$

So the claim is proved. $\qquad\square$

The following examples provide instances where the inequalities in (4.10) are strict.

Example 4.19. Assume $[\boldsymbol{D}_1, \boldsymbol{o}_1]$ and $[\boldsymbol{D}_2, \boldsymbol{o}_2]$ are unimodular discrete spaces such that $\underline{\mathrm{udim}}_M(\boldsymbol{G}_1) < \mathrm{udim}_H(\boldsymbol{G}_1)$ and $\underline{\mathrm{udim}}_M(\boldsymbol{G}_2) = \mathrm{udim}_H(\boldsymbol{G}_2)$. By Proposition 4.18, one gets

$$\mathrm{udim}_H(\boldsymbol{G}_1 \times \boldsymbol{G}_2) \geq \mathrm{udim}_H(\boldsymbol{G}_1) + \underline{\mathrm{udim}}_M(\boldsymbol{G}_2) > \mathrm{udim}_H(\boldsymbol{G}_2) + \underline{\mathrm{udim}}_M(\boldsymbol{G}_1).$$

So by swapping the roles of the two spaces, an example of strict inequality in the left hand side of (4.10) is obtained.

Example 4.20. Let J be a subset of $\mathbb{Z}^{\geq 0}$ such that $\overline{d}(J) = 1$ and $\underline{d}(J) = 0$ simultaneously (see Subsection 3.5.1 for the definitions). Let Ψ_1 and Ψ_2 be defined as in Subsection 3.5.1 corresponding to J and $\mathbb{Z}^{\geq 0} \setminus J$ respectively. Proposition 3.15 implies that $\mathrm{udim}_H(\Psi_1) = \mathrm{udim}_H(\Psi_2) = 1$. On the other hand, (3.2) implies that

$$\#N_{2^n}(\boldsymbol{o}_1 \times \boldsymbol{o}_2) \leq 2^{J_n+1} \times 2^{(n+1-J_n)+1} = 2^{n+3}.$$

This implies that $\overline{\mathrm{growth}}(N_r(\boldsymbol{o})) \leq 1$. So the unimodular Billingsley lemma (Theorem 2.6) implies that $\mathrm{udim}_H(\Psi_1 \times \Psi_2) \leq 1$ (in fact, equality holds by Proposition 4.23 below). So the rightmost inequality in (4.10) is strict here.

4.4.3 Dimension of Embedded Spaces

It is natural to think of $\mathbb{Z}$ as a subset of $\mathbb{Z}^2$. However, $[\mathbb{Z}, 0]$ is not an equivariant subspace of $[\mathbb{Z}^2, 0]$ as defined in Definition I.2.13. By the following definition, $[\mathbb{Z}, 0]$ is called *embeddable in* $[\mathbb{Z}^2, 0]$. The dimension of embedded subspaces is studied in this subsection, which happens to be non-trivial and requires the unimodular Frostman lemma.

Definition 4.21. Let $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ and $[\boldsymbol{D}, \boldsymbol{o}]$ be random pointed discrete spaces. An **embedding** of $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ in $[\boldsymbol{D}, \boldsymbol{o}]$ is a (not necessarily unimodular) random pointed marked discrete space $[\boldsymbol{D}', \boldsymbol{o}'; \boldsymbol{m}]$ with mark space $\{0, 1\}$ such that

(i) $[\boldsymbol{D}', \boldsymbol{o}']$ has the same distribution as $[\boldsymbol{D}, \boldsymbol{o}]$.

(ii) $\boldsymbol{m}(\boldsymbol{o}') = 1$ a.s. and by letting $\boldsymbol{S} := \{v \in \boldsymbol{D}' : \boldsymbol{m}(v) = 1\}$ equipped with the induced metric from $\boldsymbol{D}'$, $[\boldsymbol{S}, \boldsymbol{o}']$ has the same distribution as $[\boldsymbol{D}_0, \boldsymbol{o}_0]$.

