Wireless Spatial Birth-Death Processes and Interference Queuing Networks

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Structure of the Lecture

- **Background and Motivation**
- **I. Wireless Birth-Death Processes**
  with A. Sankararaman, IEEE Tr. IT, 2017
  1. Stability, 2. Clustering, 3. Quantitative results
- **II. Interference Queuing Networks**
  with S. Foss & A. Sankararaman, Annals AP 2019
  1. Stability, 2. Minimal solution, 3. Initial condition
- **III. Cellular Birth-Death Processes**
  with A. AlAmmouri & J. Andrews, arXiv 1906.04683
  1. Stability, 2. Metastability, 3. Quantitative results

Motivations in Wireless Networks

- **Lack of understanding and analysis of**

  Space-time interactions

  – **Static spatial setting well understood: Stochastic Geometry [FB, Blaszczyszyn 01]**

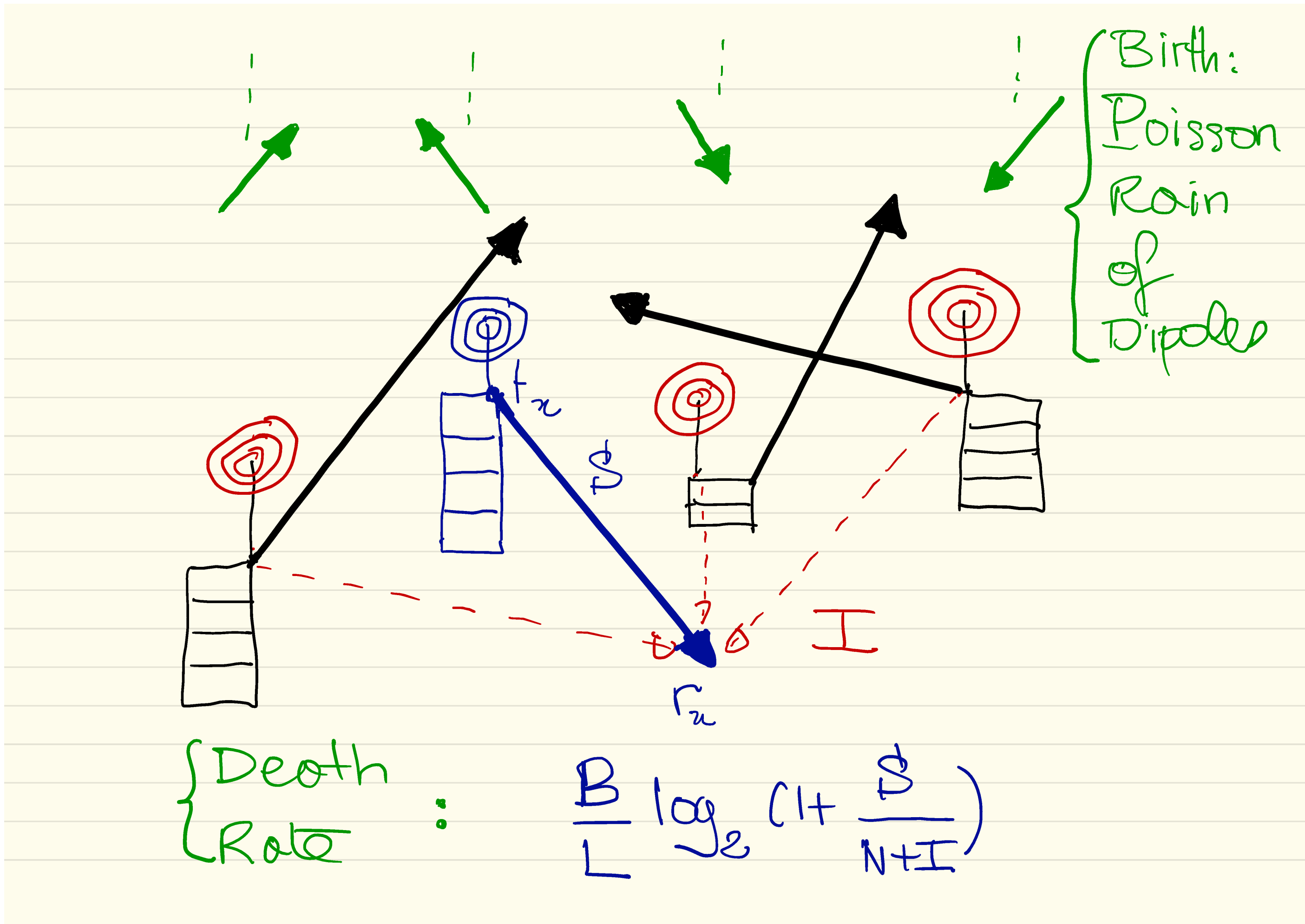
  – **Churn taken into account in flow-based queuing [Bonald, Proutiere 06], [Shakkottai, De Veciana 07] [Jiang, Walrand 09]**

- **Contents of this lecture:**

  Models with such dynamics in stochastic geometry

I. Wireless Birth-Death Processes

- **Setting: Infrastructureless Wireless Network:**
  Ad-hoc Networks, D2D Networks, IoT
- **Statistical assumptions: Markov Models:**
  Poisson, Exponential
- **Mathematical tools:**
  Point processes, Fluid, Mean-field approximation


Birth: Poisson Rain of Dipoles
t_x
S
I
r_x
Death Rate : $\frac{B}{L} \log_2 (1 + \frac{S}{N+I})$

Stochastic Network Model

- $\mathbf{S} = [-\mathbf{Q}, \mathbf{Q}] \times [-\mathbf{Q}, \mathbf{Q}]$: torus where the wireless links live
- **Links**: (Tx-Rx pairs)
- Links: arrive as a PPP on $\mathbb{R} \times \mathbf{S}$ with intensity λ:
  Prob. of a point arriving in space dx and time dt: $\lambda \mathrm{dxdt}$
- Each Tx has an i.i.d. exponential file size
  of mean $\mathbf{L}$ bits to transmit to its Rx
- A point exits after the Tx finishes transmitting its file
- Φ_t: set of locations of links present at time t:

$$\Phi_t = \{x_1, \ldots, x_{N_t}\}, \quad x_i \in \mathbf{S}$$

Interference and Service Rate

- **Interference seen at point x due to configuration Φ**

$$\mathbf{I}(\mathbf{x}, \mathbf{\Phi}) = \sum_{\mathbf{x_i} \in \mathbf{\Phi} \neq \mathbf{x}} \mathbf{l}(||\mathbf{x} - \mathbf{x_i}||)$$

  – **Distance on the torus**

  – $\mathbf{l}(\cdot)$**:** $\mathbb{R}^+ \to \mathbb{R}^+$**: path loss function**

- **The speed of file transfer by link at x in configuration Φ is**

$$\mathbf{R}(\mathbf{x}, \mathbf{\Phi}) = \mathbf{B} \log_2 \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N} + \mathbf{I}(\mathbf{x}, \mathbf{\Phi})} \right)$$

- $\mathbf{B}, \mathbf{N}$ **Positive constants**

B& D Master Equation

- **A point born at $\mathbf{x_p}$ and time $\mathbf{b_p}$ with file-size $\mathbf{L_p}$ dies at time**

$$\mathbf{d_p} = \inf \left\{ \mathbf{t} > \mathbf{b_p} : \int_{\mathbf{u}=\mathbf{b_p}}^{\mathbf{t}} \mathbf{R}(\mathbf{x_p}, \mathbf{\Phi_u})\mathbf{du} \geq \mathbf{L_p} \right\}$$

- **Spatial Birth-Death Process**
  - **Arrivals from the Poisson Rain**
  - **Departures happen at file transfer completion**

Properties of the Dynamics

- **The statistical assumptions imply that Φ_t is a Markov Process on the set of simple counting measures on S**
- **Euclidean extension of the flow-level models of [Bonald, Proutiere 06], [Shakkottai, De Veciana 07]**

Questions

- **Existence and uniqueness of the stationary regimes of Φ_t**
- **Characterization of the stationary regime(s) if existence**

Main Stability Results

$$\mathbf{a} := \int_{\mathbf{x} \in \mathbf{S}} \mathbf{l}(||\mathbf{x}||)\mathbf{dx}$$

