Generalized Robin-Neumann explicit coupling schemes for fluid-structure interaction

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Abstract

We deal with the coupled problem describing the interaction between a viscous incompressible fuid and a viscoelastic structure. For this kind of problem, the main strategies that are usually referred to in the literature are:

- **Standard explicit coupling schemes** based on Dirichlet-Neumann interface conditions, which are known to be unconditionally unstable whenever the amount of added-mass effect is large.
- **Implicit or semi-implicit** coupling schemes, which require a higher computational effort.
- **Explicit schemes** based on Robin-Robin interface coupling derived from Nitsche's method, whose accuracy demands restrictive CFL conditions or corrective iterations.

Here we present a family of **Robin-Neumann explicit coupling schemes**, involving a Robin interface condition for the fluid, which is intrinsically consistent when dealing with thin-walled structures [1]. The implicit treatment of the sole solid inertia ensures **added-mass free stability** and the explicit treatment of the solid viscoelastic contributions enables **full fluid-solid splitting**. We show that the resulting scheme with first-order extrapolation provides unconditional stability and optimal first-order accuracy.

In the case of the coupling with a **thick-walled structure** [2], we show that a consistent generalized-Robin interface condition can be recovered at the space semi-discrete level through a **mass-lumping approximation** in the solid. The methods preserve the stability properties of the thin-walled case. As regards accuracy, the splitting introduces an error perturbation whose leading term scales as $\mathcal{O}(\tau^{2^{r-1}}/\sqrt{h})$, where r stands for the extrapolation order. The $h^{-1/2}$ loss is related to the non-uniformity of the discrete solid viscoelastic operator and not to the mass-lumping approximation.

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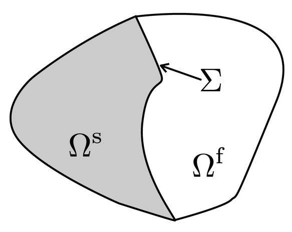
The linear model problem

For simplicity, we consider the following linear coupled problem:

- find the fluid velocity $\boldsymbol{u} : \Omega^{\mathrm{f}} \times \mathbb{R}^+ \to \mathbb{R}^d$,
- the fluid pressure $p : \Omega^{\mathrm{f}} \times \mathbb{R}^+ \to \mathbb{R}$,
- the structure displacement $\boldsymbol{d} : \Omega^{\mathrm{s}} \times \mathbb{R}^+ \to \mathbb{R}^d$
- and the structure velocity $\dot{\boldsymbol{d}} : \Omega^{\mathrm{s}} \times \mathbb{R}^+ \to \mathbb{R}^d$, such that



$$\text{(Fluid)}\begin{cases} \rho^{\mathrm{f}}\partial_t \boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{u},p) = \boldsymbol{0} & \text{in } \Omega^{\mathrm{f}},\\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega^{\mathrm{f}}, \end{cases} \quad (1)$$

$$\text{(Solid)}\begin{cases} \rho^{\mathrm{s}}\partial_t \dot{\boldsymbol{d}} - \mathbf{div}\boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{d},\dot{\boldsymbol{d}}) + \alpha\rho^{\mathrm{s}}\dot{\boldsymbol{d}} = \boldsymbol{0} & \text{in } \Omega^{\mathrm{s}},\\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{in } \Omega^{\mathrm{s}}, \end{cases} \quad (2)$$

$$\text{(Coupling)}\begin{cases} \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on } \Sigma,\\ \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{d},\dot{\boldsymbol{d}})\boldsymbol{n}^{\mathrm{s}} = -\boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{u},p)\boldsymbol{n}^{\mathrm{f}} & \text{on } \Sigma \end{cases} \quad (3)$$

Numerical investigations also include the non-linear case (e.g., Navier-Stokes, non-linear elastodynamics).