If in addition, $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ is unimodular, then $[\boldsymbol{D}', \boldsymbol{o}'; \boldsymbol{m}]$ is called an **equivariant embedding** if

(iii) The mass transport principle holds on $\boldsymbol{S}$; i.e., (I.2.2) holds for functions $g(u, v) := g(\boldsymbol{D}', u, v; \boldsymbol{m})$ such that $g(u, v)$ is zero when $\boldsymbol{m}(u) = 0$ or $\boldsymbol{m}(v) = 0$.

If an embedding (resp. an equivariant embedding) exists as above, $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ is called **embeddable** (resp. **equivariantly embeddable**) in $[\boldsymbol{D}, \boldsymbol{o}]$.

It should be noted that $[\boldsymbol{D}', \boldsymbol{o}'; \boldsymbol{m}]$ is not an equivariant process on $\boldsymbol{D}$ except in the trivial case where $\boldsymbol{m}(\cdot) = 1$ a.s.

Example 4.22. The following are instances of Definition 4.21.

(i) Let $[\boldsymbol{D}_0, \boldsymbol{o}_0] := [\mathbb{Z}, 0]$ and $[\boldsymbol{D}, \boldsymbol{o}] := [\mathbb{Z}^2, 0]$ equipped with the sup metric. Consider $m : \mathbb{Z}^2 \to \{0, 1\}$ which is equal to one on the boundary of the positive cone. Then, $[\mathbb{Z}^2, 0; m]$ is an embedding of $[\mathbb{Z}, 0]$ in $[\mathbb{Z}^2, 0]$, but is not an equivariant embedding since it does not satisfy (iii).

(ii) A point-stationary point process in $\mathbb{Z}^k$ (pointed at 0) is equivariantly embeddable in $[\mathbb{Z}^k, 0]$.

(iii) Let H be a finitely generated group equipped with the graph-distance metric of an arbitrary Cayley graph over H. Then, any subgroup of H (equipped with the induced metric) is equivariantly embeddable in H.

Proposition 4.23. *If $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ and $[\boldsymbol{D}, \boldsymbol{o}]$ are unimodular discrete spaces and the former is equivariantly embeddable in the latter, then*

$$\text{udim}_H(\boldsymbol{D}) \geq \text{udim}_H(\boldsymbol{D}_0), \tag{4.11}$$
$$\xi_M^\alpha(\boldsymbol{D}, 1) \leq \mathcal{H}_M^\alpha(\boldsymbol{D}_0), \tag{4.12}$$

for all $\alpha \geq 0$ and $M \geq 1$, where ξ_M^α is defined in Definition 4.1.

Proof. First, assume (4.12) holds. For $\alpha > \text{udim}_H(\boldsymbol{D})$, one has $\mathcal{H}_M^\alpha(\boldsymbol{D}) > 0$ (Lemma I.3.26). Therefore, Lemma 4.7 implies that $\xi_M^\alpha(\boldsymbol{D}, 1) > 0$. Hence, (4.12) implies that $\mathcal{H}_M^\alpha(\boldsymbol{D}_0) > 0$, which implies that $\text{udim}_H(\boldsymbol{D}_0) \leq \alpha$. So it is enough to prove (4.12).