- **Theorem**
  - If $\lambda > \frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{La}}$, then Φ_t admits no stationary regime.
  - If $\lambda < \frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{La}}$, and $\mathbf{r} \to \mathbf{l}(\mathbf{r})$ bounded and monotone, then Φ_t admits a unique stationary regime
- Necessary condition by Palm calculus, Stochastic intensity
- Sufficient condition by fluid limit
- **Corollary**

  For the path-loss model $\mathbf{l}(\mathbf{r}) = \mathbf{r}^{-\alpha}$, $\alpha \geq \mathbf{2}$, for all $\lambda > \mathbf{0}$, and all mean file sizes, the process Φ_t admits no stationary-regime

Main Qualitative Result

- Φ stationary point-process on S with Palm distribution $\mathbb{P}^0$
- Clustering
  Φ is clustered if for all bounded, positive, non-increasing functions $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, the shot-noise

$$F(x, \Phi) := \sum_{y \in \Phi \setminus \{x\}} f(||y - x||)$$

  satisfies

$$\mathbb{E}^0[F(0, \Phi)] \geq \mathbb{E}[F(0, \Phi)]$$

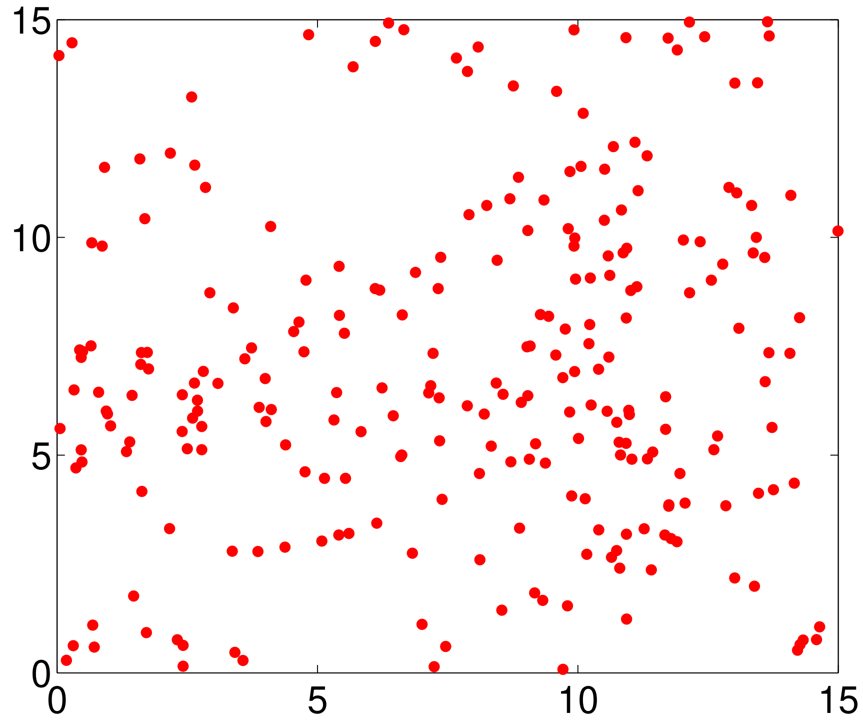
Main Qualitative Result (*continued*)

- **Theorem**
  The steady-state point process, when it exists, is clustered

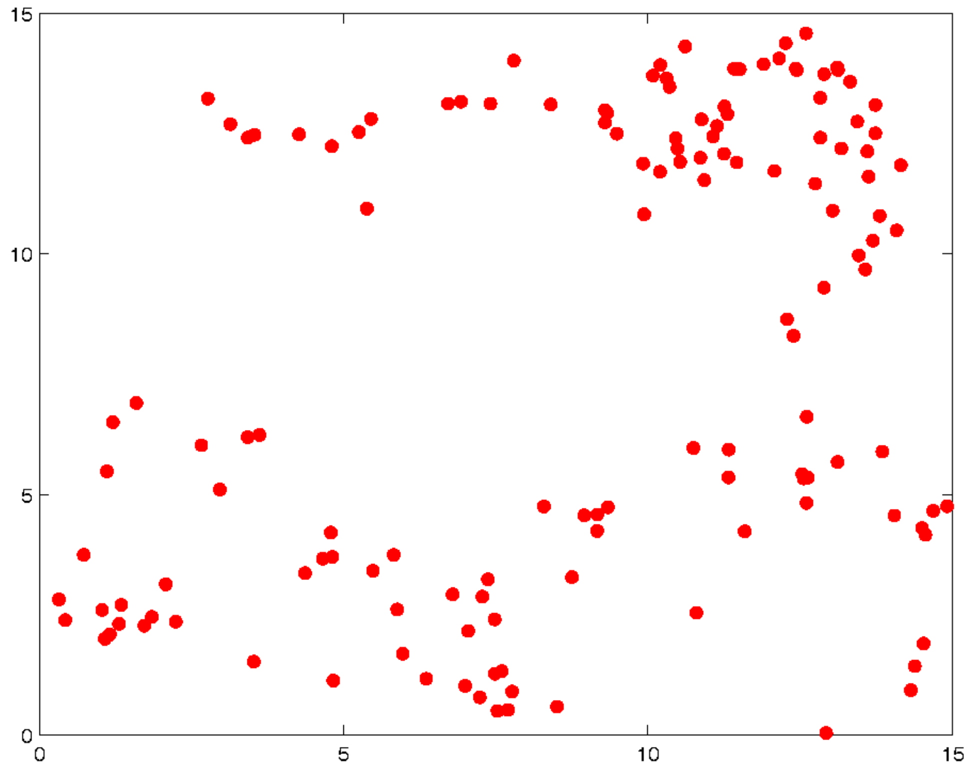
- **Follows from Palm calculus + association inequalities**

- **Interpretation of the result**
  The steady-state interference measured at a uniformly randomly chosen point of is larger in mean than that at a uniformly random location of space.

- **Key Observation**
  - **Dynamics Shapes Geometry**
  - **Geometry Shapes Dynamics**



A sample of Φ when $\lambda = 0.99$ and $\mathbf{l(r) = (r+1)^{-4}}$.



15
10
5
0
0
5
10
15

Proof of Instability in the Low SINR Case

- $\mathbf{L = 1,\ B = 1,\ N = 1}$ **to simplify notation**
- **If dynamics stationary, it must satisfy Rate-Conservation**

On average, what comes in is what goes out

- **Apply RCP to population**

$$\lambda|S| = \mathbb{E}\sum_{\mathbf{x}\in\Phi_0}\mathbf{R}(\Phi_0,\mathbf{x})$$

mean birth rate = mean death rate

- **Apply RCP to total interference**

Proof of Instability in the Low SINR Case (*continued*)

$$\text{Total Interference} := \sum_{x \in \phi_0} I(x, \phi_0) = \sum_{x \in \phi_0} \sum_{y \in \phi_0 \setminus \{x\}} l(||x - y||)$$

Average increase in total interference due to arrival = Average decrease in total interference due to departure

$$2\lambda |S| \beta \int_{x \in S} l(||x||) dx = 2\mathbb{E}\left[\sum_{x \in \phi_0} R(x, \phi_0) I(x, \phi_0)\right]$$

PASTA and Campbell | **Papangelou's Theorem**

Proof of Instability in the Low SINR Case (*continued*)

$$2\lambda|S|\beta \int_{x\in S} l(||x||)dx \quad = \quad 2\mathbb{E}\left[\sum_{x\in\phi_0} R(x,\phi_0)I(x,\phi_0)\right]$$