Variational formulation

We introduce some functional spaces, linear and bilinear forms:

$$\boldsymbol{V}^{\mathrm{f}} \stackrel{\mathrm{def}}{=} [H^1(\Omega^{\mathrm{f}})]^d, \qquad a(\boldsymbol{u},\boldsymbol{v}^{\mathrm{f}}) \stackrel{\mathrm{def}}{=} 2\mu\big(\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{v}^{\mathrm{f}})\big)_{\Omega^{\mathrm{f}}},$$

$$Q \stackrel{\mathrm{def}}{=} L^2(\Omega^{\mathrm{f}}), \qquad b(p,\boldsymbol{v}^{\mathrm{f}}) \stackrel{\mathrm{def}}{=} -(p,\operatorname{div}\boldsymbol{v}^{\mathrm{f}})_{\Omega^{\mathrm{f}}},\ l(\boldsymbol{v}^{\mathrm{f}}) \stackrel{\mathrm{def}}{=} (\boldsymbol{f}^{\Gamma},\boldsymbol{v}^{\mathrm{f}})_{\Gamma},$$

$$\boldsymbol{V}^{\mathrm{s}} \stackrel{\mathrm{def}}{=} \big\{\boldsymbol{v}^{\mathrm{s}} \in [H^1(\Omega^{\mathrm{s}})]^d \,/\, \boldsymbol{v}^{\mathrm{s}}|_{\Gamma^{\mathrm{d}}} = \boldsymbol{0}\big\}, \qquad a^{\mathrm{e}}(\boldsymbol{d},\boldsymbol{v}^{\mathrm{s}}) \stackrel{\mathrm{def}}{=} \big(\boldsymbol{\sigma}(\boldsymbol{d}),\boldsymbol{\epsilon}(\boldsymbol{v}^{\mathrm{s}})\big)_{\Omega^{\mathrm{s}}},$$

$$\boldsymbol{W} \stackrel{\mathrm{def}}{=} \big\{(\boldsymbol{v}^{\mathrm{f}},\boldsymbol{v}^{\mathrm{s}}) \in \boldsymbol{V}^{\mathrm{f}}\times\boldsymbol{V}^{\mathrm{s}} \,/\, \boldsymbol{v}^{\mathrm{f}}|_{\Sigma} = \boldsymbol{v}^{\mathrm{s}}|_{\Sigma}\big\}, \qquad a_h^{\mathrm{v}}(\dot{\boldsymbol{d}},\boldsymbol{v}^{\mathrm{s}}) \stackrel{\mathrm{def}}{=} \beta\big(\boldsymbol{\sigma}(\dot{\boldsymbol{d}}),\boldsymbol{\epsilon}(\boldsymbol{v}^{\mathrm{s}})\big)_{\Omega^{\mathrm{s}}} + \alpha\rho^{\mathrm{s}}(\dot{\boldsymbol{d}},\boldsymbol{v}^{\mathrm{s}})_{\Omega^{\mathrm{s}},h}.$$

The coupled problem (1)-(3) admits the following variational formulation, including a mass-lumping approximation in the structure inertia: for $t>0$, find $(\boldsymbol{u}(t),\dot{\boldsymbol{d}}(t)) \in \boldsymbol{W}$, $p(t) \in Q$ and $\boldsymbol{d}(t) \in \boldsymbol{V}^{\mathrm{s}}$ such that $\dot{\boldsymbol{d}} = \partial_t\boldsymbol{d}$ and, for all $(\boldsymbol{v}^{\mathrm{f}},\boldsymbol{v}^{\mathrm{s}}) \in \boldsymbol{W}$ and $q \in Q$,

$$\rho^{\mathrm{f}}\big(\partial_t\boldsymbol{u},\boldsymbol{v}^{\mathrm{f}}\big)_{\Omega^{\mathrm{f}}} + a(\boldsymbol{u},\boldsymbol{v}^{\mathrm{f}}) + b(p,\boldsymbol{v}^{\mathrm{f}}) - b(q,\boldsymbol{u}) + \rho^{\mathrm{s}}\big(\partial_t\dot{\boldsymbol{d}},\boldsymbol{v}^{\mathrm{s}}\big)_{\Omega^{\mathrm{s}},h} + a^{\mathrm{e}}(\boldsymbol{d},\boldsymbol{v}^{\mathrm{s}}) + a^{\mathrm{v}}(\dot{\boldsymbol{d}},\boldsymbol{v}^{\mathrm{s}}) = l(\boldsymbol{v}^{\mathrm{f}}) \quad (4)$$

