Reasoning about Knowledge and Strategies

Bastien Maubert and Aniello Murano

Università degli Studi di Napoli
Federico II

Program synthesis

Basic idea:

"Program synthesis is the task to automatically construct a program that satisfies a given high-level specification."

We are interested in programs that:

- Interact with an environment
- May run forever

Example: operating systems, controllers in power plants. . .

Specification language: LTL

Propositional logic +

- $\mathbf{X}\varphi$: "φ holds at next step"
- $\varphi\mathbf{U}\psi$: "φ will hold until ψ holds"
- $\mathbf{G}\varphi$: "φ always holds"
- $\mathbf{F}\varphi$: "φ eventually holds"

LTL synthesis (Pnueli and Rosner, 1989)

- I: input variables,
- O: output variables

Game between system and environment

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LTL synthesis problem

Given a specification $\varphi \in$ LTL over $I \cup O$, synthesize a strategy $\sigma : (2^I)^* \to 2^O$ such that all resulting behaviours satisfy φ.

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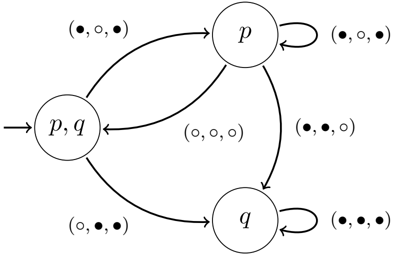
What about synthesis of distributed systems?

Distributed synthesis



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$\circ, \bullet$ are *actions*

strategies σ : Histories $\to$ Actions

Input: A concurrent game structure and a formula $\varphi \in$ LTL

Output: A distributed strategy to enforce φ

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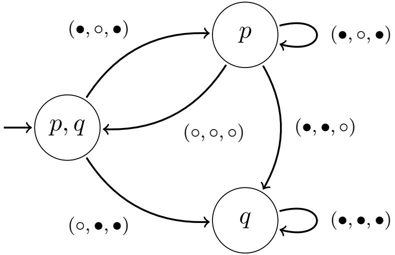

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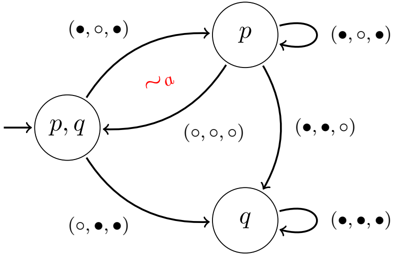

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Imperfect information

1. Strategies must be consistent with players' information

Constraint on strategies:

If $h \sim_a h'$, then $\sigma_a(h) = \sigma_a(h')$.

2. Makes epistemic reasoning meaningful and useful

Example: opacity

A system is *opaque* for property P if a spy never knows whether the current execution is in P.

Classic definition:

$$\forall h, \exists h' \text{ s.t. } h \sim_{\mathsf{spy}} h' \text{ and } h' \notin P$$

With epistemic temporal logic:

$$\mathbf{G}\neg K_{\mathsf{spy}} P$$

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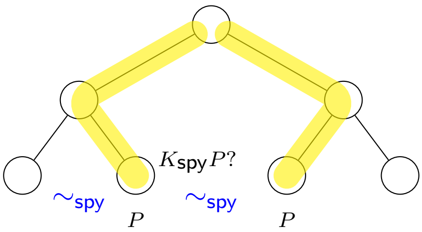
Semantics of knowledge when reasoning about strategies

Yellow subtree: controller's strategy

$\sim_{\text{spy}}$: spy's indistinguishability relation



Two possible semantics:

- spy ignores controller's strategy
  $\rightarrow K_{\text{spy}} P$ does not hold
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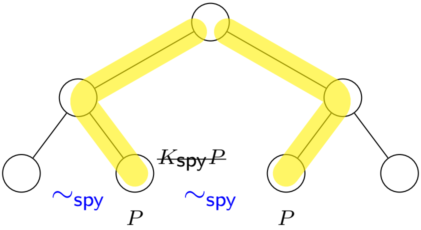

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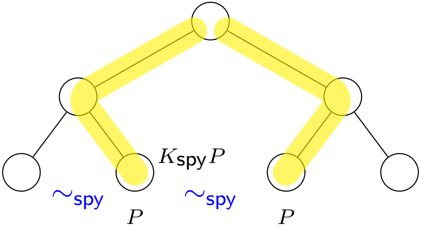

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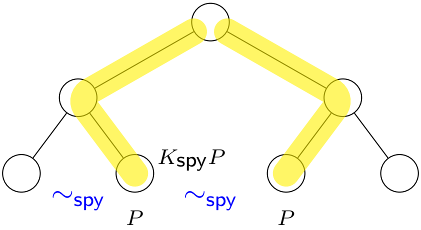

Two possible semantics:

- spy ignores controller's strategy — Uninformed semantics
  $\rightarrow K_{\textsf{spy}}P$ does not hold
- spy knows controller's strategy — Informed semantics
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In the litterature

Both semantics have been used, but implicitely.

