Splitting schemes for incompressible fluid/thin-walled structure interaction with unfitted meshes

Miguel Á. FERNÁNDEZ, Mikel LANDAJUELA

REO project-team, Inria Paris-Rocquencourt & UPMC Univ Paris VI, LJLL, France

INTRODUCTION

We deal with the coupled problem describing the interaction between a viscous incompressible fluid and a thin-walled elastic structure. For this kind of problem, significant progress has been achieved in the development and analysis of stable and accurate explicit coupling schemes when fitted meshes are used. For much applications, it is well known however that this assumption rapidly becomes cumbersome (e.g., large interface deflections, contacting structures, etc.).

Within the unfitted mesh framework splitting schemes of explicit nature are reported and analyzed in [1] using the finite element immersed boundary method, and in [2] for an unfitted Nitsche method. A major drawback of these approaches is that either stability or accuracy demands severe time-step restrictions (e.g., parabolic-CFL) or correction iterations.

Here we present two new numerical methods (semi-implicit and explicit) which bypass the above stability and accuracy issues. Their semi-implicit or explicit nature depends on the order in which the spatial and time discretizations are performed. These methods generalize the Robin-Neumann splitting paradigm of [3] to the unfitted mesh framework. A priori energy and error estimates are stated for some of the variants. Their performance is illustrated via numerical experiments in a benchmark.

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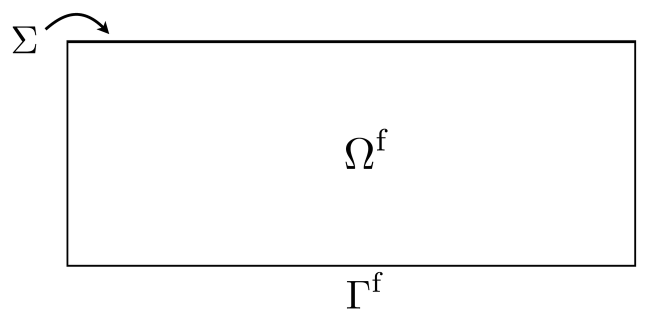
THE LINEAR COUPLED PROBLEM

We consider the following linear coupled problem:

- find the fluid velocity $\boldsymbol{u} : \Omega^{\rm f} \times \mathbb{R}^+ \to \mathbb{R}^d$,
- the fluid pressure $p : \Omega^{\rm f} \times \mathbb{R}^+ \to \mathbb{R}$,
- the structure displacement $\boldsymbol{d} : \Sigma \times \mathbb{R}^+ \to \mathbb{R}^d$
- and the structure velocity $\dot{\boldsymbol{d}} : \Sigma \times \mathbb{R}^+ \to \mathbb{R}^d$, such that



(Fluid)

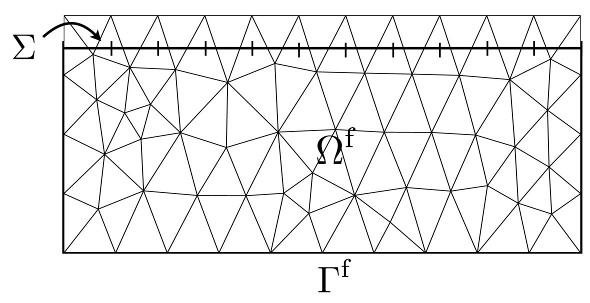
$$\begin{cases} \rho^{\rm f}\partial_t \boldsymbol{u} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{0} & \text{in} \quad \Omega^{\rm f}, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega^{\rm f}, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \Gamma^{\rm f}, \\ \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on} \quad \Sigma, \end{cases} \qquad (1)$$

(Solid)

$$\begin{cases} \rho^{\rm s}\epsilon\partial_t \dot{\boldsymbol{d}} + \boldsymbol{L}\boldsymbol{d} = -\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} & \text{in} \quad \Sigma, \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{in} \quad \Sigma, \\ \boldsymbol{d} = \boldsymbol{0} & \text{on} \quad \partial\Sigma, \end{cases} \qquad (2)$$

