Contact Processes on Point Processes

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Summary

A computational framework for evaluating the role of mobility on the propagation of epidemics on point processes

Joint ongoing work with N. Ramesan, Arxiv 2009.08515

— SIS on point processes with motion
— Structural results
— Moment measure equations
— Functional and polynomial heuristics → steady state densities
— Tentative phase diagram → survival and extinction
— First simulation validation steps

SIS Dynamics on a PPP with spatial migration

- **Individuals**

— **Form a spatial configuration Ξ_t of $\mathbb{R}^d$**

— **Have a state either I or S**

- **SIS evolution**

— **Transition from S to I in function of the local infection rate**

— **Transition from I to S with constant recovery rate**

- **Spatial evolution**

— **Migration with constant rate (keeping the SIS state)**

— **Independent (i.i.d.) random displacements on $\mathbb{R}^d$**

Basic Model - Ground Point Process

- **Initial location of individuals :**
  Ξ_0 Poisson point process of intensity λ on $\mathbb{R}^2$
- **Random waypoint motion**

— points jump from current location to another location with rate γ ;
— the displacements are random, i.i.d., independent, with symmetric distribution D on $\mathbb{R}^2$

- **This leads to a ground point process Ξ_t at time t**
- **Ξ_t is Poisson λ for all t thanks to the displacement theorem**

Basic Model

- **Interaction function f** : $\mathbb{R}^+ \to \mathbb{R}^+$

$$\mathbf{f(r)} = \alpha \mathbf{1_{r \leq a}}$$

- **The points of Ξ_t can be in one of the two states :**

— 1 (Infected) : Φ_t : point process of infected individuals
— 0 (Susceptible) : Ψ_t : point process of susceptible individuals

$$\Xi_t = \Phi_t + \Psi_t$$

- **Neither Φ_t nor Ψ_t are Poisson**

Basic Model (*continued*)

■ **SIS Dynamics**

— **Transition rate of the state of $X \in \Psi_t$ to state 1 :**

$$I_{\Phi_t}(X) = \sum_{Y \in \Phi_t} f(||X - Y||)$$

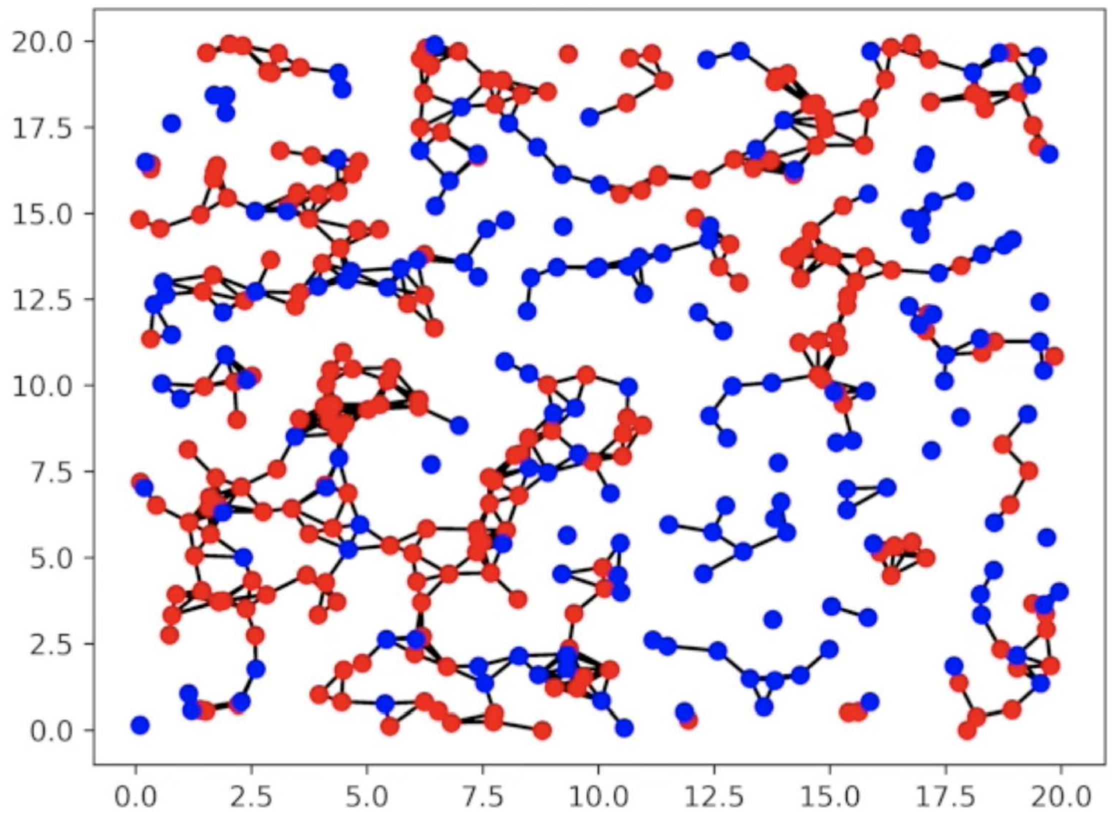
— **Transition rate of $X \in \Phi_t$ to state 0 : $\beta > 0$**

■ **Migration Dynamics Far Random Waypoint :**

— **Migration rate of each point : γ**

— **Very large i.i.d. symmetrical displacements on $\mathbb{R}^2$**

Illustration



Questions

■ **Parameters**

— α : **rate of infection**

— β : **rate of recovery**

— γ : **rate of motion**

— $\mu = \lambda \pi a^2$: **average degree of nodes**

■ **Questions**

— **Is this dynamic well defined on the whole of** $\mathbb{R}^2$ **?**

— **For what values of** $(\alpha, \beta, \gamma, \mu)$ **does the epidemic/infection survive ?**

— **While holding all else constant,**
how does varying γ **act on the epidemic's survival ?**

— **When the epidemic does survive,**
what fraction of the population is infected in the steady state ?