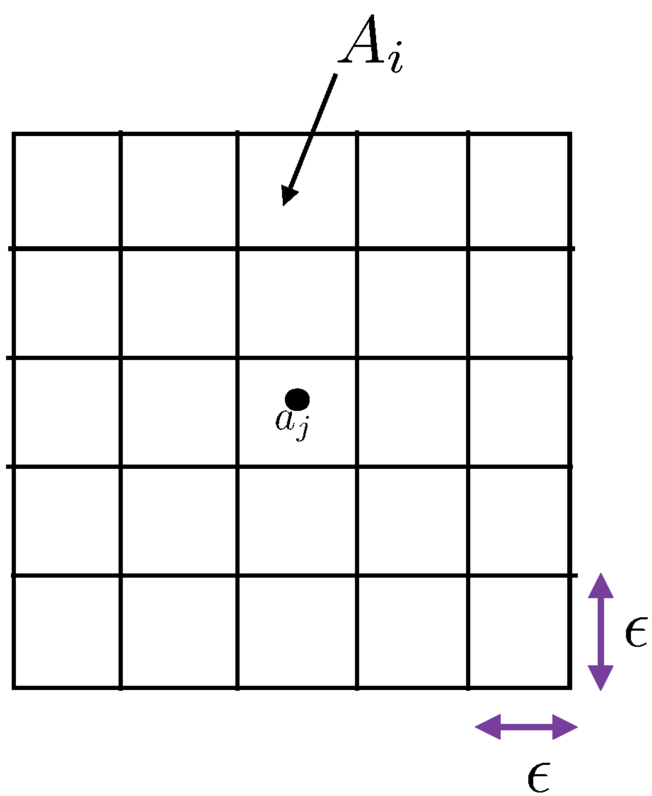
This equality implies $\lambda \int_{x\in\mathbf{S}} l(||x||)dx \leq 1$ (*)

Follows from definition $R(x,\phi) = \dfrac{1}{1+I(x,\phi)}$

Thus from (*) $\lambda > \lambda_c \implies$ unstable

Proof of Stability

Discretization



Define a path loss function l_ϵ s.t.

$$\mathbf{l}^\epsilon(\mathbf{x},\mathbf{y}) \geq \mathbf{l}(\mathbf{x},\mathbf{y}),\ \forall \mathbf{x},\mathbf{y} \in \mathbf{S}$$
$$\mathbf{l}^\epsilon(\mathbf{x},\mathbf{y}) = \mathbf{l}^\epsilon(\mathbf{a_i},\mathbf{a_j}),\ \forall \mathbf{x} \in \mathbf{A_i}, \mathbf{y} \in \mathbf{A_j}$$

$\mathbf{\Phi}^\epsilon(\mathbf{t})$**: associated SBD**

$$\mathbf{X_i}(\mathbf{t}) = \mathbf{\Phi}^\epsilon(\mathbf{t}, \mathbf{A_i})$$

Lemma When coupling arrivals and potential departures in Φ and Φ^ϵ appropriately,

$$\mathbf{\Phi_t}(\mathbf{S}) \leq ||\mathbf{X}(\mathbf{t})||_1,\ \forall \mathbf{t}$$

$\mathbf{X}(\mathbf{t})$ **stable implies $\Phi(t)$ stable**

$X_i \to X_i + 1$ at rate $\lambda\epsilon^2$

$X_i \to X_i - 1$ at rate $\dfrac{X_i}{1 + I_i^\epsilon(X)}$

$$I_i^\epsilon(X) = \sum_{j=1}^{N_\epsilon} (X_j - \mathbf{1}(j = i)) l_\epsilon(a_i, a_j)$$

Analyze this evolution through Fluid Limit techniques of [Dai 95].

$$\frac{dx_i}{dt} = \lambda\epsilon^2 - \frac{x_i(t)}{\sum_j x_j(t) l_\epsilon(a_i, a_j)}$$

Fluid Scale Evolution.

Can show that if $\lambda < \dfrac{1}{\int_{x \in \mathbf{S}} l_\epsilon(||x||) dx} \implies X(t)$ is stable.

Lyapunov Function $\max_i x_i(t)$

By letting $\epsilon \to 0$, we can conclude that
$\lambda < \lambda_c \implies \phi_t$ admits an unique stationary regime.

Quantitative Results

- **Heuristics for the intensity of the steady-state process**

  1. **Poisson or first order heuristic β_{f}**
     Derived by neglecting clustering and assuming Poisson Mean-field interpretation like in Aloha analysis
  2. **Cavity of second-order heuristic β_{s} based on a second-order cavity approximation of the dynamics**

Poisson Heuristic

- **Exact Rate Conservation Law:**

$$\lambda \mathbf{L} = \beta \mathbb{E}^{\mathbf{0}}_{\Phi}\left[\log_2\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N} + \mathbf{I}(\mathbf{0})}\right)\right].$$

Poisson Heur.: Largest solution to the fixed point equation:

$$\lambda \mathbf{L} = \frac{\beta_{\mathbf{f}}}{\ln(\mathbf{2})} \int_{\mathbf{z}=\mathbf{0}}^{\infty} \frac{\mathbf{e}^{-\mathbf{N}\mathbf{z}}(\mathbf{1} - \mathbf{e}^{-\mathbf{z}})}{\mathbf{z}} \mathbf{e}^{-\beta_{\mathbf{f}} \int_{\mathbf{x}\in\mathbf{S}}(\mathbf{1}-\mathbf{e}^{-\mathbf{z}\mathbf{l}(||\mathbf{x}||)})\mathbf{d}\mathbf{x}} \mathbf{d}\mathbf{z}$$

Ignores the Palm effect and uses the fact that if $\mathbf{X}, \mathbf{Y}$ are non-negative and independent,

$$\mathbb{E}\left[\ln\left(\mathbf{1} + \frac{\mathbf{X}}{\mathbf{Y} + \mathbf{a}}\right)\right] = \int_{\mathbf{z}=\mathbf{0}}^{\infty} \frac{\mathbf{e}^{-\mathbf{a}\mathbf{z}}}{\mathbf{z}} (\mathbf{1} - \mathbb{E}[\mathbf{e}^{-\mathbf{z}\mathbf{X}}]) \mathbb{E}[\mathbf{e}^{-\mathbf{z}\mathbf{Y}}] \mathbf{d}\mathbf{z}.$$

- **The Poisson heuristic is tight in heavy and light traffic**

Second Order Heuristic

The intensity β_s is given by

$$\beta_s = \frac{\lambda L}{B \log_2 \left(1 + \frac{1}{N + I_s}\right)}$$

where I_s is the smallest solution of the fixed-point equation

$$I_s = \lambda L \int_{x \in S} \frac{l(||x||)}{B \log_2 \left(1 + \frac{1}{N + I_s + l(||x||)}\right)} dx$$

Second Order Heuristic (*continued*)

- **Rationale** based on $\rho_2(\mathbf{x}, \mathbf{y})$: second moment measure of Φ
- **Rate Conservation for** ρ_2: when considering $\mathbf{I_s}$ as a constant

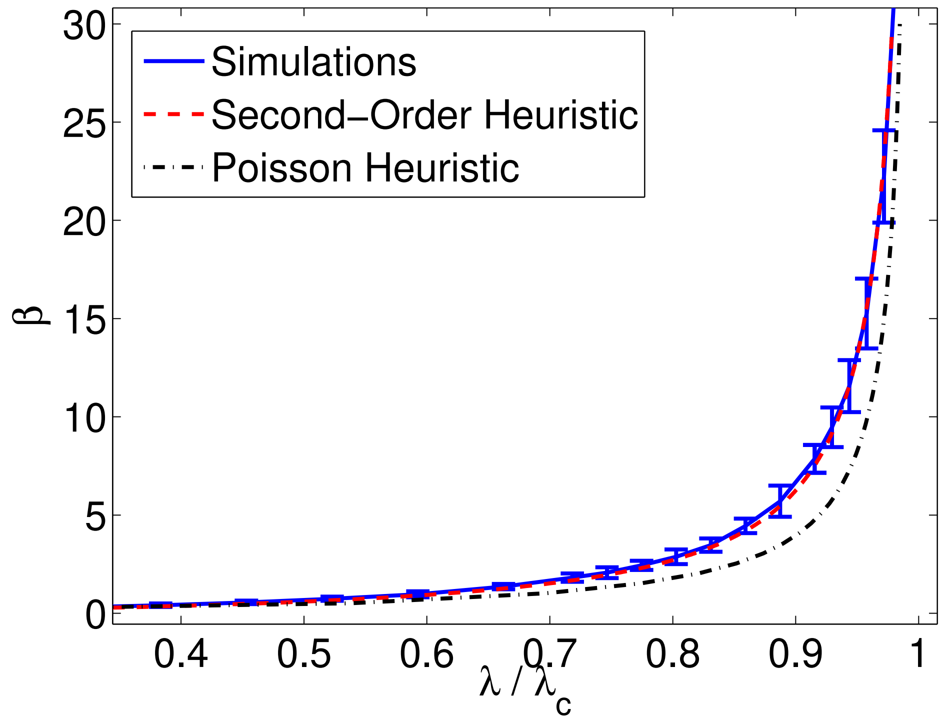
$$\rho_2(\mathbf{x}, \mathbf{y})\frac{\mathbf{1}}{\mathbf{L}}\mathbf{B}\log_2\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N} + \mathbf{I_s} + \mathbf{l}(||\mathbf{x} - \mathbf{y}||)}\right) = \lambda\beta_\mathbf{s}$$

