

Learning Stochastic Systems with Global PAC Bounds

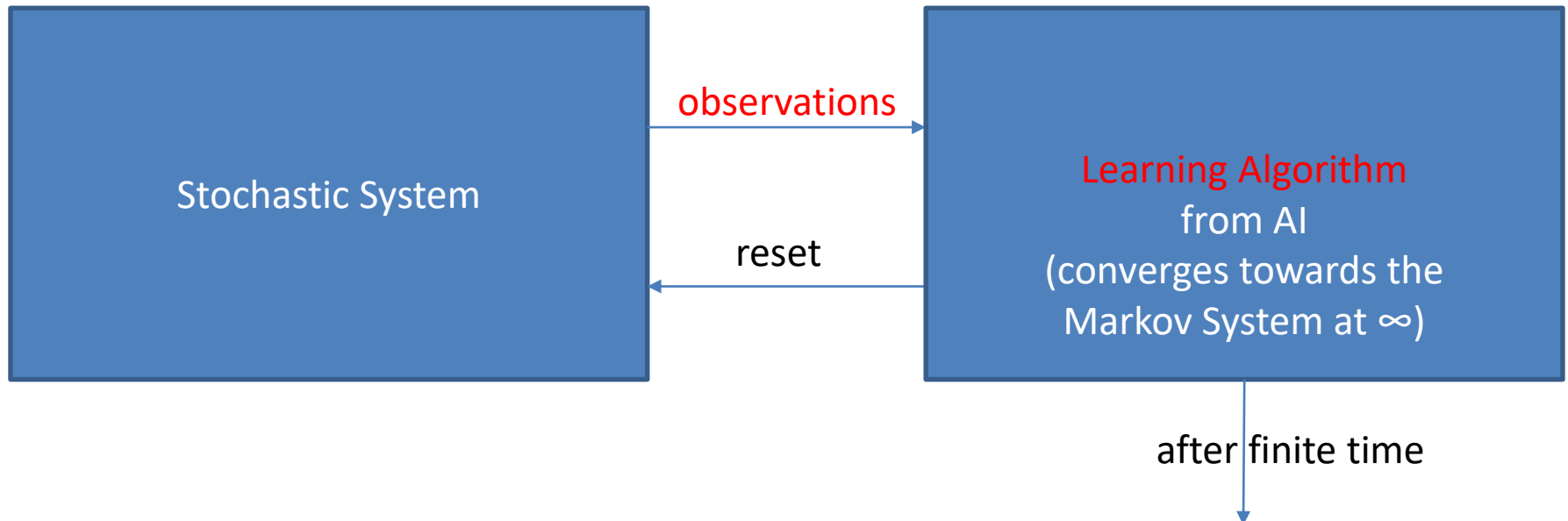
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Formalism

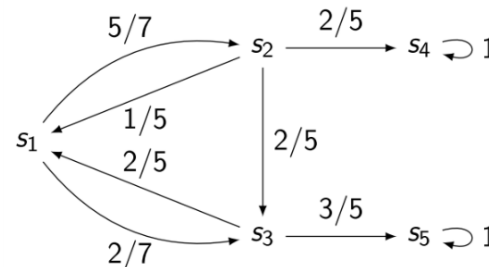


Information on the system:
Set of states – known in this talk
Support of transitions – depends
Transition probabilities – never