Mathematical Framework

- **Phase space**
  - **Steady state on compact phase spaces is degenerate : extinction is a.s. certain**
  - **Focus on dynamics on the whole of $\mathbb{R}^{d}$ which are space-time invariant**
- **Mathematical tools**
  - **Contact process mathematics developed in particle systems**
  - **Moment measure RCP methods similar to those developed for wireless and peer-to-peer**

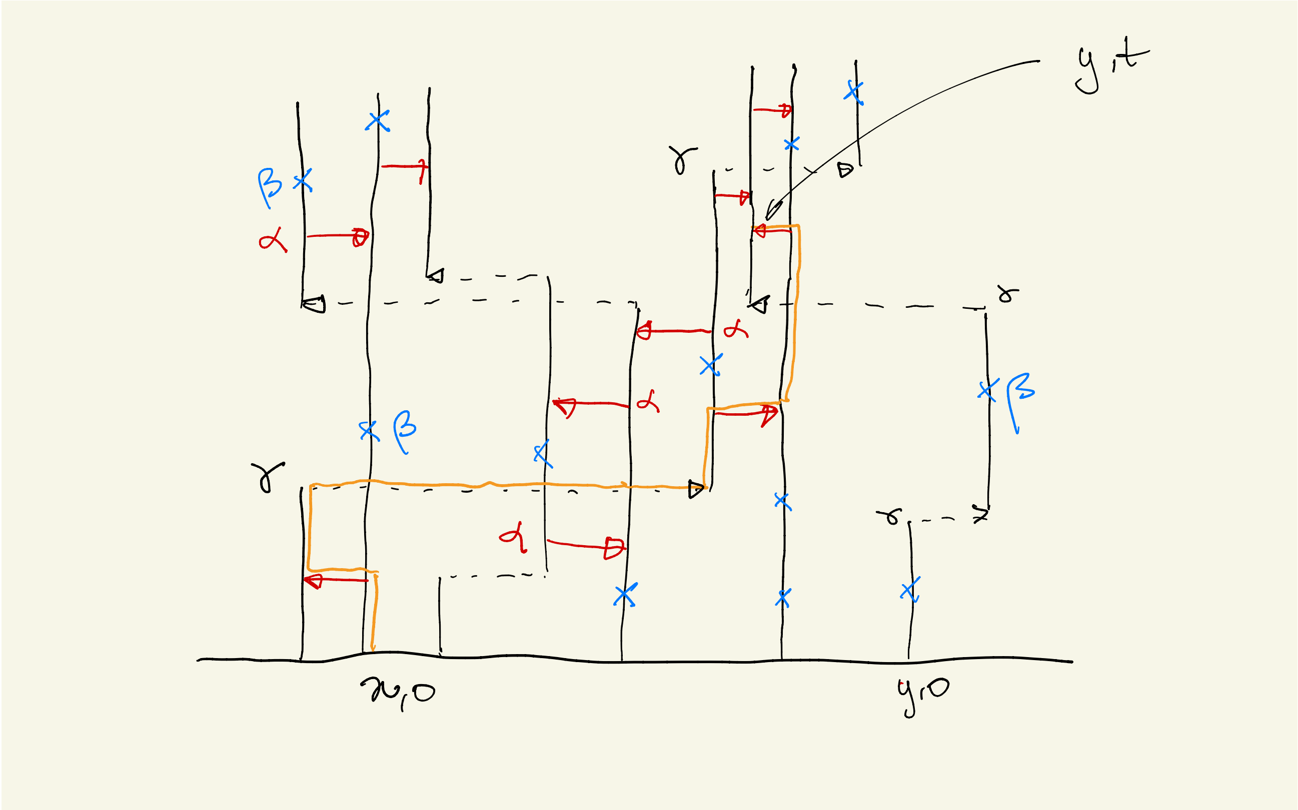
Properties of the Dynamics

- **The pair (Ψ_t, Φ_t) is Markov on the space of counting measures**
- **Construction on a compact of time**
- **Extension of basic properties of contact processes**

— **Monotonicity :** $\Phi_0 \subset \widetilde{\Phi}_0 \Rightarrow \Phi_t \subset \widetilde{\Phi}_t$ **for all** t
— **Additivity :** $\Phi_t^{A \cup B} = \Phi_t^A \cup \Phi_t^B$ **for all** A, B, t
— **Duality** $\mathbb{P}^A(\Phi_t \cap B \neq \emptyset) = \mathbb{P}^B(\Phi_t \cap A \neq \emptyset)$ **for all** A, B, t

- **Liggett**

Graphical Representation



Extinction or Survival ?

- **Extinction :** $\mathbb{P}^{\{x\}}[\Phi_t \neq \emptyset \ \forall t] = 0$
- **Survival : non-extinction**
- **These probabilities do not depend on** x

- **Theorem**

Fixing all parameters other than β**, there exists** $0 \leq \beta_c \leq \infty$ **:**

— **For** $\beta < \beta_c$ **: survival**
— **For** $\beta > \beta_c$ **: extinction**

Maximal Invariant Measure

- **There exists a maximal stationary prob. measure ν**

Take $\Phi_0 = \Xi$ and uses that Φ_t is stoch. decreasing

- **Relation between survival prob. and maximal invariant measure**

$$\mathbb{P}^{\{0\}}(\mathbf{\Phi_t} \neq \emptyset \ \forall \mathbf{t}) = \mathbf{p}$$

with

— p **the probability that the typical node is infected in the maximal invariant measure**

— $\mathbb{P}^{\{0\}}$ **the law of the dynamics starting with the point at the origin being the only one infected under Palm of the initial PPP**

First Moment RCP (RCP1)

- **Assume that there exists a time-space stationary regime**
- **Notation**

— **Φ has spatial intensity λp**
— **Ψ has intensity $\lambda(1-p)$**

- **Infection rate of locus $x \in \mathbb{R}^2$:**

$$\mathbf{I}_{\Phi}(\mathbf{x}) = \sum_{\mathbf{X} \in \widetilde{\Phi}} \mathbf{f}(||\mathbf{X} - \mathbf{x}||)$$

- **From Campbell's formula**

$$\mathbb{E}[\mathbf{I}_{\Phi}(\mathbf{x})] = \lambda \mathbf{p} \alpha \pi \mathbf{a}^2$$

First Moment RCP (RCP1) (*continued*)

- **Spatial infection rate**

$$\mathbf{i} = \mathbb{E}[\sum_{\mathbf{Y} \in \Psi \cap \mathbf{D}} \mathbf{I}_{\Phi}(\mathbf{Y})] = \lambda(\mathbf{1} - \mathbf{p})\mathbb{E}^{\mathbf{0}}_{\Psi}[\mathbf{I}_{\Phi}(\mathbf{0})]$$

with $\mathbf{D} \subset \mathbb{R}^2$ of volume 1 and $\mathbb{E}^{\mathbf{0}}_{\Psi}$ the Palm distribution of Ψ

- **Spatial recovery rate (proportional to first moment)**

$$\mathbf{r} = \mathbb{E}[\sum_{\mathbf{X} \in \Phi \cap \mathbf{D}} \beta] = \lambda \mathbf{p} \beta$$

- **Lemma [RCP : $\mathbf{i} = \mathbf{r}$]**

$$\mathbf{p}\beta = (\mathbf{1} - \mathbf{p})\mathbb{E}^{\mathbf{0}}_{\Psi}[\mathbf{I}_{\Phi}(\mathbf{0})]$$