FIRST DISCRETIZE IN SPACE AND THEN IN TIME : SEMI-IMPLICIT SCHEMES

- We consider a quasi-uniform family of meshes $\{\mathcal{T}_h^{\rm f}\}_{0<h\leq 1}$ which cover the fluid domain. Each $\mathcal{T}_h^{\rm f}$ is fitted to the boundary $\Gamma^{\rm f}$ but not to Σ.
- Problem (1)-(2) is discretized in space as follows (See [2]): for $t > 0$, find $\big(\boldsymbol{u}_h(t), p_h(t), \dot{\boldsymbol{d}}_h(t), \boldsymbol{d}_h(t)\big) \in \boldsymbol{V}_h^{\rm f} \times Q_h \times \boldsymbol{V}_h^{\rm s} \times \boldsymbol{V}_h^{\rm s}$, such that $\dot{\boldsymbol{d}}_h = \partial_t \boldsymbol{d}_h$ and



$$\begin{cases} \rho^{\rm f}\big(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h^{\rm f}\big)_{\Omega^{\rm f}} + a^{\rm f}\big((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h^{\rm f},q_h)\big) + \rho^{\rm s}\epsilon\big(\partial_t\dot{\boldsymbol{d}}_h, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + a^{\rm s}(\boldsymbol{d}_h,\boldsymbol{v}_h^{\rm s}) + s_h\big((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h^{\rm f},q_h)\big) \\ - \big(\boldsymbol{\sigma}(\boldsymbol{u}_h,p_h)\boldsymbol{n}, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma - \big(\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma + \dfrac{\gamma\mu}{h}\big(\boldsymbol{u}_h - \dot{\boldsymbol{d}}_h, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma = 0. \end{cases} \qquad (3)$$

Here, $\gamma > 0$ denotes the Nitsche's penalty parameter and s_h the stabilization bilinear form $\quad s_h\big((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h^{\rm f},q_h)\big) \stackrel{\rm def}{=} \sum_{K\in\mathcal{T}_h^{\rm f}}\Big[\frac{\gamma_{\rm p}h^2}{\mu}\big(\boldsymbol{\nabla}p_h,\boldsymbol{\nabla}q_h\big)_K + \gamma_{\rm g}\mu h\big([\![\boldsymbol{\nabla}\boldsymbol{u}_h]\!]_{\partial K}, [\![\boldsymbol{\nabla}\boldsymbol{v}_h^{\rm f}]\!]_{\partial K}\big)_{\partial K}\Big]$.

- Problem (3) is aproximated in time with an incremental displacement-correction scheme (See [3]). The resulting scheme is displayed in Algorithm 1. The implicit treatment of the solid inertia, through an intermediate solid velocity, guarantees (added-mass free) stability, while the extrapolation of the solid elastic terms introduces a certain degree of fluid-solid splitting. In the following, $x^\star$ denotes the r-th order extrapolation, that is, $x = 0$ if $r = 0$, $x^\star = x^{n-1}$ if $r = 1$ and $x^\star = 2x^{n-1} - x^{n-2}$ if $r = 2$.

Algorithm 1: Semi-implicit coupling scheme

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1. Fluid with solid inertia sub-step: find $\big(\boldsymbol{u}_h^n, p_h^n, \dot{\boldsymbol{d}}_h^{n-\frac{1}{2}}\big) \in \boldsymbol{V}_h^{\rm f} \times Q_h \times \boldsymbol{V}_h^{\rm s}$ such that

$$\begin{cases} \rho^{\rm f}\big(\partial_\tau \boldsymbol{u}_h^n, \boldsymbol{v}_h^{\rm f}\big)_{\Omega^{\rm f}} + a^{\rm f}\big((\boldsymbol{u}_h^n,p_h^n),(\boldsymbol{v}_h^{\rm f},q_h)\big) + \dfrac{\rho^{\rm s}\epsilon}{\tau}\big(\dot{\boldsymbol{d}}_h^{n-\frac{1}{2}} - \dot{\boldsymbol{d}}_h^{n-1}, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + a^{\rm s}(\boldsymbol{d}_h^\star,\boldsymbol{v}_h^{\rm s}) + s_h\big((\boldsymbol{u}_h^n,p_h^n),(\boldsymbol{v}_h^{\rm f},q_h)\big) \\ - \big(\boldsymbol{\sigma}(\boldsymbol{u}_h^n,p_h^n)\boldsymbol{n}, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma - \big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-\frac{1}{2}}, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma + \dfrac{\gamma\mu}{h}\big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma = 0 \end{cases}$$

