Alternative axiomatisations of common knowledge

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Epistemic language

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{E}\varphi \mid \mathbf{C}\varphi$$

(only one, fixed group)

- $\mathbf{K}_i\varphi$ = "agent i knows that φ"
- $\mathbf{E}\varphi = \bigwedge_{i \in Agt} \mathbf{K}_i\varphi$ = "it is shared knowledge that φ"
- $\mathbf{C}\varphi = \bigwedge_{k \geq 0} \mathbf{E}^k\varphi$ = "it is common knowledge that φ"

S5 individual knowledge

$$\boxed{\mathrm{S5}(\mathbf{K}) \;=\; \text{modal logic S5 for the modal operators } \mathbf{K}_i}$$

- truth axiom:

$$\mathbf{K}_i\varphi \to \varphi$$

- positive introspection axiom:

$$\mathbf{K}_i\varphi \to \mathbf{K}_i\mathbf{K}_i\varphi$$

- negative introspection axiom:

$$\neg\mathbf{K}_i\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\varphi$$

Shared knowledge: definition

$$\boxed{\texttt{Def}(\mathbf{E})\text{: } \mathbf{E}\varphi \leftrightarrow \bigwedge_{i \in Agt} \mathbf{K}_i\varphi}$$

- normal modal operator:
  - axiom $\texttt{K}(\mathbf{E})$ provable
  - rule of necessitation $\texttt{RN}(\mathbf{E})$ derivable
- truth axiom provable:

$$\mathbf{E}\varphi \to \varphi$$

- neither positive nor negative introspection provable
- axiom $\texttt{B}(\mathbf{E})$ provable:

$$\varphi \to \mathbf{E}\neg\mathbf{E}\neg\varphi$$

Common knowledge: basic desiderata

- truth axiom:

$$\mathbf{C}\varphi \to \varphi \text{ should be valid}$$

- positive introspection axioms:

$$\mathbf{C}\varphi \to \mathbf{E}\mathbf{C}\varphi \text{ should be valid}$$
$$\mathbf{C}\varphi \to \mathbf{C}\mathbf{C}\varphi \text{ should be valid}$$

- negative introspection axioms:

$$\neg\mathbf{C}\varphi \to \mathbf{E}\neg\mathbf{C}\varphi \text{ should be valid}$$
$$\neg\mathbf{C}\varphi \to \mathbf{C}\neg\mathbf{C}\varphi \text{ should be valid}$$

- fixed-point axiom follows:

$$\mathtt{FP} \quad \mathbf{C}\varphi \to \mathbf{E}(\varphi \wedge \mathbf{C}\varphi)$$

Common knowledge: induction principles

- two versions
  - induction axiom schema:

    $$\texttt{GFP} \quad \mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\varphi \to \mathbf{C}\varphi)$$

  - induction rule:

    $$\texttt{RGFP} \quad \text{from } \varphi \to \mathbf{E}(\varphi \wedge \psi), \text{ infer } \varphi \to \mathbf{C}\psi$$

- intuitive in temporal logics (well-founded orderings)
- doesn't 'talk' in epistemic logics

  > *"If it is the case that φ is 'self-evident', in the sense that if it is true, then everyone knows it, and, in addition, if φ is true, then everyone knows ψ, we can show by induction that if φ is true, then so is $\mathbf{E}^k(\psi \wedge \varphi)$ for all k." [vDHvdHK15]*

- aim: find a more intuitive axiom

Recap: the axiom system with induction rule

$\mathtt{S5}(\mathbf{K})$ and $\mathtt{Def}(\mathbf{E})$, plus:

$$\begin{array}{ll} \mathtt{FP} & \mathbf{C}\varphi \to \mathbf{E}(\varphi \wedge \mathbf{C}\varphi) \\ \mathtt{RGFP} & \text{from } \varphi \to \mathbf{E}(\varphi \wedge \psi)\text{, infer } \varphi \to \mathbf{C}\psi \end{array}$$

[HM92, FHMV95]