Implications of RCP1

- **A natural conjecture is that there is repulsion between Φ and Ψ :**

$$\mathbb{E}^{\mathbf{0}}_{\boldsymbol{\Psi}}[\mathbf{I}_{\boldsymbol{\Phi}}(\mathbf{0})] \leq \mathbb{E}[\mathbf{I}_{\boldsymbol{\Phi}}(\mathbf{0})]$$

- **Let $\mu = \lambda\pi\mathbf{a^2}$**

- **Lemma Under the (Φ, Ψ) repulsion conjecture,**

— If $\alpha\mu \leq \beta$, then extinction

— If $\alpha\mu > \beta$, then one may have a non-degenerate stationary regime

Implications of RCP1 (*continued*)

- **In the last case, the fraction of infected nodes satisfies**

$$\mathbf{0 < p \leq 1 - \frac{\beta}{\alpha\mu}}$$

- **The last bound is conjectured to be reached in the high motion mean-field regime (simulation and discrete time evidence)**

Pair-Correlation Function Representation of RCP1

$$\mathbb{E}^{0}_{\Psi}[\mathbf{I}_{\Phi(0)}] = \lambda \mathbf{p} \int\limits_{\mathbb{R}^2} \mathbf{f}(\mathbf{x}) \xi_{\Phi,\Psi}(\mathbf{x}) \mathbf{dx}$$

with $\xi_{\Phi,\Psi}(\mathrm{x})$ the pair correlation function of processes Φ and Ψ

- **RCP 1 in integral form :**

$$\beta = (\mathbf{1} - \mathbf{p}) \lambda \int\limits_{\mathbb{R}^2} \xi_{\Psi,\Phi}(\mathbf{x}) \mathbf{f}(||\mathbf{x}||) \mathbf{dx}$$

- **Makes the relation between first and second moment explicit**

Second Moment RCP (RCP2)

- **Setting : any space time invariant measure**
- **Integral equations on unknown (isotropic) pair correlation functions :**

$$\xi_{\Phi,\Phi}(\mathbf{r}), \quad \xi_{\Psi,\Psi}(\mathbf{r}), \quad \xi_{\Phi,\Psi}(\mathbf{r})$$

- **Related by**

$$\mathbf{p}^2\xi_{\Phi,\Phi}(\mathbf{r}) + (\mathbf{1}-\mathbf{p})^2\xi_{\Psi,\Psi}(\mathbf{r}) + \mathbf{2p}(\mathbf{1}-\mathbf{p})\xi_{\Psi,\Phi}(\mathbf{r}) = \xi_{\Xi,\Xi}(\mathbf{r}) = \mathbf{1}$$

Second Moment RCP (RCP2) (*continued*)

Three point conditional densities :

— $\mu(\Phi)^{0,r}_{\Psi,\Phi}(x)$ conditional density of Φ at x given that Ψ has a points at $(0,0)$ and Φ a point at $(r,0)$

— $\mu(\Phi)^{0,r}_{\Psi,\Psi}(x)$ conditional density of Φ at x given that Ψ has a points at $(0,0)$ and Ψ a point at $(r,0)$

Theorem

$$p\xi_{\Phi,\Phi}(r)(\beta+\gamma) \;=\; p\gamma+(1-p)\xi_{\Psi,\Phi}(r)\left(f(r)+\int_{\mathbb{R}^2}\mu(\Phi)^{0,r}_{\Psi,\Phi}(x)f(||x||)dx\right)$$

$$p\xi_{\Psi,\Phi}(r)\beta+(1-p)\gamma \;=\; (1-p)\xi_{\Psi,\Psi}(r)\left(\gamma+\int_{\mathbb{R}^2}\mu(\Phi)^{0,r}_{\Psi,\Psi}(x)f(||x||)dx\right)$$

Second Moment RCP (RCP2) (*continued*)

RCP for (I, I) points separated by distance r:

$$\underbrace{\lambda \mathbf{p} \lambda \mathbf{p} \xi_{\Phi,\Phi}(\mathbf{r})\beta}_{\text{death by recovery}} + \underbrace{\lambda \mathbf{p} \lambda \mathbf{p} \xi_{\Phi,\Phi}(\mathbf{r})\gamma}_{\text{death by motion}}$$

$$= \underbrace{\lambda \mathbf{p} \lambda \mathbf{p} \gamma}_{\text{birth by motion}} + \underbrace{\lambda \mathbf{p} \lambda (\mathbf{1} - \mathbf{p}) \xi_{\Psi,\Phi}(\mathbf{r}) \left(\mathbf{f}(\mathbf{r}) + \int_{\mathbb{R}^2} \mu(\Phi)^{\mathbf{0},\mathbf{r}}_{\Psi,\Phi}(\mathbf{x}) \mathbf{f}(||\mathbf{x}||) \mathbf{dx} \right)}_{\text{birth by infection}}$$

RCP for (S, S) points separated by distance r:

$$\underbrace{\lambda \mathbf{p} \lambda (\mathbf{1} - \mathbf{p}) \xi_{\Psi,\Phi}(\mathbf{r})\beta}_{\text{birth by recovery}} + \underbrace{\lambda (\mathbf{1} - \mathbf{p}) \lambda (\mathbf{1} - \mathbf{p})\gamma}_{\text{birth by motion}}$$

$$= \underbrace{\lambda (\mathbf{1} - \mathbf{p}) \lambda (\mathbf{1} - \mathbf{p}) \xi_{\Psi,\Psi}(\mathbf{r}) \Big(\gamma}_{\text{death by motion}} + \underbrace{\int_{\mathbb{R}^2} \mu(\Phi)^{\mathbf{0},\mathbf{r}}_{\Psi,\Psi}(\mathbf{x}) \mathbf{f}(||\mathbf{x}||) \mathbf{dx} \Big)}_{\text{death by infection}}$$

RCP2 Heuristic Factorizations

Heuristic computational track

Closure of the integral relations on moment measure of order 1 and 2 by a factorization of moment measures of order 3 based on

— either Bayes' rule and conditional independence heuristic

— or Mean value heuristic

Mean Value Heuristics

- **The Geometric Mean heuristic of parameter** $0 \leq \eta \leq 1$

$$\mu(\Phi)^{\mathbf{0},\mathbf{r}}_{\Psi,\Psi}(\mathbf{x}) = \lambda \mathbf{p} \xi_{\Psi,\Phi}(||\mathbf{x}||)^{\eta} \xi_{\Psi,\Phi}(||\mathbf{x}-\mathbf{r}||)^{\mathbf{1}-\eta}$$

and

$$\mu(\Phi)^{\mathbf{0},\mathbf{r}}_{\Psi,\Phi}(\mathbf{x}) = \lambda \mathbf{p} \xi_{\Psi,\Phi}(||\mathbf{x}||)^{\eta} \xi_{\Phi,\Phi}(||\mathbf{x}-\mathbf{r}||)^{\mathbf{1}-\eta}$$