2. Solid sub-step: find $\big(\dot{\boldsymbol{d}}_h^n, \boldsymbol{d}_h^n\big) \in \boldsymbol{V}_h^{\rm s} \times \boldsymbol{V}_h^{\rm s}$ such that $\dot{\boldsymbol{d}}_h^n = \partial_\tau \boldsymbol{d}_h^n$ and

$$\rho^{\rm s}\epsilon\big(\partial_\tau\dot{\boldsymbol{d}}_h^n, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + a^{\rm s}(\boldsymbol{d}_h^n,\boldsymbol{v}_h^{\rm s}) + \big(\boldsymbol{\sigma}(\boldsymbol{u}_h^n,p_h^n)\boldsymbol{n}, \boldsymbol{v}_h^{\rm s}\big)_\Sigma - \frac{\gamma\mu}{h}\big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-\frac{1}{2}}, \boldsymbol{v}_h^{\rm s}\big)_\Sigma = 0$$

- The following result states the stability and convergence properties of Algorithm 1. Note that with $r = 1$, it delivers unconditional optimal overall first-order accuracy. This is a significant progress with respect to the splitting schemes reported in [2], whose accuracy is known to be non-uniform in h.

Proposition 1. *Let $\{(\boldsymbol{u}_h^n, p_h^n, \dot{\boldsymbol{d}}_h^n, \boldsymbol{d}_h^n)\}_{n\geq 1}$ be given by Algorithm 1. For $\gamma > 0$ sufficiently large, there holds $E_h^n \lesssim E_h^0$ unconditionally for $r \in \{0,1\}$ and under the condition $\tau = \mathcal{O}(h^{\frac{6}{5}})$ for $r = 2$. Moreover, under the same conditions, there holds $\mathcal{E}_h^n \lesssim h + \tau + \tau^{2^{r-1}}$.*

FIRST DISCRETIZE IN TIME AND THEN IN SPACE : EXPLICIT SCHEMES

- We consider the time semi-discretization of problem (1)-(2) proposed in [3]. It is based on the following explicit Robin-Neumann time splitting on the interface:

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n} + \kappa\boldsymbol{u}^n = \kappa\dot{\boldsymbol{d}}^{n-1} + \boldsymbol{g}^\star & \text{on} \quad \Sigma, \\ \rho^{\rm s}\epsilon\partial_\tau\dot{\boldsymbol{d}}^n + \boldsymbol{L}\boldsymbol{d}^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n} & \text{on} \quad \Sigma, \end{cases} \qquad (4)$$

with $\boldsymbol{g}^\star \stackrel{\rm def}{=} \rho^{\rm s}\epsilon\partial_\tau\boldsymbol{d}^\star + \boldsymbol{\sigma}(\boldsymbol{u}^\star,p^\star)\boldsymbol{n}$ and $\kappa \stackrel{\rm def}{=} \rho^{\rm s}\epsilon/\tau$.

- The fundamental idea consists in performing directly an unfitted interface treatment (*à la* Nitsche) of the Robin-Neumann time splitting (4), by extending the arguments introduced in [2, 4]. The proposed fully discrete scheme is based on the following result:

Proposition 2. *Let $\{(\boldsymbol{u}^n, p^n, \dot{\boldsymbol{d}}^n, \boldsymbol{d}^n)\}_{n\geq 1}$ be given by the time semi-discretization of problem (1)-(2) with the transmission conditions (4). Then, there holds*

$$\begin{cases} \rho^{\rm f}\big(\partial_\tau\boldsymbol{u}^n, \boldsymbol{v}_h^{\rm f}\big)_{\Omega^{\rm f}} + a^{\rm f}\big((\boldsymbol{u}^n,p^n),(\boldsymbol{v}_h^{\rm f},q_h)\big) + \rho^{\rm s}\epsilon\big(\partial_\tau\dot{\boldsymbol{d}}^n, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + a^{\rm s}(\boldsymbol{d}^n,\boldsymbol{v}_h^{\rm s}) \\ + \dfrac{\gamma\kappa\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{u}^n - \dot{\boldsymbol{d}}^{n-1}, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma - \dfrac{\kappa h}{\gamma\mu+\kappa h}\Big[\big(\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n}, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma + \big(\boldsymbol{u}^n - \dot{\boldsymbol{d}}^{n-1}, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma\Big] \\ - \dfrac{h}{\gamma\mu+\kappa h}\big(\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n}, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma - \dfrac{\gamma\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{g}^\star, \boldsymbol{v}_h^{\rm f} - \boldsymbol{v}_h^{\rm s}\big)_\Sigma + \dfrac{h}{\gamma\mu+\kappa h}\big(\boldsymbol{g}^\star, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma = 0 \end{cases} \qquad (5)$$

- A salient feature of (5) is the fact that for $\kappa \to \infty$ (i.e. $\tau \to 0$) we formally retrieve the unfitted formulation (3). Taking successively $\boldsymbol{v}_h^{\rm s} = \boldsymbol{0}$ and $(\boldsymbol{v}_h^{\rm f}, q_h) = (\boldsymbol{0}, 0)$ in (5) we obtain the fully discrete method reported in Algorithm 2.