Generalized Robin-Neumann schemes

Piecewise affine finite element spaces are denoted by $\boldsymbol{V}_h^{\mathrm{f}} \subset \boldsymbol{V}^{\mathrm{f}}$, $Q_h \subset Q$, $\boldsymbol{V}_h^{\mathrm{s}} \subset \boldsymbol{V}^{\mathrm{s}}$, where the subscript $h>0$ indicates the level of spatial refinement. We will consider the standard solid- and fluid-sided discrete lifting operators, $\mathcal{L}_h^{\mathrm{s}}$ and $\mathcal{L}_h^{\mathrm{f}}$ and the interface operator defined by $\boldsymbol{B}_h \stackrel{\mathrm{def}}{=} (\mathcal{L}_h^{\mathrm{s}})^*\mathcal{L}_h^{\mathrm{s}}$, where $(\mathcal{L}_h^{\mathrm{s}})^*$ stands for the adjoint operator of $\mathcal{L}_h^{\mathrm{s}}$ with respect to the lumped-mass inner product in $\boldsymbol{V}_h^{\mathrm{s}}$, denoted by $(\cdot,\cdot)_{\Omega^{\mathrm{s}},h}$. In the specific case of the **coupling with a thin solid** (i.e. $\Omega^{\mathrm{s}} = \Sigma$), we simply have $\boldsymbol{B}_h = \boldsymbol{Id}_{\Sigma}$.

The notation $x^\star$ denotes the r-th order extrapolation of x, namely $x^\star = 0$ if $r=0$, $x^\star = x^{n-1}$ if $r=1$ and $x^\star = 2x^{n-1} - x^{n-2}$ if $r=2$.

Algorithm 1: Explicit Generalized-Robin-Neumann scheme

For $n > r$,

1. Fluid step (generalized Robin): find $(\boldsymbol{u}_h^n,p_h^n) \in \boldsymbol{V}_h^{\mathrm{f}} \times Q_h$ such that, for all $(\boldsymbol{v}_h^{\mathrm{f}},q_h) \in \boldsymbol{V}_h^{\mathrm{f}} \times Q_h$,

$$\begin{cases} \rho^{\mathrm{f}}\big(\partial_\tau\boldsymbol{u}_h^n,\boldsymbol{v}_h^{\mathrm{f}}\big)_{\Omega^{\mathrm{f}}} + a(\boldsymbol{u}_h^n,\boldsymbol{v}_h^{\mathrm{f}}) + b(p_h^n,\boldsymbol{v}_h^{\mathrm{f}}) - b(q_h,\boldsymbol{u}_h^n) + s_h(p_h^n,q_h) + \dfrac{\rho^{\mathrm{s}}}{\tau}\big(\boldsymbol{B}_h\boldsymbol{u}_h^n,\boldsymbol{v}_h^{\mathrm{f}}\big)_{\Sigma}\\ \quad = \dfrac{\rho^{\mathrm{s}}}{\tau}\big(\boldsymbol{B}_h(\dot{\boldsymbol{d}}_h^{n-1} + \tau\partial_\tau\dot{\boldsymbol{d}}_h^\star),\boldsymbol{v}_h^{\mathrm{f}}\big)_{\Sigma} + \rho^{\mathrm{f}}\big(\partial_\tau\boldsymbol{u}_h^\star,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{f}}\big)_{\Omega^{\mathrm{f}}} + a(\boldsymbol{u}_h^\star,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{f}}) + b(p_h^\star,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{f}}) + l(\boldsymbol{v}_h^{\mathrm{f}}) \end{cases}$$