- **Example G1 :** $\eta = \frac{1}{2}$
- **The theorem and e.g. G1 lead to an integral equation satisfied by the 3 pair correlation functions**

Bayes and Conditional Independence Heuristics

- **Bayes' rule rewritten in three different ways**

$$\mu(\Phi)^{0,r}_{\Psi,\Psi}(x)\xi_{\Psi,\Psi}(r)\lambda^2(1-p)^2 =$$
$$\mu(\Psi,\Psi)^{x}_{\Phi}(0,r)\lambda p = \mu(\Psi,\Phi)^{r}_{\Psi}(0,x)\lambda(1-p) = \mu(\Psi,\Phi)^{0}_{\Psi}(r,x)\lambda(1-p)$$

- **Hence**

$$\left(\mu(\Phi)^{0,r}_{\Psi,\Psi}(x)\xi_{\Psi,\Psi}(r)\lambda^2(1-p)^2\right)^3 =$$
$$(\mu(\Psi,\Psi)^{x}_{\Phi}(0,r)\lambda p)(\mu(\Psi,\Phi)^{r}_{\Psi}(0,x)\lambda(1-p))\left(\mu(\Psi,\Phi)^{0}_{\Psi}(r,x)\lambda(1-p)\right)$$

- **Conditional independence heuristic e.g.**

$$(\mu(\Psi,\Psi)^{x}_{\Phi}(0,r)) = \xi_{\Psi,\Phi}(||x||)\lambda(1-p)\xi_{\Psi,\Phi}(||x-(r,0)||)\lambda(1-p)$$

Bayes Independent BI1 Example

- **Hence**

$$\left(\mu(\Phi)^{0,r}_{\Psi,\Psi}(x)\xi_{\Psi,\Psi}(r)\lambda^2(1-p)^2\right)^3 =$$
$$= (\xi_{\Psi,\Phi}(||x||)\lambda(1-p)\xi_{\Psi,\Phi}(||x-(r,0)||)\lambda(1-p)\lambda p)$$
$$(\xi_{\Psi,\Psi}(r)\lambda(1-p)\xi_{\Psi,\Phi}(||x-(r,0)||)\lambda p\lambda(1-p))$$
$$(\xi_{\Psi,\Psi}(r)\lambda(1-p)\xi_{\Psi,\Phi}(||x||)\lambda p\lambda(1-p))$$

- **Example Heuristic B1I :**

$$\mu(\Phi)^{0,r}_{\Psi,\Psi}(x) = \lambda p \frac{\xi_{\Psi,\Phi}(||x||)^{\frac{2}{3}}\xi_{\Psi,\Phi}(||x-(r,0)||)^{\frac{2}{3}}}{\xi_{\Psi,\Psi}(r)^{\frac{1}{3}}}$$

$$\mu(\Phi)^{0,r}_{\Psi,\Phi}(x) = \lambda p \frac{\xi_{\Psi,\Phi}(||x||)^{\frac{2}{3}}\xi_{\Phi,\Phi}(||x-(r,0)||)^{\frac{2}{3}}}{\xi_{\Psi,\Phi}(r)^{\frac{1}{3}}}$$

- **+ a collection of other heuristics : b1g1, m2bi, . . .**

RCP2 Integral Equations - B1I

$$(\beta+\gamma)\mathbf{p}\xi_{\Phi,\Phi}(\mathbf{r}) = \mathbf{p}\gamma + (\mathbf{1}-\mathbf{p})\xi_{\Psi,\Phi}(\mathbf{r})\mathbf{f}(\mathbf{r})$$

$$+\lambda(\mathbf{1}-\mathbf{p})\,\mathbf{p}\xi_{\Psi,\Phi}(\mathbf{r})^{\frac{2}{3}}\int_{\mathbb{R}^2}\xi_{\Psi,\Phi}(||\mathbf{x}||)^{\frac{2}{3}}\xi_{\Phi,\Phi}(||\mathbf{x}-(\mathbf{r},\mathbf{0})||)^{\frac{2}{3}}\mathbf{f}(||\mathbf{x}||)\mathbf{dx}$$

$$\beta\mathbf{p}\xi_{\Psi,\Phi}(\mathbf{r}) = (\mathbf{1}-\mathbf{p})\gamma\left(\xi_{\Psi,\Psi}(\mathbf{r})-\mathbf{1}\right)$$

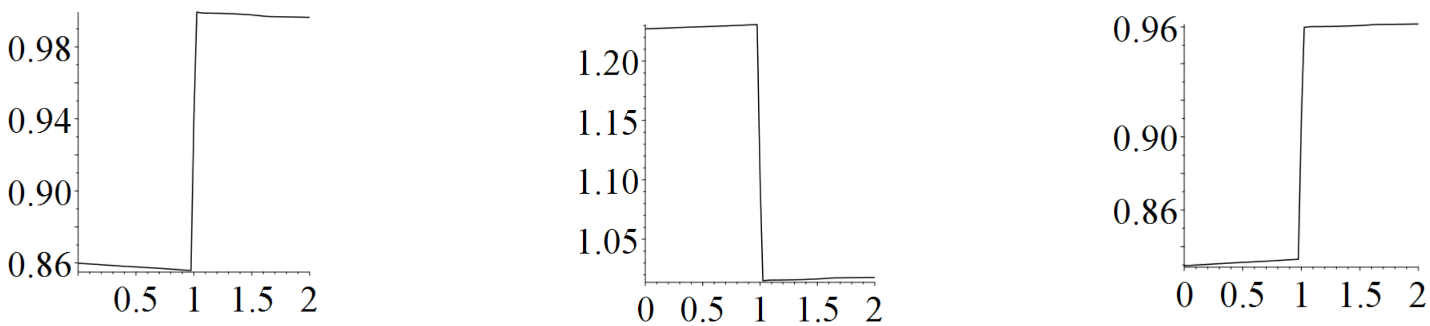
$$+\lambda(\mathbf{1}-\mathbf{p})\,\mathbf{p}\xi_{\Psi,\Psi}(\mathbf{r})^{\frac{2}{3}}\int_{\mathbb{R}^2}\xi_{\Psi,\Phi}(||\mathbf{x}||)^{\frac{2}{3}}\xi_{\Psi,\Phi}(||\mathbf{x}-(\mathbf{r},\mathbf{0})||)^{\frac{2}{3}}\mathbf{f}(||\mathbf{x}||)\mathbf{dx}$$

$$\mathbf{p} = \mathbf{1} - \frac{\beta}{\lambda\mathbf{2}\pi\int_{\mathbb{R}^+}\xi_{\Psi,\Phi}(\mathbf{r})\mathbf{f}(\mathbf{r})\mathbf{rdr}}$$

$$\xi_{\Psi,\Psi}(\mathbf{r}) = \frac{\mathbf{1}}{(\mathbf{1}-\mathbf{p})^{\mathbf{2}}}\left(\mathbf{1}-(\mathbf{p})^{\mathbf{2}}\xi_{\Phi,\Phi}(\mathbf{r}) - \mathbf{2p}(\mathbf{1}-\mathbf{p})\,\xi_{\Psi,\Phi}(\mathbf{r})\right)$$