- sound and complete for S5 models
  - rule of necessitation $\mathtt{RN}(\mathbf{C})$ derivable
  - axioms $\mathtt{K}(\mathbf{C})$, $\mathtt{T}(\mathbf{C})$, $\mathtt{4}(\mathbf{C})$, $\mathtt{5}(\mathbf{C})$ provable
  - induction axiom schema $\mathtt{GFP}$ provable

Recap: the axiom system with induction axiom

$\mathtt{S5}(\mathbf{K})$ and $\mathtt{Def}(\mathbf{E})$, plus:

$$
\begin{array}{ll}
\mathtt{K}(\mathbf{C}) & \text{system K for } \mathbf{C} \\
\mathtt{FP} & \mathbf{C}\varphi \to \mathbf{E}(\varphi \wedge \mathbf{C}\varphi) \\
\mathtt{GFP} & \mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\varphi \to \mathbf{C}\varphi)
\end{array}
$$

[Leh84, HM85]

- sound and complete for S5 models
  - induction rule $\mathtt{RGFP}$ provable
  - original presentation has moreover axioms $\mathtt{T}(\mathbf{C})$, $\mathtt{4}(\mathbf{C})$, $\mathtt{5}(\mathbf{C})$
    $\Rightarrow$ redundant!

A new axiomatisation of S5 common knowledge

$\mathtt{S5}(\mathbf{K})$ and $\mathtt{Def}(\mathbf{E})$, plus:

$\mathtt{S4}(\mathbf{C})$	an axiomatisation of S4 for $\mathbf{C}$
$\mathtt{FP}_1$	$\mathbf{C}\varphi \to \mathbf{E}\varphi$
$\mathtt{GFP}_1$	$\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$

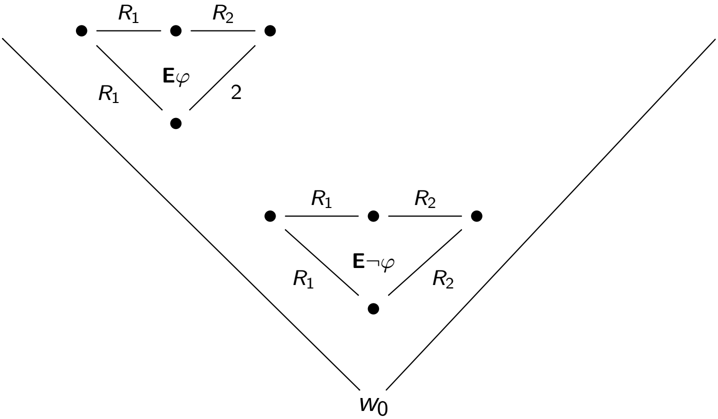
- sound for S5 models
  - $\mathtt{GFP}_1$ provable in the axiom system with induction axiom $\mathtt{GFP}$
- complete for S5 models
  - induction axiom $\mathtt{GFP}$ provable
  - proof uses $\mathtt{K}(\mathbf{C})$, $\mathtt{RN}(\mathbf{C})$,$\mathtt{T}(\mathbf{C})$ $\mathtt{4}(\mathbf{C})$
- given $\mathtt{S5}(\mathbf{K})$, $\mathtt{GFP}_1$ equivalent to the (a priori stronger):

$$\mathtt{GFP}_2 \quad \mathbf{C}\Big(\bigwedge_{i\in Agt} \mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi\Big) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$$

Semantics of the new axiom

$$\boxed{\texttt{GFP}_1: \quad \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)}$$

excludes:



(big cone $= R_{\mathbf{E}}^*(w_0)$; triangles $= R_{\mathbf{E}}$-accessible worlds)

Soundness: proof of $\mathtt{GFP}_1$

Proposition

$\mathtt{GFP}_1$ *is provable from* $\mathtt{GFP}$.

Proof.