By the unimodular Frostman lemma (Theorem 4.2), there is a bounded function $w : \mathcal{D}_* \to \mathbb{R}^{\geq 0}$ such that $\mathbb{E}[w(\boldsymbol{o})] = \xi_M^\alpha(\boldsymbol{D}, 1)$, and almost surely, $w(N_r(\boldsymbol{o})) \leq r^\alpha$ for all $r \geq M$. Assume $[\boldsymbol{D}', \boldsymbol{o}'; \boldsymbol{m}]$ is an equivariant embedding as in Definition 4.21. For $x \in \boldsymbol{D}'$, let $w'(x) := w'_{\boldsymbol{D}'}(x) := w[\boldsymbol{D}', x]$. Consider the random pointed marked discrete space $[\boldsymbol{S}, \boldsymbol{o}'; w']$ obtained by restricting w' to $\boldsymbol{S}$. By the definition of equivariant embeddings and by directly verifying the mass transport principle, the reader can obtain that $[\boldsymbol{S}, \boldsymbol{o}'; w']$ is unimodular. Since $[\boldsymbol{S}, \boldsymbol{o}']$ has the same distribution as $[\boldsymbol{D}_0, \boldsymbol{o}_0]$, there exists an equivariant process $\boldsymbol{w}_0$ on $\boldsymbol{D}_0$ such that $[\boldsymbol{S}, \boldsymbol{o}'; w']$ has the same distribution as $[\boldsymbol{D}_0, \boldsymbol{o}_0; \boldsymbol{w}_0]$ (see Proposition B.1 in the arXiv version of Part I, which is a converse to Lemma I.2.11). According to the above discussion, one has

$$\forall r \geq M : w'(N_r(\boldsymbol{S}, \boldsymbol{o}')) \leq w'(N_r(\boldsymbol{D}', \boldsymbol{o}')) \leq r^\alpha, \quad a.s.$$

This implies that $\boldsymbol{w}_0(N_r(\boldsymbol{o}_0)) \leq r^\alpha$ a.s. Therefore, the mass distribution principle (Theorem 2.2) implies that $\mathbb{E}[\boldsymbol{w}_0(\boldsymbol{o}_0)] \leq \mathcal{H}_M^\alpha(\boldsymbol{D}_0)$. One the other hand,

$$\mathbb{E}[\boldsymbol{w}_0(\boldsymbol{o}_0)] = \mathbb{E}[w'(\boldsymbol{o}')] = \mathbb{E}[w(\boldsymbol{o})] = \xi_M^\alpha(\boldsymbol{D}, 1),$$

where the last equality is by the assumption on w. This implies that $\mathcal{H}_M^\alpha(\boldsymbol{D}_0) \geq \xi_M^\alpha(\boldsymbol{D}, 1)$ and the claim is proved. $\square$

It is natural to expect that an embedded space has a smaller Hausdorff measure. This is stated in the following conjecture.

Conjecture 4.24. *Under the setting of Proposition 4.23, for all $\alpha > 0$, one has $\mathcal{M}^\alpha(\boldsymbol{D}) \geq \mathcal{M}^\alpha(\boldsymbol{D}_0)$.*

Note that in the case $\alpha = 0$, the conjecture is implied by Proposition I.3.29. Also, in the general case, the conjecture is implied by (4.12) and Conjecture 4.4.

Another problem is the validity of Proposition 4.23 under the weaker assumption of being non-equivariantly embeddable. As a partial answer, if growth $(\#N_r(\boldsymbol{o}))$

exists, then (4.11) holds. This is proved as follows:

$$\text{udim}_H(\boldsymbol{D}_0) \leq \text{ess inf } \overline{\text{growth}}\,(\#N_r(\boldsymbol{o}_0)) \leq \text{ess inf } \overline{\text{growth}}\,(\#N_r(\boldsymbol{o}))$$
$$= \text{ess inf growth}\,(\#N_r(\boldsymbol{o})) = \text{udim}_H(\boldsymbol{D}),$$

where the first inequality and the last equality are implied by the unimodular Billingsley lemma (Theorem 2.6).