RCP2 Integral Equations

■ Numerical Illustration



Left : $\xi_{\Psi,\Phi}(r)$ **; Center : the** $\xi_{\Phi,\Phi}(r)$ **; Right :** $\xi_{\Psi,\Psi}(r)$
$\beta = 1$, $a = 1$, $\lambda = 1$, $\gamma = 1$, **and** $\alpha = 1$

Polynomial Heuristic - B1I

- **Assume that**

— $\xi_{\Psi,\Phi}(.)$ **is almost constant on** $(0, a)$ **and equal to** $w < 1$
— $\xi_{\Phi,\Phi}(.)$ **is almost constant on** $(0, a)$ **and equal to** $v > 1$
— $\xi_{\Psi,\Psi}(.)$ **is almost constant on** $(0, a)$ **and equal to** z

- **B1I Polynomial System**

$$\begin{aligned}
(\gamma + \beta)pv &= \gamma p + \alpha(1 - p)w + \beta p v^{\frac{2}{3}} w^{\frac{1}{3}} \\
\beta p w &= (1 - p)\gamma(z - 1) + \beta p z^{\frac{2}{3}} w^{\frac{1}{3}} \\
\beta &= (1 - p)\alpha\mu w \\
1 &= (1 - p)^2 z + 2p(1 - p)w + p^2 v
\end{aligned}$$

Polynomial Heuristic - B1I - Numerical Example

- **Boolean supercritical :** $\mu = \mathbf{12.56}$

γ	.2	1	5	∞
p_{sim}	0.28	0.29	0.33	
$p_{\text{p-bli}}$	0.32	0.32	0.34	0.36

Fraction of infected nodes for $\beta = 8$**,** $a = 2$**,** $\lambda = 1$ **and** $\alpha = 1$

- **Boolean subcritical :** $\mu = 3$

γ	.5	1	2	5	∞
p_{sim}	0.14	0.20	0.23	0.28	
$p_{\text{p-m2bi}}$	0.21	0.22	0.25	0.28	0.33

Fraction of infected nodes for $\beta = 2$**,** $a = 1$**,** $\lambda \sim 0.955$ **and** $\alpha = 1$

Factorization Heuristics Predict p Reasonably Well

γ	0	.2	1	5	∞
p_{sim}	0.26	0.28	0.29	0.33	0.36
$p_{\text{p-b1i}}$	0.313	0.315	0.323	0.341	0.363
$p_{\text{p-b1g1}}$	0.325	0.326	0.331	0.343	0.363
$p_{\text{p-m2bi}}$	0.328	0.328	0.329	0.341	0.363
$p_{\text{p-m}\infty}\text{bi}$	0.33	0.33	0.33	0.34	0.36
$p_{\text{f-h0}}$	0.23	0.28	0.29	0.32	0.36
$p_{\text{p-h0}}$	0.23	0.25	0.27	0.32	0.36

Figure: $\beta = 8, a = 2, \lambda = 1, \alpha = 1$

γ	0+	0.01	0.1	.2	1	5	100
p_{sim}				0.54	0.61	0.66	0.68
$p_{\text{p-b1i}}$	0.478		0.503	0.523	0.599	0.657	0.680
$p_{\text{p-b1g1}}$	0.530		0.544	0.557	0.609	0.658	0.680
$p_{\text{p-m2bi}}$	0.523		0.538	0.551	0.605	0.656	0.680
$p_{\text{p-m}\infty\text{bi}}$		0.54	0.55	0.56	0.61	0.66	0.68

Figure: $\beta = 1, a = 1, \lambda = 1, \alpha = 1$

Criticality by Polynomial Heuristic - B1I

- **p $\sim$ 0 only possible if**

$$\mathbf{2}(\mu\alpha-\beta)\gamma^{\mathbf{2}}+(\mathbf{2}\beta(\mu\alpha-\beta)+\beta^{\mathbf{2}}(\rho-\mathbf{1})-\beta\alpha)\gamma+\beta^{\mathbf{3}}(\rho-\mathbf{1})=\mathbf{0}$$

with $\rho=\left(\frac{\alpha\mu}{\beta}\right)^{\frac{2}{3}}>1$

- **There are real roots, which are both positive iff**

$$\mathbf{\Delta}:=(\mathbf{2}\beta(\mu\alpha-\beta)+\beta^{\mathbf{2}}(\rho-\mathbf{1})-\beta\alpha)^{\mathbf{2}}-\mathbf{8}(\mu\alpha-\beta)\beta^{\mathbf{3}}(\rho-\mathbf{1})>\mathbf{0}$$

Criticality by Polynomial Heuristic - B1I (*continued*)

- **In this case, there exist**

$$\gamma_{\mathbf{c}}^{+} = \frac{\beta(\alpha - \mathbf{2}(\mu\alpha - \beta)) - \beta^{\mathbf{2}}(\rho - \mathbf{1}) + \sqrt{\Delta}}{\mathbf{4}(\mu\alpha - \beta)}$$

$$\gamma_{\mathbf{c}}^{-} = \frac{\beta(\alpha - \mathbf{2}(\mu\alpha - \beta)) - \beta^{\mathbf{2}}(\rho - \mathbf{1}) - \sqrt{\Delta}}{\mathbf{4}(\mu\alpha - \beta)}$$

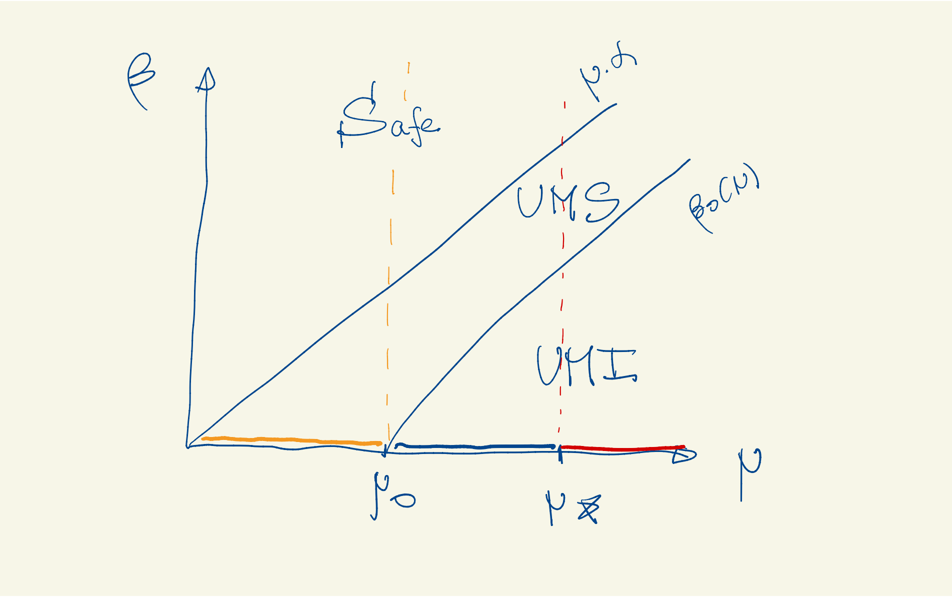
- **For** $\gamma < \gamma_c^-$ **or** $\gamma > \gamma_c^+$**, survival**
- **For** $\gamma_c^- < \gamma < \gamma_c^+$**, extinction**
- **+ a collection of similar results based on other heuristics : b1g1, m2bi, . . .**