- From the definition of second moment measure,

$$\mathbf{I_s} = \int_{\mathbf{x}\in\mathbf{S}} \mathbf{l}(||\mathbf{x}||)\frac{\rho_2(\mathbf{0}, \mathbf{x})}{\beta_\mathbf{s}}\mathbf{dx}$$

which gives the fixed point equation for $\mathbf{I_s}$

- The formula for $\beta_\mathbf{s}$ follows from **Rate Conservation for** $\rho_1 = \beta_\mathbf{s}$



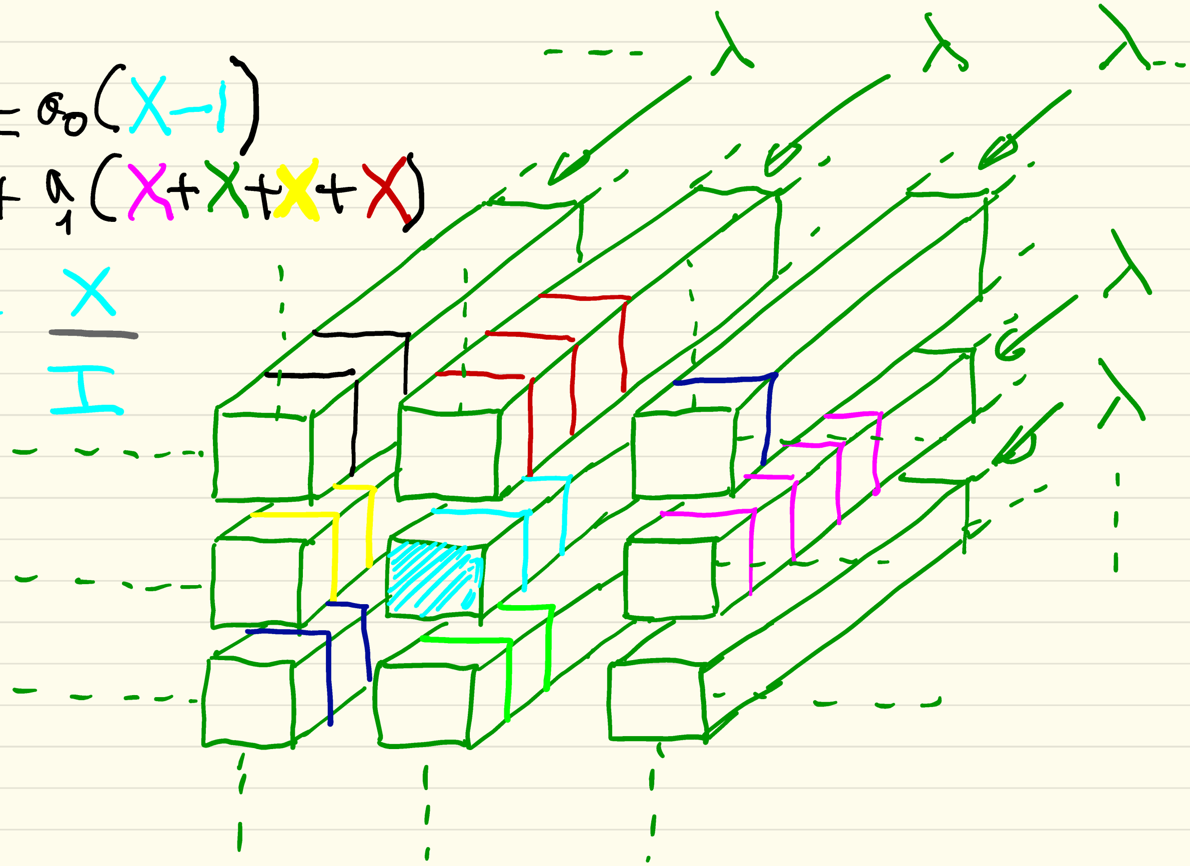
95% **confidence interval when** $\mathbf{l(r) = (r+1)^{-4}}$

II. Interference Queuing Networks

- **Aim**: extension of dynamics to $\mathbb{R}^2$ (scalability)
- **Setting**
  - **Discretization**: queuing dynamics on a grid
  - **Low SINR**: linearization of the log
- **Mathematical Tools**
  - **Interacting particle systems**
  - **Coupling from the past**
  - **Rate conservation principle**

$$I = \theta_0 (X - 1) + \theta_1 (X + X + X + X)$$

$$\rho = \frac{X}{I}$$



Assumptions, Notation

- Queue i at $\mathbf{i} \in \mathbb{Z}^d$ has state $\mathbf{x_i(t)} \in \mathbb{N}$ at time t
- **Arrivals to queues:**
  i.i.d. **Poisson processes of rate** $\lambda > 0$
- **Interference sequence:** $\{\mathbf{a_i}\}_{\mathbf{i} \in \mathbb{Z}^d}$
  non-negative, $\mathbf{a_0} = 1$, symmetric ($\mathbf{a_i} = \mathbf{a_{-i}}$), irreducible and with finite support.
- **Service discipline:**
  generalized processor-sharing with rate of queue i at time t:

$$\frac{\mathbf{x_i(t)}}{\sum_{\mathbf{j} \in \mathbb{Z}^d} \mathbf{a_j x_{i-j}(t)}}$$

Construction

$(\mathcal{A}_i, \mathcal{D}_i)$, $i \in \mathbb{Z}^d$, i.i.d.
$\mathcal{A}_i$ and $\mathcal{D}_i$ independent

$\mathcal{A}_i$, PPP intensity λ
$\mathcal{D}_i$, PPP intensity 1

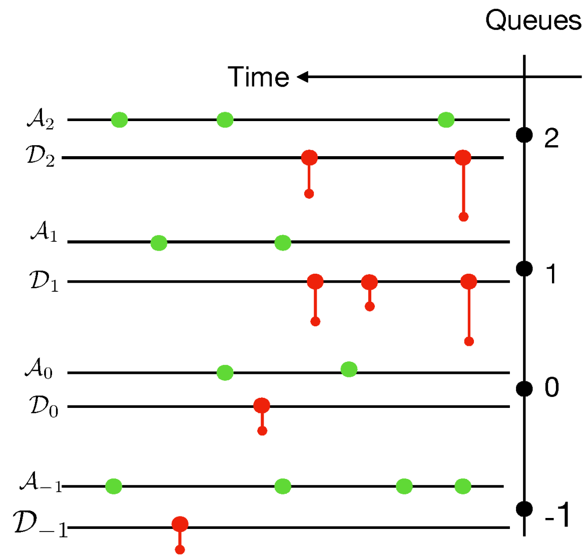
Translation Invariant Dynamics

$x_i(t^+) \leftarrow x_i(t) + 1$ if $t \in \mathcal{A}_i$

$x_i(t^+) \leftarrow x_i(t) - 1$ if

$(t, u) \in \mathcal{D}_i$ and

$$u \leq \frac{x_i(t)}{\sum_{j \in \mathbb{Z}^d} a_j x_{i-j}(t)} \mathbf{1}(x_i(t) > 0)$$



For all finite t, construction of $\{x_i(t)\}_i$ as a factor of $\{\mathcal{A}_i, \mathcal{D}_i\}_i$