- premise of $\mathtt{GFP}_1$ logically stronger than that of $\mathtt{GFP}$
  - relies on $\mathtt{T}(\mathbf{K})$ (or rather, its consequence $\mathtt{T}(\mathbf{E})$)
- case analysis:
  1. $\big(\varphi \wedge \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)\big) \to (\varphi \wedge \mathbf{C}(\varphi \to \mathbf{E}\varphi))$ — $\mathtt{T}(\mathbf{E})$, $\mathtt{K}(\mathbf{C})$
  2. $\big(\varphi \wedge \mathbf{C}(\varphi \to \mathbf{E}\varphi)\big) \to \mathbf{C}\varphi$ — $\mathtt{GFP}$
  3. $\big(\varphi \wedge \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\varphi)\big) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — from 1, 2
  4. $\big(\neg\varphi \wedge \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)\big) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — from 3 by uniform substitution of φ by $\neg\varphi$
  5. $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — from 3, 4

Completeness: a key lemma

Lemma

The schema $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ *is provable from the axiom schemas* $\mathtt{K}(\mathbf{C})$, $\mathtt{4}(\mathbf{C})$, $\mathtt{RN}(\mathbf{C})$, $\mathtt{T}(\mathbf{C})$, *and* $\mathtt{FP}$.

Proof.

1. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{E}(\varphi \to \mathbf{E}\varphi)$ — by $\mathtt{FP}$, $\mathtt{T}(\mathbf{C})$, $\mathtt{K}(\mathbf{E})$
2. $\mathbf{E}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{E}\neg\mathbf{E}\varphi \to \mathbf{E}\neg\varphi)$ — by $\mathtt{K}(\mathbf{E})$
3. $\neg\varphi \to \mathbf{E}\neg\mathbf{E}\varphi$ — $\mathtt{B}(\mathbf{E})$
4. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\neg\varphi \to \mathbf{E}\neg\varphi)$ — from 1, 2, 3
5. $\mathbf{CC}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ — from 4 by $\mathtt{RN}(\mathbf{C})$ and $\mathtt{K}(\mathbf{C})$
6. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{CC}(\varphi \to \mathbf{E}\varphi)$ — $\mathtt{4}(\mathbf{C})$
7. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ — from 5 and 6

Completeness: proof of GFP

Proposition

GFP *is provable from* $\mathtt{GFP}_1$.

Proof.

1. $\mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — $\mathtt{GFP}_1$
2. $\big(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)\big) \to \mathbf{C}(\mathbf{E}\varphi \vee \mathbf{E}\neg\varphi)$ — by $\mathtt{RN}(\mathbf{C})$ and $\mathtt{K}(\mathbf{C})$
3. $\big(\mathbf{C}(\varphi \to \mathbf{E}\varphi) \wedge \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)\big) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — from 1 and 2
4. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to \mathbf{C}(\neg\varphi \to \mathbf{E}\neg\varphi)$ — key lemma
5. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{C}\varphi \vee \mathbf{C}\neg\varphi)$ — from 3, 4
6. $\mathbf{C}(\varphi \to \mathbf{E}\varphi) \to (\mathbf{C}\varphi \vee \neg\varphi)$ — from 5 by $\mathtt{T}(\mathbf{C})$

Example: compatriots [LH15]

- in a conference break, two Dutch talk together in English, not knowing that they are compatriots

$$\neg(\mathbf{K}_1 d_2 \vee \mathbf{K}_1 \neg d_2) \wedge \neg(\mathbf{K}_2 d_1 \vee \mathbf{K}_2 \neg d_1) \tag{1}$$

- a third person tells them: "hey, you're compatriots"

$$\mathbf{C}(d_1 \leftrightarrow d_2) \tag{2}$$

- background knowledge:

$$\mathbf{C} \bigwedge_i \left((d_i \to \mathbf{K}_i d_i) \wedge (\neg d_i \to \mathbf{K}_i \neg d_i)\right) \tag{3}$$

- implies common knowledge that both are compatriots:

$$(2) \wedge (3) \to (\mathbf{C}(d_1 \wedge d_2) \vee \mathbf{C}(\neg d_1 \wedge \neg d_2)) \tag{4}$$

Example: compatriots (ctd.)