2. Solid step (Neumann): find $(\dot{\boldsymbol{d}}_h^n,\boldsymbol{d}_h^n) \in \boldsymbol{V}_h^{\mathrm{s}} \times \boldsymbol{V}_h^{\mathrm{s}}$ such that $\dot{\boldsymbol{d}}_h^n = \partial_\tau\boldsymbol{d}_h^n$ and, for all $\boldsymbol{v}_h^{\mathrm{s}} \in \boldsymbol{V}_h^{\mathrm{s}}$,

$$\rho^{\mathrm{s}}\big(\partial_\tau\dot{\boldsymbol{d}}_h^n,\boldsymbol{v}_h^{\mathrm{s}}\big)_{\Omega^{\mathrm{s}},h} + a^{\mathrm{e}}(\boldsymbol{d}_h^n,\boldsymbol{v}_h^{\mathrm{s}}) + a_h^{\mathrm{v}}(\dot{\boldsymbol{d}}_h^n,\boldsymbol{v}_h^{\mathrm{s}}) = -\rho^{\mathrm{f}}\big(\partial_\tau\boldsymbol{u}_h^n,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{s}}\big)_{\Omega^{\mathrm{f}}} - a(\boldsymbol{u}_h^n,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{s}}) - b(p_h^n,\mathcal{L}_h^{\mathrm{f}}\boldsymbol{v}_h^{\mathrm{s}})$$

The main features of these schemes are:

- The so-called **Robin consistency** of the generalized-Robin interface condition for the fluid, since it can formally be interpreted as the discrete counterpart of $\boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{u},p)\boldsymbol{n}^{\mathrm{f}} + \rho^{\mathrm{s}}\boldsymbol{B}_h\partial_t\boldsymbol{u} = \rho^{\mathrm{s}}\boldsymbol{B}_h\partial_t\dot{\boldsymbol{d}} - \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{d},\dot{\boldsymbol{d}})\boldsymbol{n}^{\mathrm{s}}$.
- A full fluid-solid splitting obtain through appropriate extrapolations of the solid velocity and stress on the interface and mass-lumping approximation in the structure inertia. The implicit treatment of the solid inertia ensures added-mass free stability.
- A standard solid Neumann step.

Stability and convergence analysis

Theorem 1. *Assume that $\boldsymbol{f}^{\Gamma} = \boldsymbol{0}$ (free system) and let $\{(\boldsymbol{u}_h^n,p_h^n,\boldsymbol{d}_h^n,\dot{\boldsymbol{d}}_h^n)\}_{n>r}$ be the sequence given by Algorithm 1. Suppose, if $r=2$, that the following condition hold:*

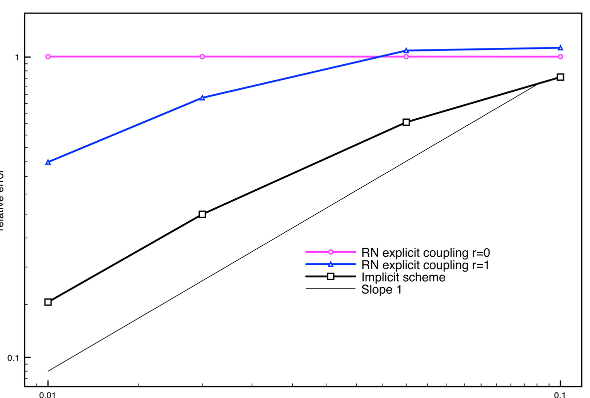
$$\begin{cases} \tau\left(\alpha + \beta\left(\dfrac{\omega_{\mathrm{e}}}{h}\right)^2\right) < \delta,\\ \tau^5\left(\dfrac{\omega_{\mathrm{e}}}{h}\right)^6 + \tau^2\left(\dfrac{\omega_{\mathrm{e}}}{h}\right)^2\left(\alpha + \beta\left(\dfrac{\omega_{\mathrm{e}}}{h}\right)^2\right) < \gamma, \end{cases}$$