Boolean percolation argument

decomposition of $[0, t]$ in small enough intervals

Stability, Minimal Stationary Regime

- **Stability**: when starting the system empty at time 0, weak convergence of the state of any finite set of queues as $t \to \infty$
- **Theorem** If

$$\lambda < \frac{1}{\sum_{i \in \mathbb{Z}^d} a_i}$$

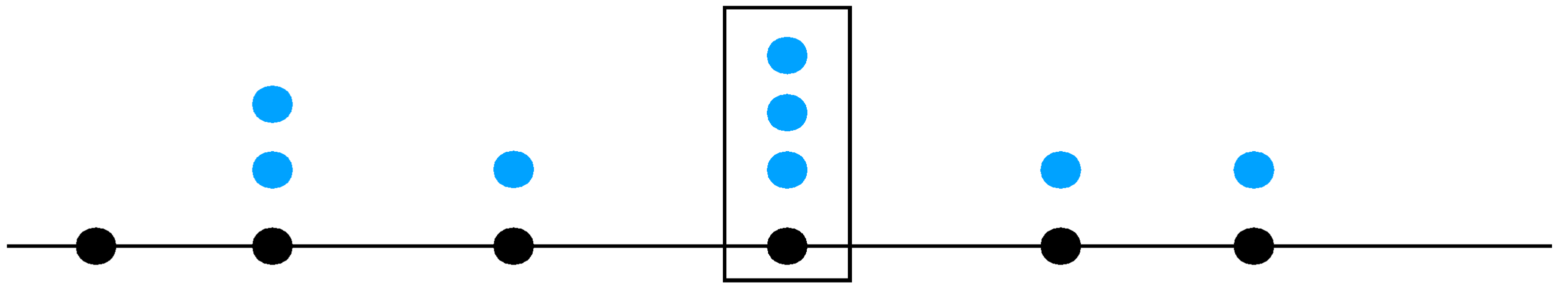
  – The network is stable

  – The weak limit is the minimal stationary regime

- **Proof**: CFP
- The stability condition $\lambda < \frac{1}{\sum_{i \in \mathbb{Z}^d} a_i}$ is sharp (current proof in special cases only)

Idea of Proof

- **Intuition**
- **Monotonicity**
- **Infinite space, Torus, and Cube**
- **Coupling from the Past**



Consider any local maximum queue i, i.e. $x_i(t) = \max\{x_{i-j}(t) : a_j > 0\}$

Its instantaneous departure rate is $\dfrac{x_i(t)}{\sum_{j\in\mathbb{Z}^d} a_j x_{i-j}(t)} \geq \dfrac{1}{\sum_{j\in\mathbb{Z}^d} a_j}$

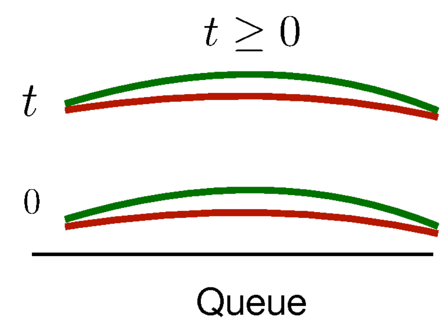
The arrival rate at every queue is λ

if $\lambda \sum\limits_{j\in\mathbb{Z}^d} a_j < 1$, then this local maximum queue has negative drift

Idea of Proof (*continued*)

Monotonicity

If two initial conditions $\{x_i(0)\}_{i\in\mathbb{Z}^d}$ and $\{y_i(0)\}_{i\in\mathbb{Z}^d}$ s.t. for all $i \in \mathbb{Z}^d$ $x_i(0) \le y_i(0)$, then there exists a coupling such that almost-surely $\forall t \ge 0$, $\forall i \in \mathbb{Z}^d$ $x_i(t) \le y_i(t)$.



Proof Induction

Arrivals retain the ordering

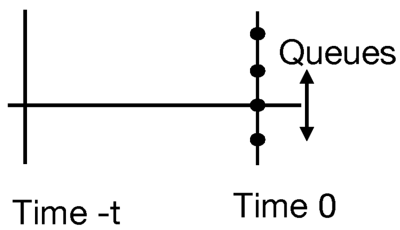
Two queues are equal - higher interference system has smaller departure

Unequal queues - Retains ordering as at-most one customer departs

Idea of Proof (*continued*)

Backward Construction

$x_{i;t}(0)$ Queue length of i at time 0 ***given*** the entire system was started empty at time -t



Monotonicity => $t \to x_{i;t}(0)$ is non-decreasing

$$x_{i,\infty}(0) := \lim_{t\to\infty} x_{i;t}(0) \quad \text{a.s.}$$

0-1 Law $\mathbb{P}[\cap_{i\in\mathbb{Z}^d} x_{i,\infty}(0) < \infty] \in \{0,1\}$

If $x_{i,\infty}(0) < \infty$ a.s. => System is stable

$\{x_{i,\infty}(0)\}_{i\in\mathbb{Z}^d}$ is a *minimal stationary solution* to the dynamics.

Idea of Proof ($continued$)

Infinite, Finite, Torus

Two systems on $B \subset \mathbb{Z}^d$ with the same dynamics.
All queues in $B^{\complement}$ are frozen without activity.

- $\{y_i(t)\}_{i \in B}$: the set B is a torus.

- $\{z_i(t)\}_{i \in B}$: the set B has boundary effects.

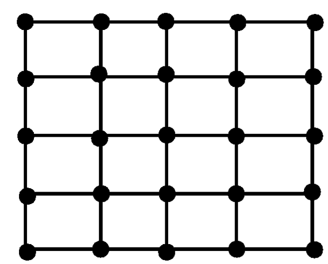
Interference is lower at the boundaries.

$\forall t \; \forall i \in B$

1) $x_i(t) \geq z_i(t)$

2) $y_i(t) \geq z_i(t)$

Monotonicity



Idea of Proof (*continued*)

Torus

<u>Theorem</u> - If $\lambda \sum_{j \in \mathbb{Z}^d} a_j < 1$, then $\{y_i(t)\}_{i \in B}$ is Positive Recurrent and the stationary distribution possess exponential moments. Furthermore, the mean queue length satisfies $\mathbb{E}[y_0(t)] = \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$

<u>Proof Idea of Stability</u>

$$\frac{d}{dt} y_i = \lambda - \frac{y_i}{\sum_{j \in \mathbb{Z}^d} a_j y_{(i-j)/B}(t)}$$

Fluid scale equation

Consider the maximal queue $i^*(t) := \arg\max_{i \in B} y_i(t)$

$$\frac{d}{dt} y_{i^*(t)} = \lambda - \frac{y_{i^*(t)}}{\sum_{j \in \mathbb{Z}^d} a_j y_{i^*(t)-j}(t)}$$

This has negative drift

$$\leq \lambda - \frac{1}{\sum_{j \in \mathbb{Z}^d} a_j} \quad < -\epsilon$$

Can upper bound by a stable Single server queue.

Idea of Proof (*continued*)

Mean Queue Length in Torus

Rate Conservation - "On Average what comes in is what goes out".

For Ex. $$\lambda = \mathbb{E}\left[\frac{y_0(t)}{\sum_{j\in\mathbb{Z}^d} a_j y_{j/B}(t)} \mathbf{1}_{y_0(t)>0}\right]$$

Avg arrival rate equals avg departure rate.

Consider $I(t) := y_0(t) \sum_{j\in\mathbb{Z}^d} a_j y_j(t)$ in stationarity and solve $\frac{d}{dt}\mathbb{E}[I(t)] = 0$

Average increase due to arrivals -	$\lambda + \lambda(\sum_{j\in\mathbb{Z}^d} a_j)\mathbb{E}[y_0(t)]$
Average decrease due to departures -	$\mathbb{E}[y_0(t)]$

Equating the two yields $\mathbb{E}[y_0(t)] \in \left\{\frac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}, \infty\right\}$

Idea of Proof (*continued*)

Cube

Monotonicity => $x_i(t) \geq z_i(t)$ and $y_i(t) \geq z_i(t)$

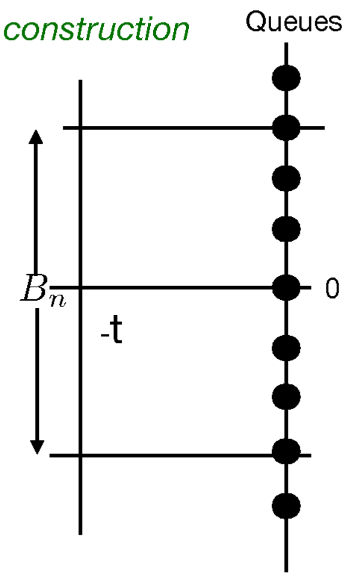
Thus $\mathbb{E}[z_0(t)] \leq \dfrac{\lambda}{1 - \lambda \sum_{j \in \mathbb{Z}^d} a_j}$ ***Uniformly in the size of*** B

Consider $B_n \nearrow \mathbb{Z}^d$ and corresponding stationary $z_0^{(n)}(0)$

Idea of Proof (*continued*)

Let $B_n \nearrow \mathbb{Z}^d$. $z_{0,t}^{(n)}(0)$ - the queue length of queue 0 at time 0, when the truncated B_n system is started empty at time -t.