- common knowledge obtained through deduction, using $\mathtt{GFP}_1$
- proof:
  1. $(2) \wedge (3) \to \mathbf{C}\big(\mathbf{E}(d_1 \wedge d_2) \vee \mathbf{E}(\neg d_1 \wedge \neg d_2)\big)$
     by $\mathtt{4}(\mathbf{C})$, $\mathtt{FP}_1$, $\mathtt{K}(\mathbf{C})$
  2. $\mathbf{C}\big(\mathbf{E}(d_1 \wedge d_2) \vee \mathbf{E}(\neg d_1 \wedge \neg d_2)\big) \to \big(\mathbf{C}(d_1 \wedge d_2) \vee \mathbf{C}(\neg d_1 \wedge \neg d_2)\big)$
     consequence of $\mathtt{GFP}_1$
  3. $(2) \wedge (3) \to \big(\mathbf{C}(d_1 \wedge d_2) \vee \mathbf{C}(\neg d_1 \wedge \neg d_2)\big)$
     from 1, 2

- induction principles not so easy to apply
  - group version of the omniscience problem?
  - implicit vs. explicit common knowledge [LH15]
- often common knowledge cannot be deduced
  - cf. consecutive numbers puzzle

Commonly knowing whether

- definable from 'knowing that' operators:
  - $\mathbf{Kif}_i\varphi = \mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi$ "i knows whether φ"
  - $\mathbf{Eif}\varphi = \mathbf{E}\varphi \vee \mathbf{E}\neg\varphi$ "it is shared knowledge whether φ"
  - $\mathbf{Cif}\varphi = \mathbf{C}\varphi \vee \mathbf{C}\neg\varphi$ "it is common knowledge whether φ"
- the other way round:
  - $\mathbf{C}\varphi = \varphi \wedge \mathbf{Cif}\varphi$
  - $\mathbf{E}\varphi = \ldots$
  - $\mathbf{C}\varphi = \ldots$
- easy axiomatisation
  1. axiomatisation of $\mathbf{Kif}_i$ [FWvD15]
  2. $\mathbf{Eif}\varphi \leftrightarrow \bigwedge_{i \in Agt} \mathbf{Kif}_i\varphi$ (N.B.: does not hold for belief!)
  3. standard axiomatisation of $\mathbf{C}$, substituting $\mathbf{C}\varphi$ by $\varphi \wedge \mathbf{Cif}\varphi$
     - induction axiom GFP becomes:

$$\big((\varphi \to \mathbf{Eif}\varphi) \wedge \mathbf{Cif}(\varphi \to \mathbf{Eif}\varphi)\big) \to (\varphi \to \mathbf{Cif}\varphi)$$

$\Rightarrow$ not easy to parse

A nicer axiomatisation of commonly knowing whether

- $\mathtt{FP}_1$ and $\mathtt{GFP}_1$ become:

$$\begin{array}{ll} \mathtt{FP}_2 & \textbf{Cif}\varphi \to \textbf{Eif}\varphi \\ \mathtt{GFP}_2 & \textbf{Cif}\,\textbf{Eif}\varphi \to \big(\textbf{Eif}\varphi \to \textbf{Cif}\varphi\big) \end{array}$$

Conclusion

- $\mathtt{GFP}_1$ 'talks to us'
  - more intuitive than the standard induction principles
- what about logics weaker than S5?
  - $\mathtt{GFP}_2$ surely not a reasonable principle of common belief:
    - common belief that each of us has an opinion about φ does not imply common belief about φ
  - the weaker $\mathtt{GFP}_1$ not reasonable either!
  - counterexample: misunderstanding in conversation
    - $\mathbf{B}_1\,\mathbf{CB}\,p \wedge \mathbf{B}_2\,\mathbf{CB}\,\neg p$
    - consequence: $\mathbf{B}_1\,\mathbf{CB}\,\mathbf{EB}\,p \wedge \mathbf{B}_2\,\mathbf{CB}\,\mathbf{EB}\,\neg p$
    - consequence: $\mathbf{B}_1\,\mathbf{CB}\,(\mathbf{EB}\,p \vee \mathbf{EB}\,\neg p) \wedge \mathbf{B}_2\,\mathbf{CB}\,(\mathbf{EB}\,p \vee \mathbf{EB}\,\neg p)$
    - consequence: $\mathbf{CB}\,(\mathbf{EB}\,p \vee \mathbf{EB}\,\neg p)$

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