Algorithm 2: Explicit coupling scheme

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1. Fluid sub-step: find $\big(\boldsymbol{u}_h^n, p_h^n\big) \in \boldsymbol{V}_h^{\rm f} \times Q_h$ such that

$$\begin{cases} \rho^{\rm f}\big(\partial_\tau\boldsymbol{u}_h^n, \boldsymbol{v}_h^{\rm f}\big)_{\Omega^{\rm f}} + a^{\rm f}\big((\boldsymbol{u}_h^n,p_h^n),(\boldsymbol{v}_h^{\rm f},q_h)\big) + s_h\big((\boldsymbol{u}_h,p_h),(\boldsymbol{v}_h^{\rm f},q_h)\big) + \dfrac{\gamma\kappa\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-1}, \boldsymbol{v}_h^{\rm f}\big)_\Sigma \\ - \dfrac{\kappa h}{\gamma\mu+\kappa h}\big[\big(\boldsymbol{\sigma}(\boldsymbol{u}_h^n,p_h^n)\boldsymbol{n}, \boldsymbol{v}_h^{\rm f}\big)_\Sigma + \big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-1}, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma\big] - \dfrac{\gamma\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{g}_h^\star, \boldsymbol{v}_h^{\rm f}\big)_\Sigma \\ - \dfrac{h}{\gamma\mu+\kappa h}\big(\boldsymbol{\sigma}(\boldsymbol{u}_h^n,p_h^n)\boldsymbol{n}, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma + \dfrac{h}{\gamma\mu+\kappa h}\big(\boldsymbol{g}_h^\star, \boldsymbol{\sigma}(\boldsymbol{v}_h^{\rm f},-q_h)\boldsymbol{n}\big)_\Sigma = 0 \end{cases}$$

2. Solid sub-step: find $\big(\dot{\boldsymbol{d}}_h^n, \boldsymbol{d}_h^n\big) \in \boldsymbol{V}_h^{\rm s} \times \boldsymbol{V}_h^{\rm s}$ such that $\dot{\boldsymbol{d}}_h^n = \partial_\tau\boldsymbol{d}_h^n$ and

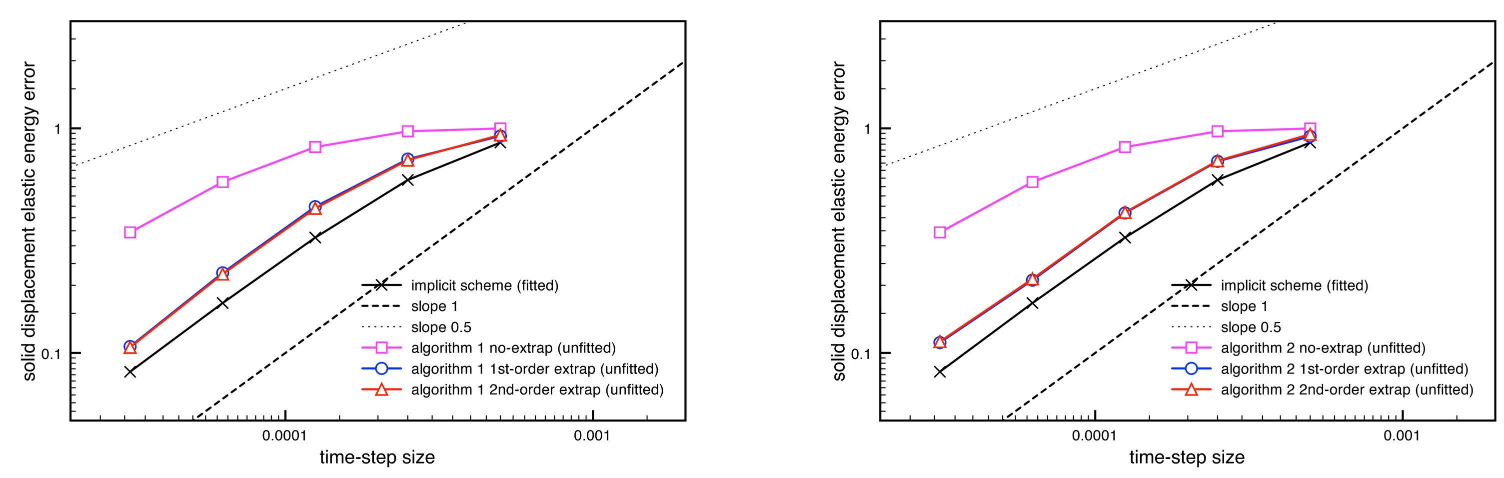
$$\begin{cases} \rho^{\rm s}\epsilon\big(\partial_\tau\dot{\boldsymbol{d}}_h^n, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + a^{\rm s}(\boldsymbol{d}_h^n,\boldsymbol{v}_h^{\rm s}) + \dfrac{\kappa h}{\gamma\mu+\kappa h}\big(\boldsymbol{\sigma}(\boldsymbol{u}_h^n,p_h^n)\boldsymbol{n}, \boldsymbol{v}_h^{\rm s}\big)_\Sigma \\ - \dfrac{\gamma\kappa\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{u}_h^n - \dot{\boldsymbol{d}}_h^{n-1}, \boldsymbol{v}_h^{\rm s}\big)_\Sigma + \dfrac{\gamma\mu}{\gamma\mu+\kappa h}\big(\boldsymbol{g}_h^\star, \boldsymbol{v}_h^{\rm s}\big)_\Sigma = 0 \end{cases}$$