Notice $\forall t \geq 0 \quad \lim_{n\to\infty} z_{0,t}^{(n)}(0) = x_{0,t}(0)$ *Corollary of the construction*



Monotonicity =>

$\lim_{t\to\infty} z_{0,t}^{(n)} := z_{0,\infty}^{(n)}$ and $\lim_{n\to\infty} z_{0,\infty}^{(n)} := z_{0,\infty}^{(\infty)}$ a.s.

We know $\sup_{n\in\mathbb{N}} \mathbb{E}[z_{0,\infty}^{(n)}] \leq \dfrac{\lambda}{1-\lambda\sum_{j\in\mathbb{Z}^d} a_j}$

thus, $\mathbb{E}[z_{0,\infty}^{(\infty)}] < \infty$

Quantitative Properties of Minimal Stationary Regime

- **Lemma** The weak limit, when it exists, satisfies

$$\mathbb{E}[\mathbf{x_0}] = \frac{\lambda \mathbf{a_0}}{\mathbf{1} - \lambda \sum_{i \in \mathbb{Z}^d} \mathbf{a_i}}$$

In addition its coordinates $(\mathbf{x_i})_{i \in \mathbb{Z}^d}$ are associated

- **Association**: analogue of of clustering in the continuum
- **Remarkable fact**: closed form for the mean for this infinite-dimensional, non-reversible, non-asymptotically independent particle system

Uniqueness

- Below, assume that $\lambda < \frac{1}{\sum_{i \in \mathbb{Z}^d} a_i}$
- **Proposition** If $\mathbb{E}[x_0^2] < \infty$, then the minimal solution is the **unique** stationary solution with finite second moment
- **Proposition** If

$$\lambda < \frac{2}{3} \frac{1 + c}{\sum_{j \in \mathbb{Z}^d} a_j} \quad \text{where} \quad c = \frac{\sqrt{a_0^2 + a_0 \sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j} - a_0}{\sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j}$$

then $\mathbb{E}[x_0^2] < \infty$

Domain of Attraction of the Minimal Solution

- **Theorem** If $\lambda < \frac{2}{3}\frac{1+c}{\sum_{j\in\mathbb{Z}^d} a_j}$ and the initial condition satisfies

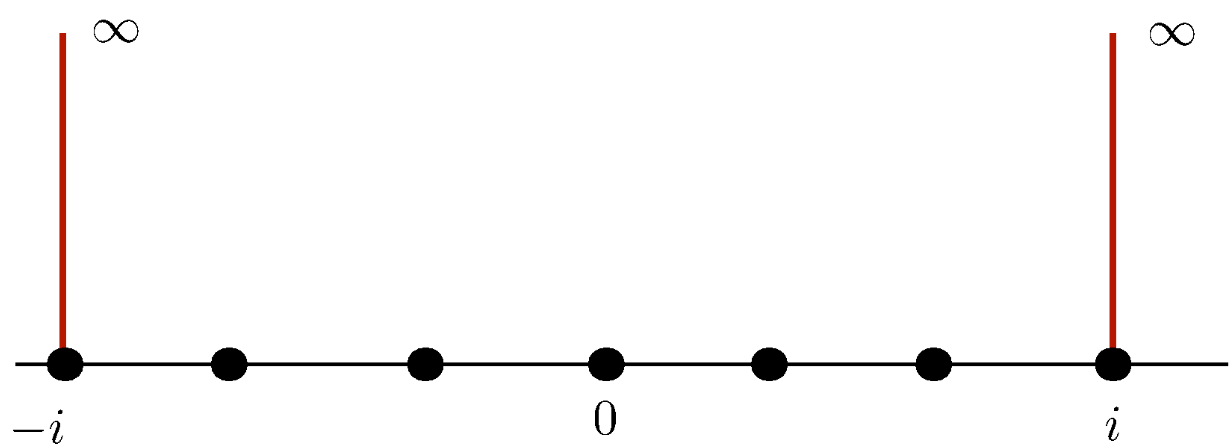
$$\sup_{i\in\mathbb{Z}^d} x_i(0) < \infty$$

then $\{x_i(\cdot)\}_{i\in\mathbb{Z}^d}$ **converges weakly to the minimal stationary solution**

- **Theorem** For $d = 1$, for all $\lambda > 0$, there exists

1. A deterministic sequence $(\alpha_i)_{i\in\mathbb{Z}}$ such that if $x_i(0) \geq \alpha_i$ for all $i \in \mathbb{Z}$, then $\lim_{t\to\infty} x_0(t) = \infty$ a.s.
2. A distribution ξ on $\mathbb{N}$ s.t. if $\{x_i(0)\}_{i\in\mathbb{Z}}$ is i.i.d. with marginal distr. ξ, then $\lim_{t\to\infty} x_0(t) = \infty$ a.s.

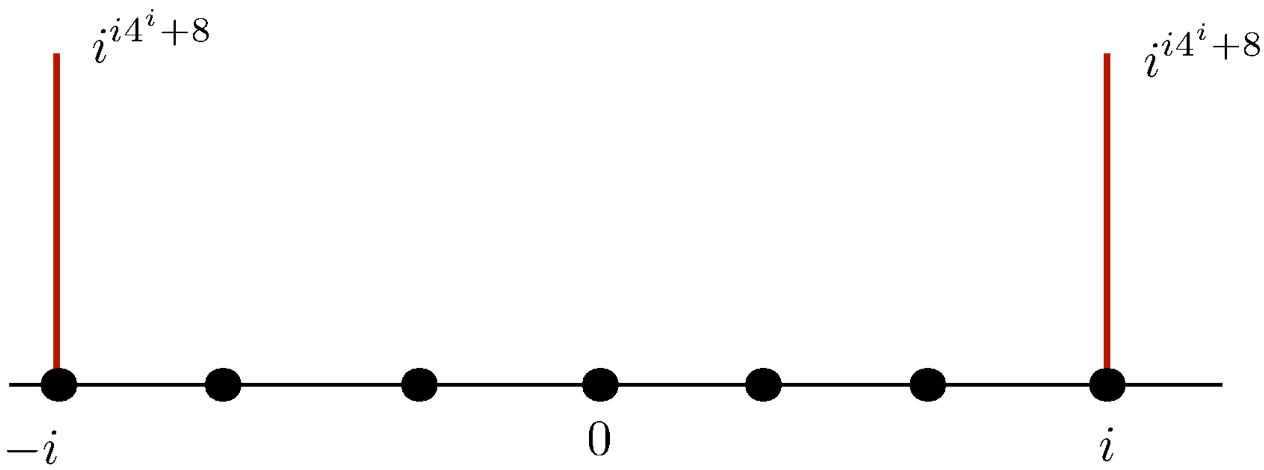
Divergence Examples



$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \le i^{-4}$$

Because of the infinite barrier, all queues diverge to infinity at a linear rate.

Divergence Examples (*continued*)



$$\exists (t_i)_{i\in\mathbb{N}} \text{ s.t. } t_i \to \infty \text{ s.t. } \mathbb{P}[x_0(t_i) < i] \leq i^{-3}$$

Since interested only in finite time t_i, can bring down the barrier to a finite value at a small penalty in probability.

Borel-Cantelli to conclude the proof.