Observation W:
Sequence of States observed
e.g.: $s_1 s_2 s_3 s_1 s_3 s_5$

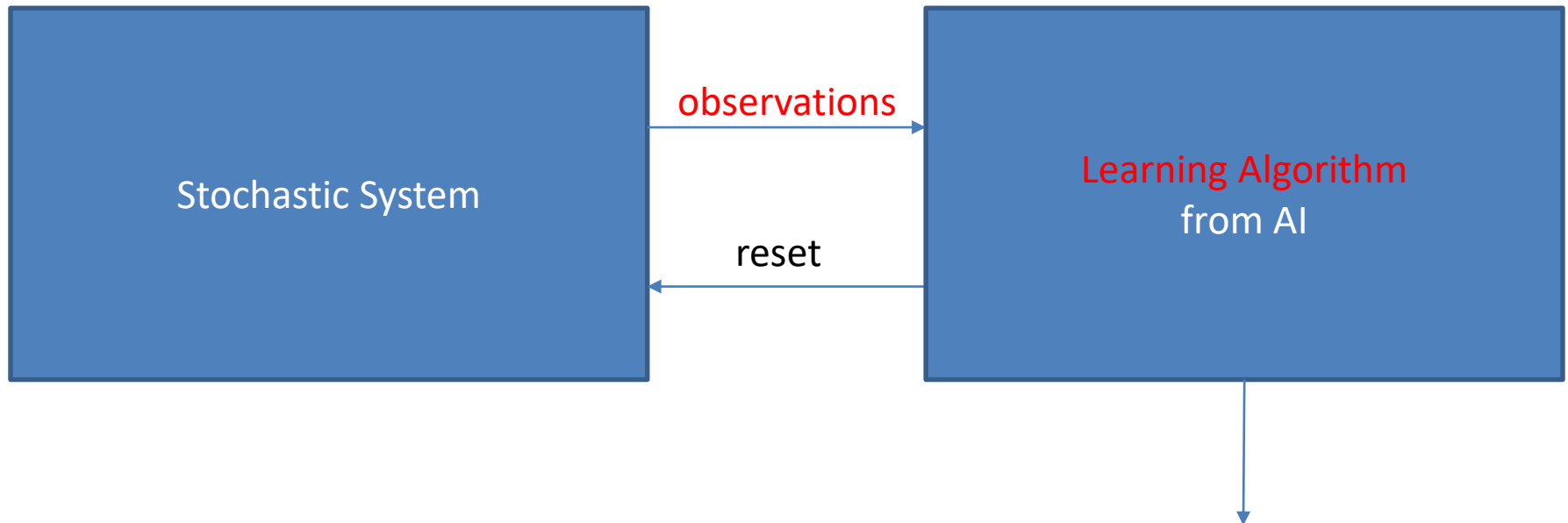
Stochastic Model
(e.g. Markov Chain)

+ PAC bounds:



with probability 97%,
error < 3%

Formalism



Learning algorithms:

Frequency estimator:

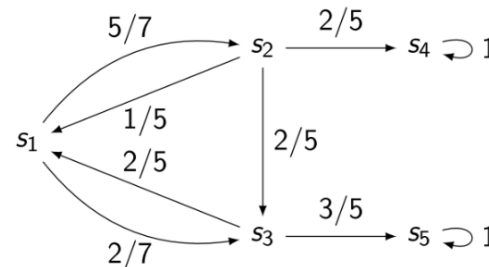
$$\text{Estimated_Proba}(s_1, s_2) = \frac{nb(s_1, s_2)}{nb(s_1)}$$

β -Laplace smoothing:

$$\text{Estimated_Proba}(s_1, s_2) = \frac{nb(s_1, s_2) + \beta}{nb(s_1) + k_1 \beta}$$

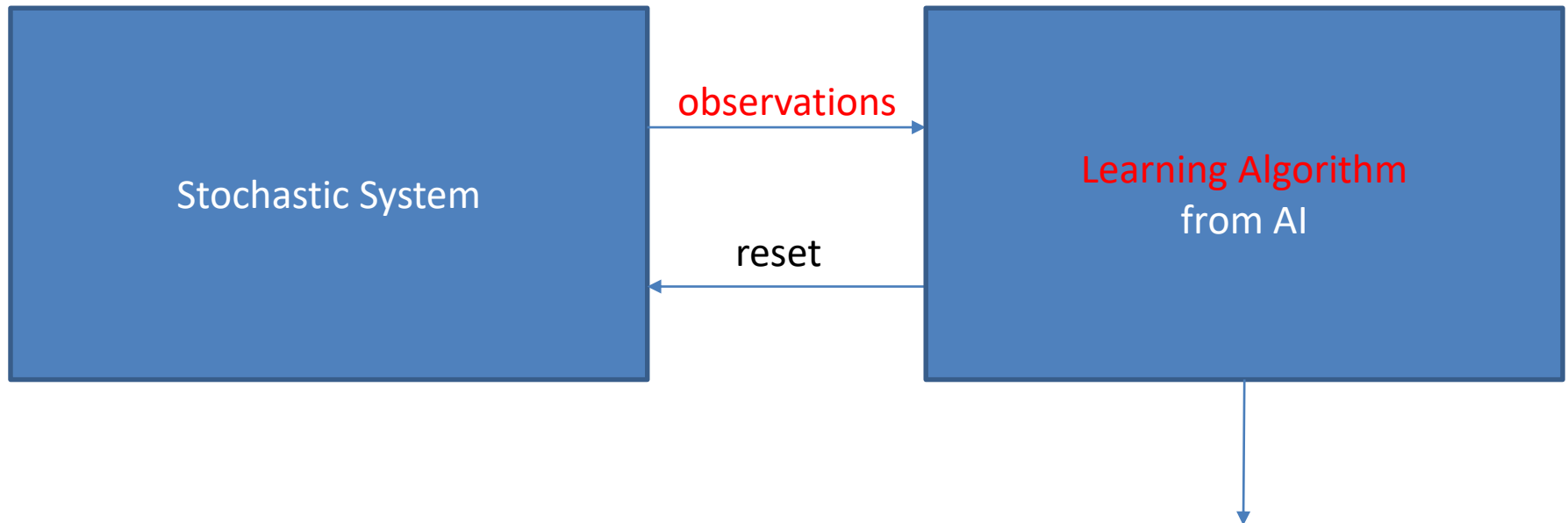
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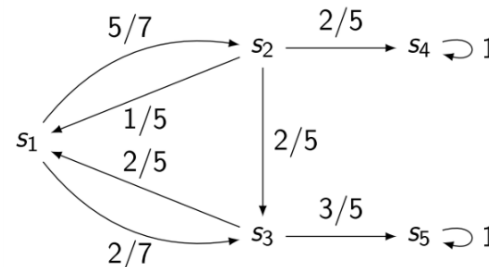
Formalism