Tentative Phase Diagram obtained by Polynomial Systems

- **Based on analysis of the roots of the polynomials around $p = 0$**
- **Similar results with numerical variations across heuristics**
- **Small variations depending on the chosen heuristic**

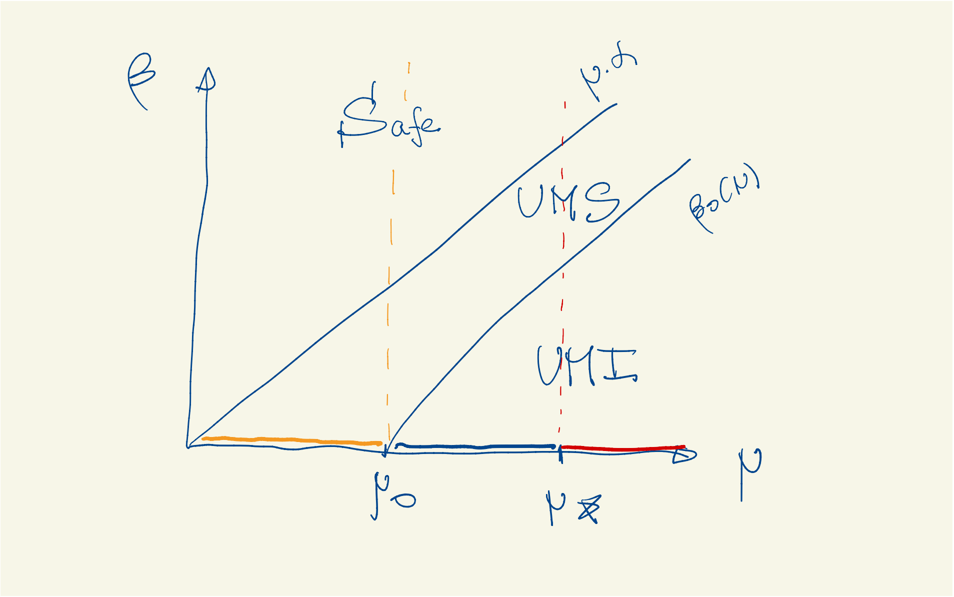
The Phase Diagram : Safe - Unsafe

- **Safe region**
  Extinction regardless of γ
  Wedge $\beta > \alpha\mu$
- **Unsafe region**
  $\exists\gamma$ **with survival**



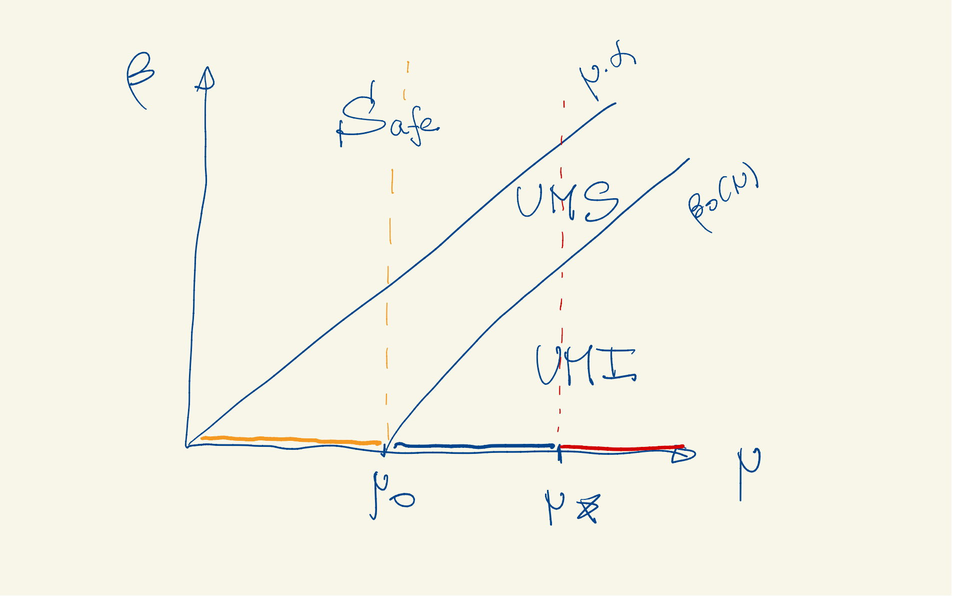
The Phase Diagram : Sensitive - Insensitive

- **Partition of Unsafe region**
  - — **UMI**
    motion insensitive
  - — **UMS**
    motion sensitive



The Phase Diagram : Thresholds on μ

- **Yellow segment**
  motion subcritical
- **Red semi-line**
  Boolean supercritical
- **Blue segment**
  motion supercrical
  and
  Boolean subcritical



$\mu_0 \sim 0.343$ **(m2bi)**, $\mu_* \sim 4.5$

UMS-Boolean Subcritical Region

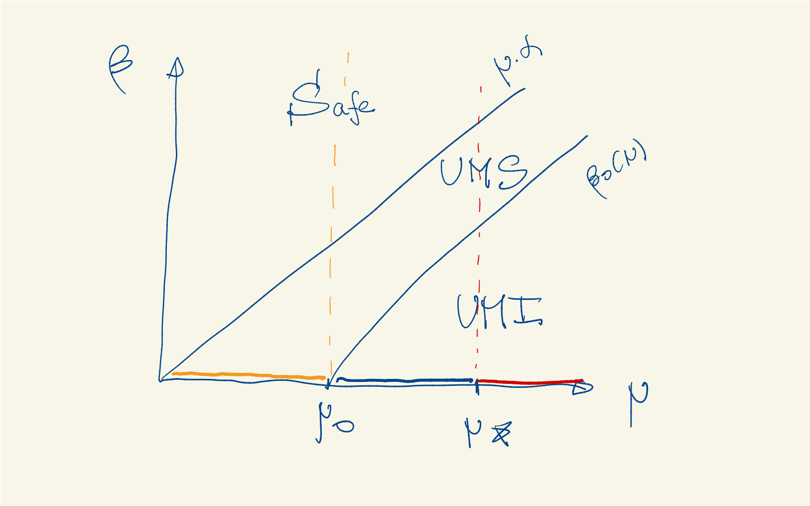
- $\{0 < \mu < \mu_0\} \cap \{0 < \beta < \alpha\mu\}$
- $\{\mu_0 < \mu < \mu_*\} \cap \{\beta_0 < \beta < \alpha\mu\}$
- $\subset$ **UMS**