III. Cellular Birth-Death Processes

- **Aim**: extend the dynamics of I. to cellular
- **Setting**
  - Single Cell: uplink
  - Low SINR: linearization of the log
- **Mathematical Tools**
  - Stability, metastability
  - Rate conservation principle
  - First and second order heuristics

System Model

- **A single BS at the origin with association**
  - $\mathcal{D}$ **compact e.g. ball of fixed radius centered at the BS**
  - **Users arrive in $\mathcal{D}$ according to a Poisson rain of intensity λ**
  - **Each user transmits a file to the BS**
  - **Once the file is transmitted, the user leaves the system**
  - **Files are exponential of mean length $\mathbf{L} = \frac{1}{\mu}$**

Dynamics

- **Markov Spatial Birth-Death Process Φ_t**
  with state space counting measures Φ on $\mathcal{D}$
- **Interference seen at BS for point x in configuration Φ**

$$\mathbf{I}(\mathbf{x}, \mathbf{\Phi}) = \sum_{\mathbf{x_i} \in \mathbf{\Phi} \neq \mathbf{x}} \mathbf{P_{x_i} l}(||\mathbf{x_i}||)$$

  – **Attenuation l: continuous, non-increasing and bounded**
  – **Power control: $\mathbf{P_x} = \mathbf{l}(||\mathbf{x}||)^{-\beta}$, $\beta \in [\mathbf{0}, \mathbf{1}]$**
- **The speed of file transfer on link between x and BS in Φ**

$$\mathbf{R}(\mathbf{x}, \mathbf{\Phi}) = \mathbf{B} \log_{\mathbf{2}} \left(\mathbf{1} + \frac{\mathbf{P_x l}(||\mathbf{x}||}{\mathbf{N} + \mathbf{I}(\mathbf{x}, \mathbf{\Phi})} \right)$$

- **Death rate of x proportional to $\mathrm{R}(\mathbf{x}, \mathbf{\Phi})$**

Stability Condition

$$\lambda_{\mathbf{c}} = \frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{L}}$$

- **Theorem Under assumptions on l,**
  - **If $\lambda > \lambda_{\mathbf{c}}$, then $\Phi_{\mathbf{t}}$ is transient**
  - **If $\lambda < \lambda_{\mathbf{c}}$, then $\Phi_{\mathbf{t}}$ is ergodic (unique stationary regime)**
- **Stability condition oblivious of**
  - **thermal noise**
  - **attenuation function and power control**

Steady State Distribution

- **For all λ such that the system is stable, let $\gamma(\mathbf{x})$ be the density of the steady state point process Φ**
- **From RCP, for all x $\in \mathcal{D}$, in the low SINR case**

$$\lambda \mathbf{L} = \gamma(\mathbf{x}) \frac{\mathbf{B}}{\ln(\mathbf{2})} \mathbb{E}^{\mathbf{0}}_{\mathbf{x}} \left[\frac{\mathbf{l}(||\mathbf{x}||)^{\mathbf{1}-\beta}}{\sum_{\mathbf{y}\in\mathbf{\Phi},\ \mathbf{y}\neq\mathbf{x}} \mathbf{l}(||\mathbf{y}||)^{\mathbf{1}-\beta}} \right]$$

Poisson Heuristic

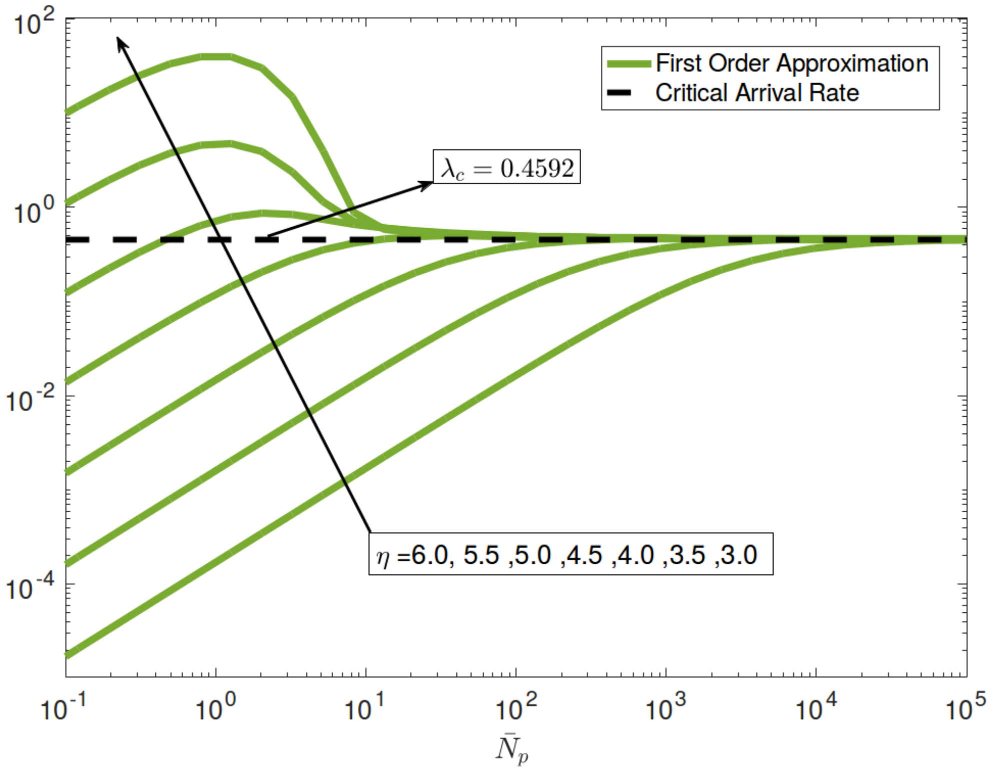
- **Under the Poisson heuristic, if the system is stable, then**

$$\gamma_{\mathbf{f}}(\mathbf{x}) = \frac{\mathbf{Z}^*}{\mathbf{l}(||\mathbf{x}||)^{\mathbf{1}-\beta}}$$

with $\mathbf{Z}^*$ solution of the consistency equation

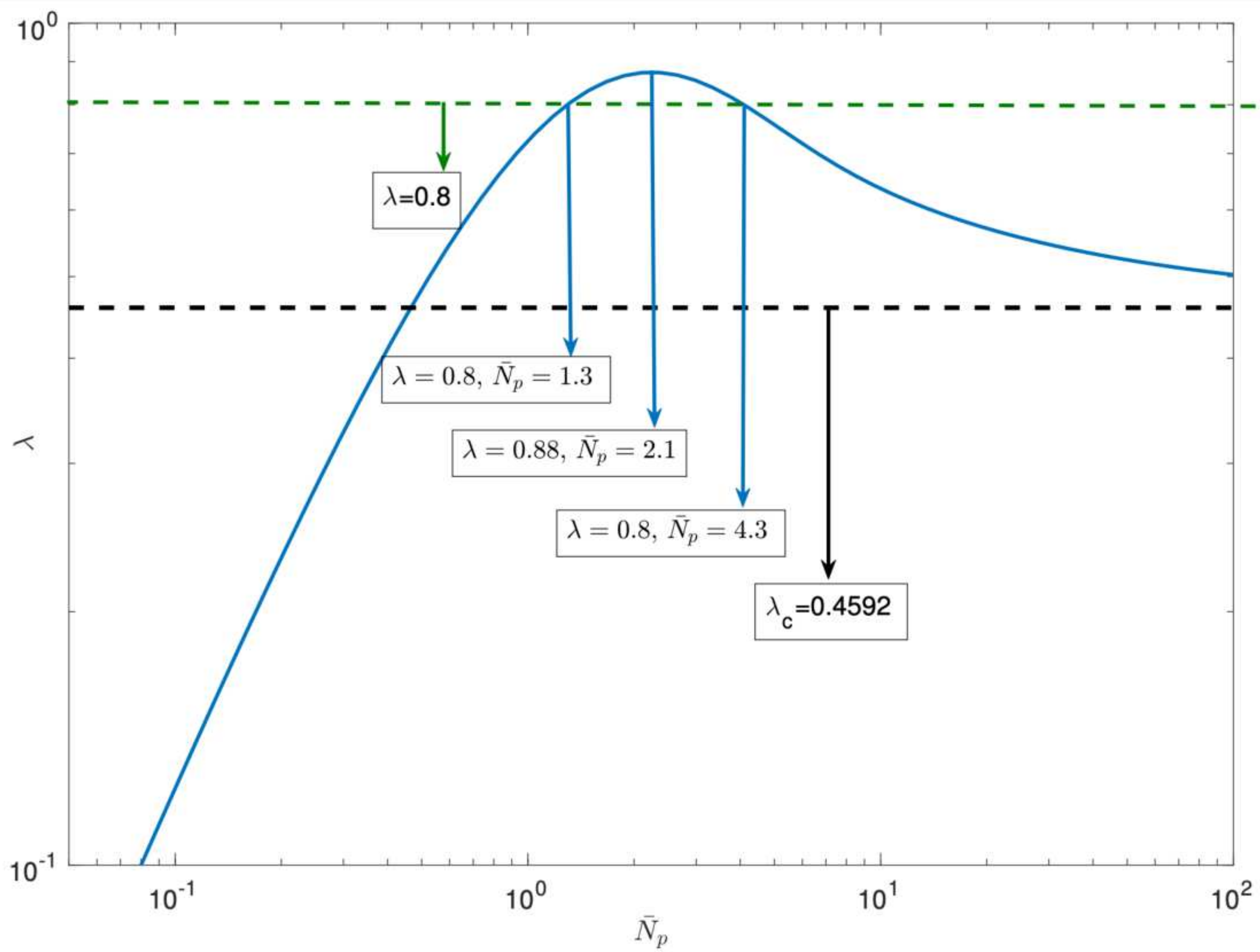
$$\frac{\lambda \mathbf{L} \ln(\mathbf{2})}{\mathbf{B}} = \mathbf{Z} \int_{\mathbf{0}}^{\infty} \mathbf{e}^{-\mathbf{tN}} \exp\left(-\mathbf{Z} \int_{\mathcal{D}} \left(\mathbf{1} - \mathbf{e}^{-\mathbf{t}\mathbf{l}(||\mathbf{y}||)^{\mathbf{1}-\beta}} \right) \mathbf{l}(||\mathbf{y}||)^{\mathbf{1}-\beta} \mathbf{dy} \right) \mathbf{dt}$$