Informed semantics:

Distributed synthesis from epistemic temporal specifications

- van der Meyden and Vardi, 1998
- van der Meyden and Wilke, 2005

Uninformed semantics:

All epistemic extensions of ATL and SL (that we know of)

One paper talks about this issue: Puchala, 2010

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What is known about distributed synthesis?

Peterson and Reif (1979), Pnueli and Rosner (1990)

Distributed synthesis for reachability objective is undecidable.

Two known ways of retrieving decidability for temporal objectives:

1. Public actions
2. Hierarchical information

For epistemic temporal objectives and

- informed semantics:
  - decidable for public actions
  - undecidable for hierarchical information

  [van der Meyden and Wilke, 2005]
- uninformed semantics:
  - decidable for public actions [Belardinelli et al., 2017]
  - decidable for hierarchical information [Puchala, 2010]

Overview

Peterson and Reif (1979), Pnueli and Rosner (1990)

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SL with imperfect information and knowledge

SL (Chatterjee et al. 2010, Mogavero et al. 2014)

LTL +

- $\exists \sigma\, \varphi$

  "there exists a strategy σ s.t. φ"

- $(a, \sigma)\varphi$

  "when player a plays strategy σ, φ"

SL with imperfect information and knowledge

SL$_{\mathrm{ii}}$ (Berthon et al. 2017)

LTL $+$

- $\exists^{o}\sigma\,\varphi$
  "there exists a strategy σ with observational power o s.t. φ"
- $(a,\sigma)\varphi$
  "when player a plays strategy σ, φ"

SL with imperfect information and knowledge

ESL (M. and Murano, 2018)

LTL +

- $\exists^{o}\sigma\,\varphi$
  "there exists a strategy σ with observational power o s.t. φ"
- $(a, \sigma)\varphi$
  "when player a plays strategy σ, φ"
- $K_a\varphi$
  "player a knows that φ"
- $\mathbf{A}\varphi$
  "in all outcomes, φ"

What can ESL express?

- Distributed synthesis:

$$\exists^{o_1} x_1 \, \dots \exists^{o_n} x_n \, (a_1, x_1) \dots (a_n, x_n) \forall^{o_e} y \, (e, y) \, \psi$$

- Existence of Nash equilibria:

$$\exists^{o_a} x \, \exists^{o_b} y \, \exists^{o_c} z \, (a, x)(b, y)(c, z) \bigwedge_{d \in \{a,b,c\}} \exists^{o_d} x' \, (d, x') \, \mathsf{Win}_d \to \mathsf{Win}_d$$

- Players changing observation:

$$\exists^{o_1} x_1 \, (a, x_1) \mathbf{AF} \exists^{o_2} x_2 \, (a, x_2) \mathbf{AF} \, \mathsf{Win}_a$$

"First I find my glasses, then I play for real."

Main result

Hierarchical instances

An ESL formula Φ is *hierarchical* if:

- innermost strategies observe better than outermost ones
- epistemic subformulas do not talk about current strategies

Considering the uninformed semantics of knowledge:

Theorem

Model-checking hierarchical instances of ESL is decidable.

Corollaries:

On systems with hierarchical information,
for epistemic temporal specifications with uninformed semantics,

We can solve

- distributed synthesis,
- module checking,
- synthesis of Nash equilibria,
- rational synthesis,
- ...

Interested? Come to Napoli!

Rational distributed synthesis

Rational synthesis

Fisman et al. (2010), Condurache et al. (2016), Kupferman et al. (2016)

- Environment made of several components $\{e_1, \ldots, e_m\}$
- individual LTL goals ψ_i
- System made of one component a
- LTL specification ψ_g

$$\Phi_{\text{c-RS}} := \exists x\, \exists y_1 \ldots \exists y_m\, (\boldsymbol{a}, \boldsymbol{x})(\boldsymbol{e}, \boldsymbol{y})\, \varphi_{\mathsf{NE}} \wedge \mathbf{A}\psi_g$$
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