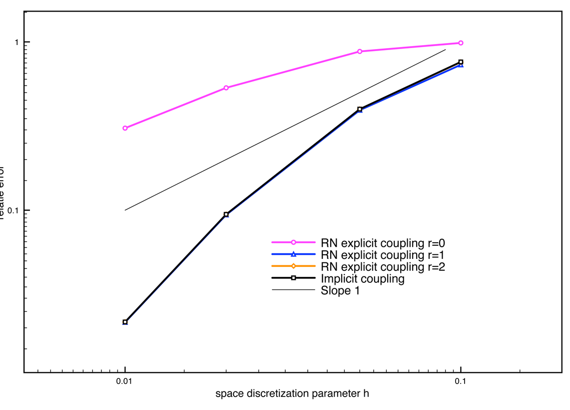
where $\omega_{\mathrm{e}} \stackrel{\mathrm{def}}{=} C_{\mathrm{inv}}\sqrt{\beta_{\mathrm{e}}/\rho^{\mathrm{s}}}$, $0 \le \delta \le 1$ and $\tau\gamma < 1$. Then the schemes are energy stable.

Theorem 2. *Let $(\boldsymbol{u},p,\boldsymbol{d},\dot{\boldsymbol{d}})$ be the solution of the coupled problem (4) and $\{(\boldsymbol{u}_h^n,p_h^n,\boldsymbol{d}_h^n,\dot{\boldsymbol{d}}_h^n)\}_{n>r}$ be the discrete solution given by Algorithm 1 with appropriate initial data. We assume that the exact solution has enough regularity. Then, we have the following error estimates, for $n>r$ such as $n\tau < T$:*

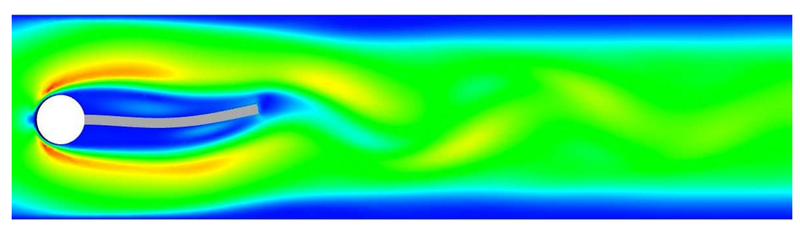
$$\mathcal{E}_h^n \lesssim c_1 h + c_2\tau + c_3\begin{cases} \tau^{2^{r-1}} & \text{coupling with thin-walled structures,}\\ \tau^{2^{r-1}}/\sqrt{h} & \text{coupling with thick-walled structures.} \end{cases}$$