  $\exists$ **non-degenerate critical functions** $\gamma_c^{-,+}(\mu, \beta)$ **s.t.**

  — $\gamma < \gamma_c^-$ **: survival**

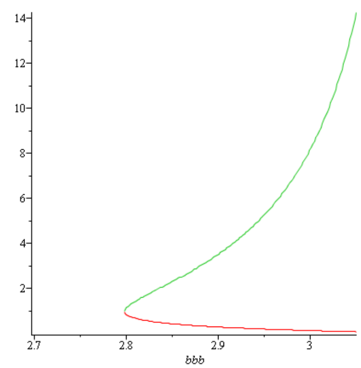
  — $\gamma_c^- < \gamma < \gamma_c^+$ **: extinction**

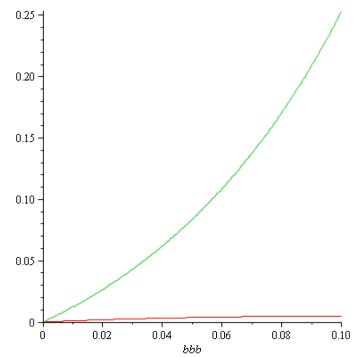
  — $\gamma > \gamma_c^+$ **: survival**



for no motion, extinction ; for small motions, survival
for intermediate motion, extinction
for high motions, survival

M2BI Example : γ_c



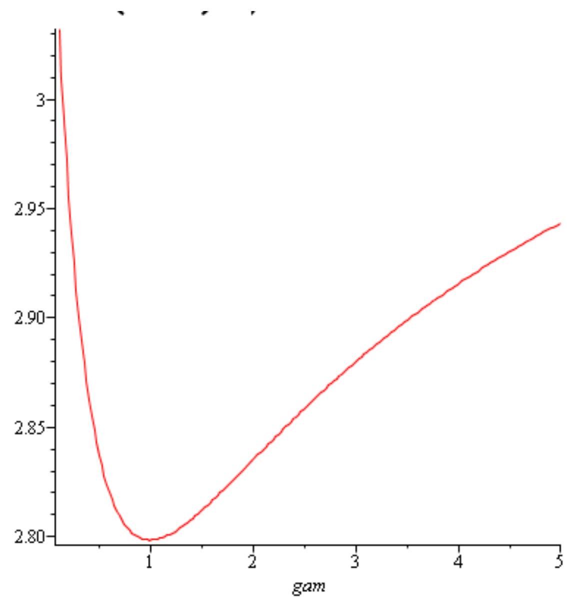


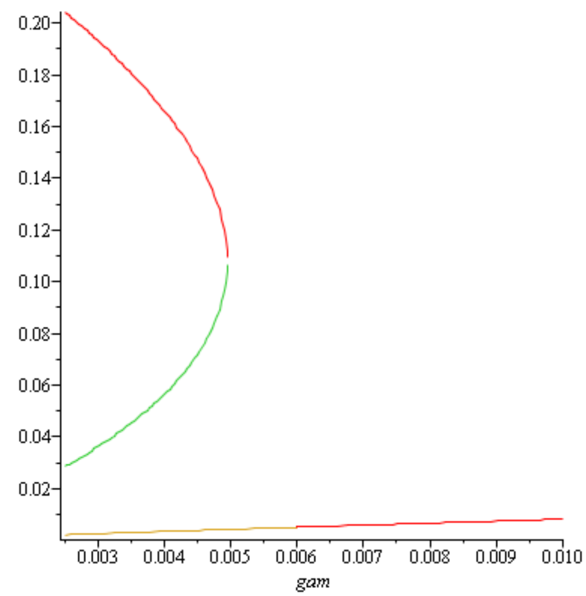
$\alpha = 1, \mu = 3.2416$ $\alpha = 1, \mu = 0.25$

$$8(\mu\alpha - \beta)\gamma^2 + 2\beta(3(\mu\alpha - \beta) - 2\alpha)\gamma + \beta^2(\mu\alpha - \beta) = 0$$

$$\beta_0 = \mu\alpha - \mu_0, \quad \mu_0 = \alpha\frac{2}{3 + \sqrt{8}}$$

M2BI Example : β_c





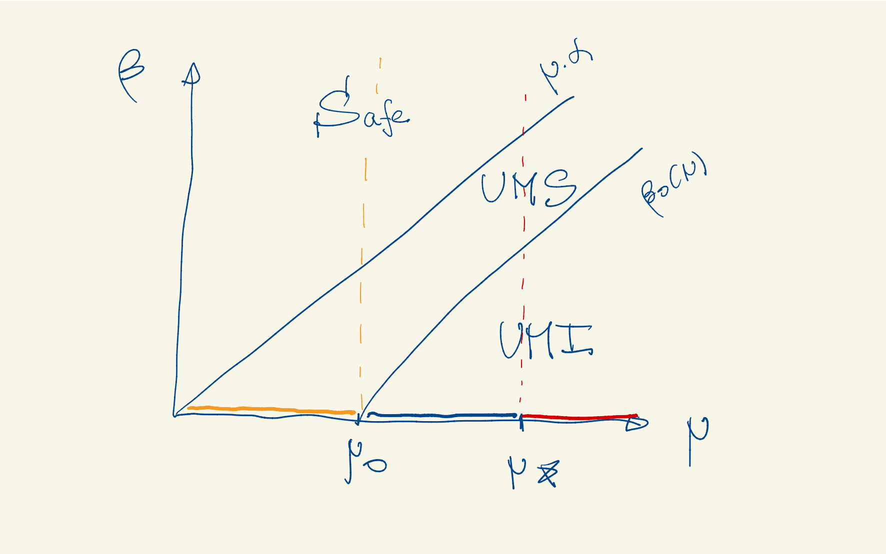
$\alpha = \mathbf{1}, \mu = \mathbf{3.2416}$ $\qquad$ $\alpha = \mathbf{1}, \mu = \mathbf{0.25}$

$$\beta^{\mathbf{3}} + \beta^{\mathbf{2}}(\mathbf{6}\gamma - \alpha\mu) + \beta\mathbf{2}\gamma(\mathbf{2}\alpha - \mathbf{3}\mu\alpha + \mathbf{4}\gamma) - \mathbf{8}\mu\alpha\gamma^{\mathbf{2}} = \mathbf{0}$$