Remark 4.25. Another possible way to prove Proposition 4.23 and Conjecture 4.24 is to consider an arbitrary equivariant covering of $\boldsymbol{D}_0$ and try to extend it to an equivariant covering of $\boldsymbol{D}$ by adding some balls (without adding a ball centered at the root). More generally, given an equivariant processes $\boldsymbol{Z}_0$ on $\boldsymbol{D}_0$, one might try to extend it to an equivariant process on $\boldsymbol{D}$ without changing the mark of the root. But at least the latter is not always possible. A counter example is when $[\boldsymbol{D}_0, \boldsymbol{o}_0]$ is K_2 (the complete graph with two vertices), $[\boldsymbol{D}, \boldsymbol{o}]$ is K_3, $\boldsymbol{Z}_0(\boldsymbol{o}_0) = \pm 1$ chosen uniformly at random, and the mark of the other vertex of $\boldsymbol{D}_0$ is $-\boldsymbol{Z}_0(\boldsymbol{o}_0)$.

4.5 Notes and Bibliographical Comments

The unimodular Frostman lemma (Theorem 4.2) is analogous to Frostman's lemma in the continuum setting (see e.g., Thm 8.17 of [18]). The proof of Theorem 4.2 is also inspired by that of [18], but there are substantial differences. For instance, the proof of Lemma 4.7 and also the use of the duality of L_1 and L_∞ in the proof of Theorem 4.2 are new. The Euclidean version of the unimodular Frostman's lemma (Theorem 4.15) and its proof are inspired by the continuum analogue (see e.g., [8]).

As already explained, the unimodular max-flow min-cut theorem (Theorem 4.13) is inspired by the max-flow min-cut theorem for finite trees. Also, the results and examples of Subsection 4.4.2 on product spaces are inspired by analogous in the continuum setting; e.g., Theorem 3.2.1 of [8].

Acknowledgements

References

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12:no. 54, 1454–1508, 2007.

[2] D. Aldous and J. M. Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 1–72. Springer, Berlin, 2004.

[3] S. Asmussen and H. Hering. *Branching processes*, volume 3 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1983.

[4] F. Baccelli, M.-O. Haji-Mirsadeghi, and A. Khezeli. On the dimension of unimodular discrete spaces. *preprint,*`https://www.researchgate.net/publication/328137256_On_the_Dimension_of_Unimodular_Discrete_Spaces`.

[5] F. Baccelli, M.-O. Haji-Mirsadeghi, and A. Khezeli. Unimodular Hausdorff and Minkowski Dimensions. *preprint*.

[6] F. Baccelli, M. O. Haji-Mirsadeghi, and A. Khezeli. Eternal Family Trees and dynamics on unimodular random graphs. In *Unimodularity in randomly generated graphs*, volume 719 of *Contemp. Math.*, pages 85–127. Amer. Math. Soc., 2018.

[7] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc. (3)*, 25:603–614, 1972.

[8] C. J. Bishop and Y. Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017.

[9] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2003. Elementary theory and methods.

[10] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II*. Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[11] L. R. Ford, Jr. and D. R. Fulkerson. *Flows in networks*. Princeton University Press, Princeton, N.J., 1962.

[12] B. E. Fristedt and W. E. Pruitt. Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 18:167–182, 1971.

[13] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[14] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc. (2)*, 24(2):373–384, 1981.

[15] H. Kesten and B. P. Stigum. A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Statist.*, 37:1211–1223, 1966.

[16] A. Khezeli. Shift-coupling of random rooted graphs and networks. In *Unimodularity in randomly generated graphs*, volume 719 of *Contemp. Math.*, pages 175–211. Amer. Math. Soc., Providence, RI, 2018.

[17] R. Lyons and Y. Peres. *Probability on trees and networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016.

[18] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[19] K. Petersen. *Ergodic theory*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989. Corrected reprint of the 1983 original.

[20] W. Rudin. *Functional analysis*. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. McGraw-Hill Series in Higher Mathematics.

[21] S. J. Taylor and J. G. Wendel. The exact Hausdorff measure of the zero set of a stable process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 6:170–180, 1966.

[22] Á. Timár. A stationary random graph of no growth rate. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(4):1161–1164, 2014.

[23] L. van den Dries and A. J. Wilkie. Gromov's theorem on groups of polynomial growth and elementary logic. *J. Algebra*, 89(2):349–374, 1984.