Consistency Equations: Numerical Results

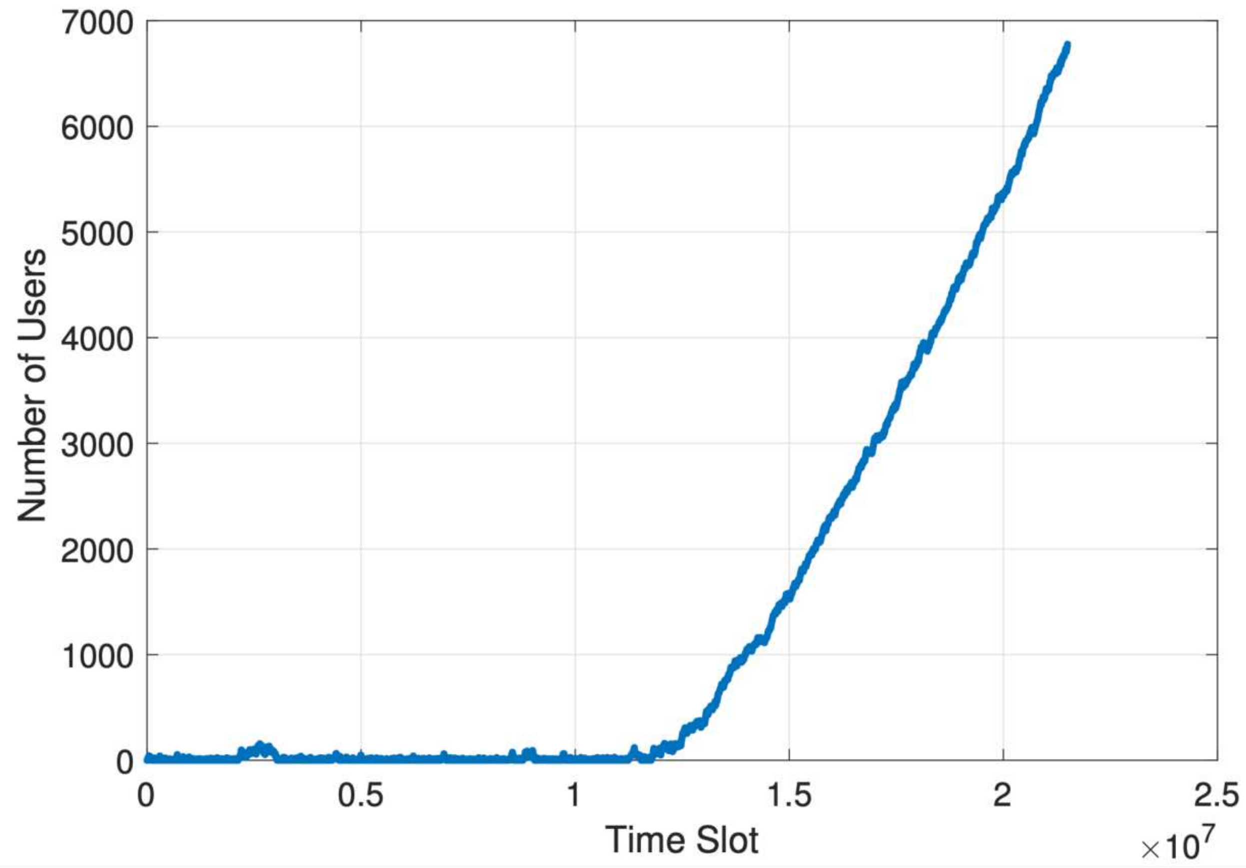


- $\mathbf{l(r)} = \frac{\mathbf{1}}{\mathbf{(1+r)}^{\eta}}$
- $\mathbf{L} = \mathbf{100}$ **bits**
- $\mathbf{B} = \mathbf{1}$ **MHz**
- $\mathbf{N} = -\mathbf{50}$ **dBm**
- **For** $\lambda < \lambda_c$ **unique solution**
- **For** $\lambda_c < \lambda < \lambda_m$ **two solutions**

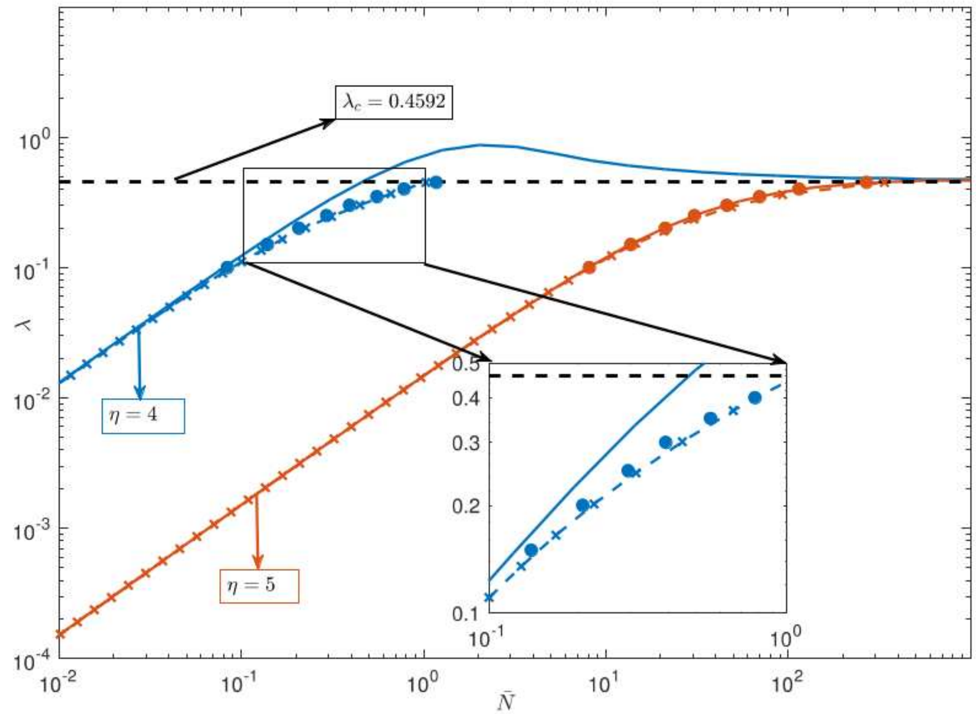
Metastability through Mean-Field



Metastability through Dynamics



First and Second Order Heuristics



First order heuristic by consistency equation
Second order heuristic by cavity method

Conclusions

- Representation of space-time interactions in wireless netw.
- No reversibility, no asymptotic independence
- Dynamic notion of capacity involving both queuing and IT
- Generative model for clustering
- Good Mean-field heuristics in general
- Exact analytical results in the low SINR case
- Metastability in the cellular extension.
- Particle system version of dynamics with closed forms