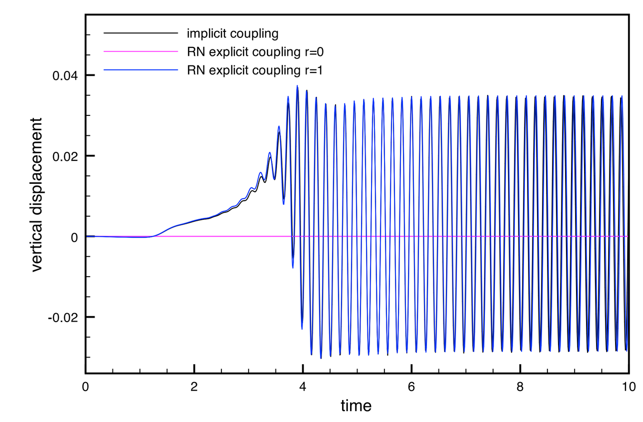
Numerical results



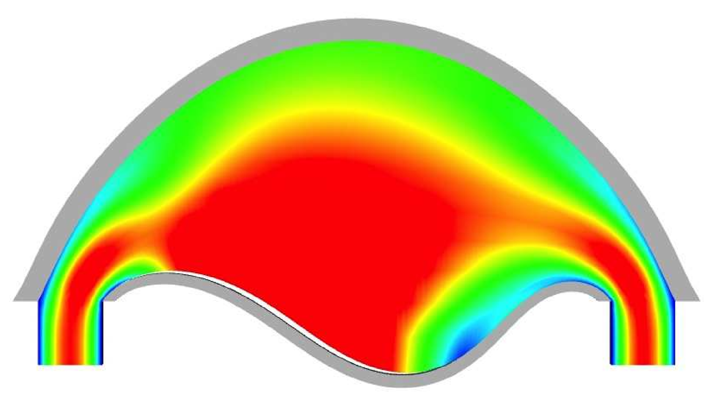


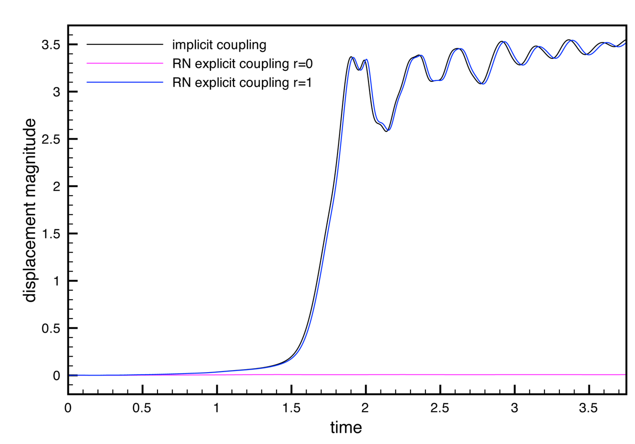
The left hand side figure reports the convergence history for a thick-walled solid and for $\tau = \mathcal{O}(h)$ whereas the right hand side figure uses the scaling $\tau = \mathcal{O}(h^2)$ with an undamped solid (i.e. $\alpha = \beta = 0$).



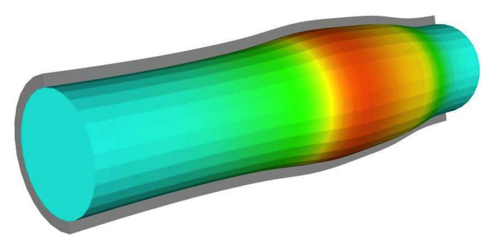


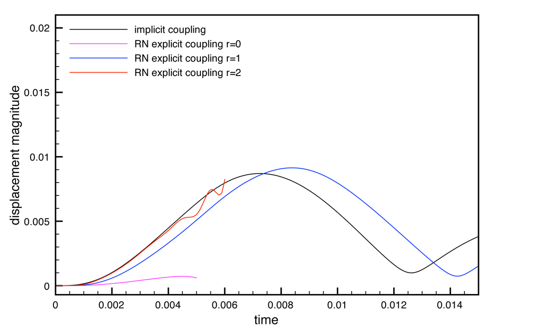
In this simulation of a flow around an elastic structure (using 3D Navier-Stokes equation in ALE formulation for the fluid), the solution given by our scheme with $r=1$ remains very close to the one given by the implicit scheme ($\tau = 10^{-3}$).

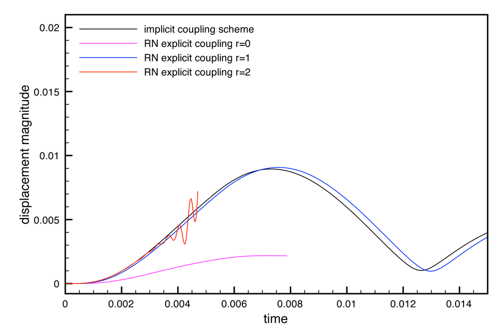




In this 3D-"baloon-type" example, where a parabolic velocity is enforced at both boundaries, we have reported the interface mid-point displacement magnitude of the bottom structure ($\tau = 0.025$).







This is the well-known example of the flow within a straight damped elastic tube ($\alpha = 1$, $\beta = 10^{-3}$), showing the propagation of a pressure-wave. Left: snapshot, center: $\tau = 10^{-4}$, right: damped solid $\tau = 4.5 \cdot 10^{-5}$. The scheme retrieves the overall dynamics of the solution provided by implicit method.

References

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[2] M. Fernández, J. Mullaert, M. Vidrascu, Generalized Robin-Neumann explicit coupling schemes for incompressible fluid-structure interaction: stability analysis and numerics, Research Report RR-8384, Inria, `http://hal.inria.fr/hal-00875819` (2013).