- The following result guarantees the energy stability of Algorithm 2 for $r = 0$.

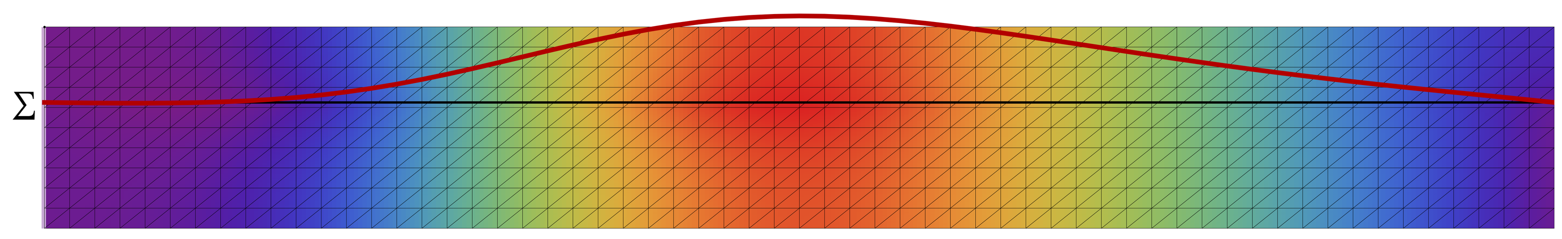
Proposition 3. *Let $\{(\boldsymbol{u}_h^n, p_h^n, \dot{\boldsymbol{d}}_h^n, \boldsymbol{d}_h^n)\}_{n\geq 1}$ be given by Algorithm 2 with $r = 0$. For $\gamma > 0$ sufficiently large, we have $E_h^n \leq E_h^0$.*

NUMERICAL RESULTS

- We consider the benchmark example of the propagation of a pressure-wave within an elastic tube (2D Stokes equations with a 1D generalized string model).
- We compare the results obtained with Algorithms 1 and 2 with those obtained with a first-order fully implicit scheme using fitted meshes. In order to highlight the uniformity of the convergence in h, we have refined both in time and in space at the same rate, $\tau = \mathcal{O}(h)$. In spite of their different semi-implicit and explicit nature, Algorithms 1 and 2 deliver practically the same behavior: stability is obtained with all the variants, optimal first-order convergence is obtained with the extrapolated variants ($r = 1, 2$, unfitted) and the implicit (fitted) scheme while a sub-optimal convergence rate is exhibited by the non-extrapolated ones ($r = 0$, unfitted).



Time convergence history of the solid displacement in the relative elastic energy norm using Algorithm 1 (left) and Algorithm 2 (right) with $\tau = \mathcal{O}(h)$.



Snapshot at time $t = 0.01$ of the fluid pressure (over the whole fictitious domain) and solid displacement (red line) obtained with Algorithm 1.
Recall that the physical fluid domain extends below the interface Σ (black line).

References

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