UMI Region

- **Motion Insensitive region $\cap$ Boolean-supercritical case**
  — **no-motion, survival since $\beta_0 < \alpha\mu$**
  — **motion, survival**

- **Motion Insensitive region $\cap$ Boolean-subcritical case**
  — **no-motion, extinction**
  — **motion, survival**



no impact of motion on survival
no, low and high motion are all equivalent

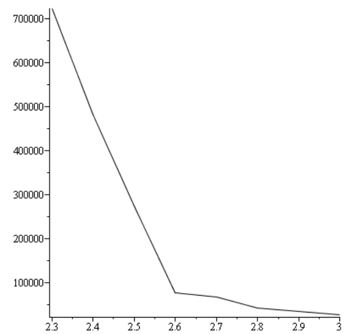
Simulation Validation of Phase Diagram

- **Simulation close to criticality is computationally challenging**
- **Methodology : mean time till absorption MTTA**

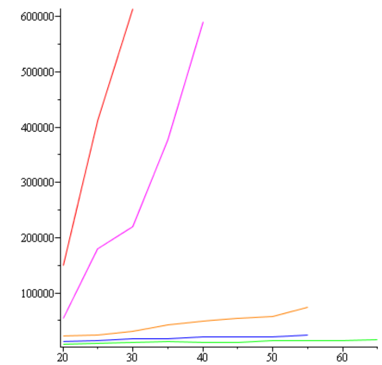
— Method 1 : inflection point w.r.t. β for fixed L
— Method 2 : dependency on the torus side L for fixed β

- **Partial validation at this stage**

Example 1 : β_c for $\gamma = \mathbf{1}$, $\alpha = \mathbf{1}$, $\mu = \mathbf{3.14}$



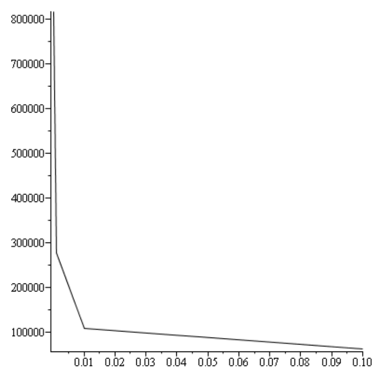
M1 : x axis : β ;
y axis : mean MTTA



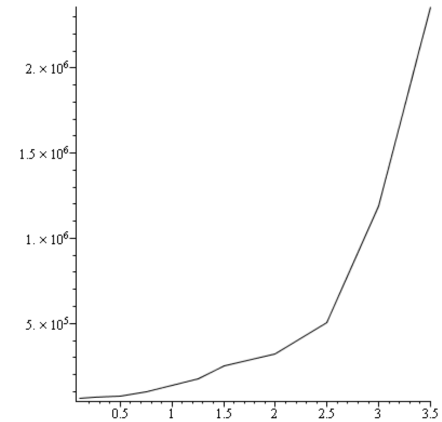
M2 : x : L ; y : mean MTTA,
$\beta = 3, 2.8, 2.4, 2.3$

p-b1i, $\beta_c = \beta_0 \sim 2.94$ **,** **p-b1g1,** $\beta_c = \beta_0 \sim 2.93$
p-m2bi, $\beta_c = \beta_0 \sim 2.82$**,** **MTTA** $\beta_c \sim 2.6$

Example 2 : γ_c for $\beta = 1/5$, $\alpha = 1$, $\mu = 1/4$



M1 estimate of γ_c^- : MTTA(γ)



M1 estimate of γ_c^+ : MTTA(γ)

p-b1i : $\gamma_c^+ \sim 1.73$, $\gamma_c^- \sim 0.007$; **p-m2bi,** $\gamma_c^+ \sim 1.84$, $\gamma_c^- \sim 0.003$
MTTA $\gamma_c^+ \sim 2.5$, $\gamma_c^- \sim 0.01$

More on Method 1

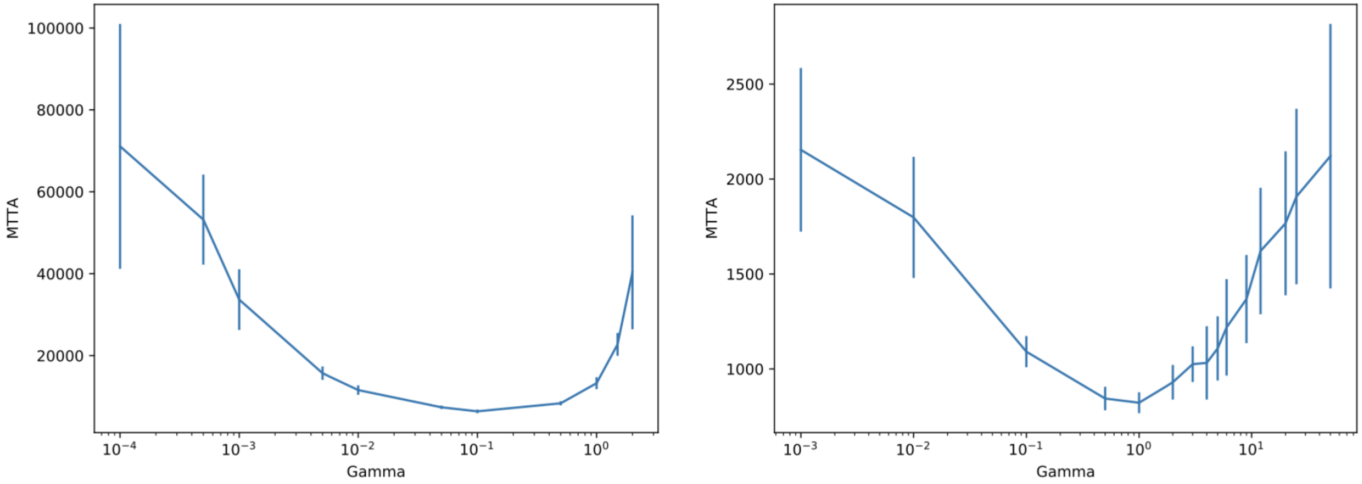


Figure: Method 1. Left: $\beta = 4.8, \alpha = 1, \mu = 5$. Right: $\beta = 0.2, \alpha = 1, \mu = 0.25$.

Fair prediction of parameter regions where the MTTA decreases (res. increases) with motion rate

To be Done

- **Prove repulsion conjecture**
- **Prove high mobility mean-field conjecture**
- **Assess monotonicity in** γ
- **Justify-prove the phase diagram**
- **Study more systematically the heuristics**
- **Extend the analysis to more SIS realistic variants**
- **Study other types of motions than far random waypoint**