Error analysis:

In AI, bounds on **transition probabilities**:
 $P(\text{Estimated_Proba}(s,t) - \text{Prob}(s,t) < 3\%) > 97\%$
using statistical « Okamoto Bounds » with
enough observations of transitions.

Stochastic Model
(e.g. Markov Chain)



+ PAC bounds:

with probability 97%,
error < 3%

Goals

Understand learning algorithms from AI.

e.g., where β -Laplace smoothing useful?

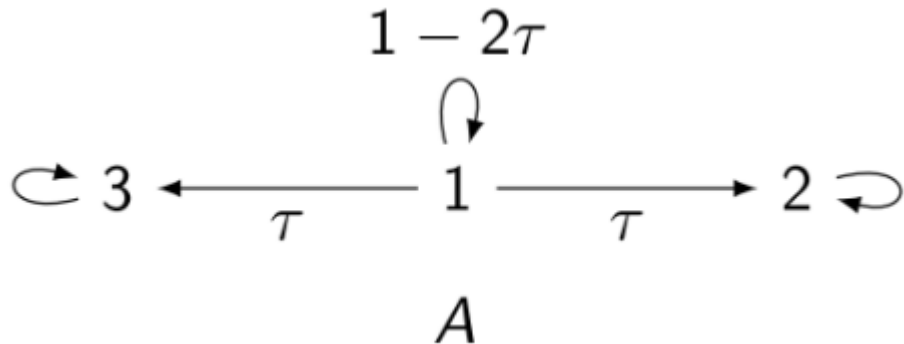
Which β to use?

Provide more meaningful bounds for model learnt:

Global bounds on behaviors of the model rather than
local transition probabilities.

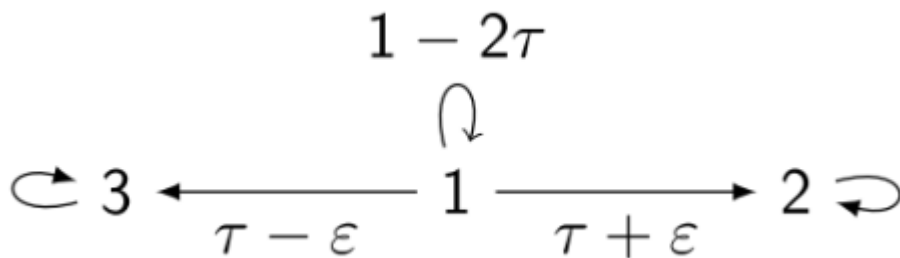
How? Use logics LTL, CTL... to specify global behavior.
We want that the probabilities to fulfil
logical formula in model and in system must be close.

Local vs Global probabilities



Global Property φ : reach state 3.

From state 1: $P^A(\varphi) = \frac{1}{2}$.



When learning A_w , small statistical errors possible
local error on proba: ε

From state 1: φ has probability $(\tau - \varepsilon)/2\tau$

Global error depends on **conditioning** τ of system

e.g.: $\tau = 2\varepsilon$, $P^{A_w}(\varphi) = \frac{1}{4}$.

Result 1:

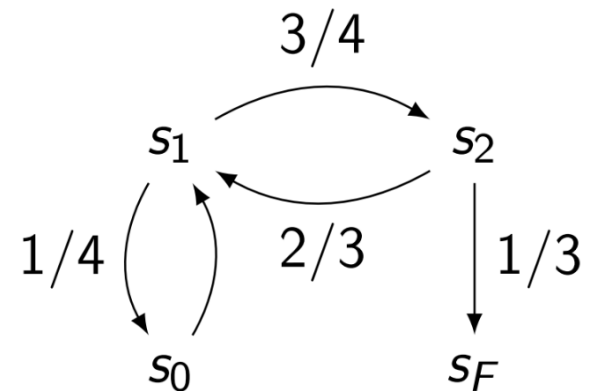
One fixed « Time Before failure » objective

Probability to see fault before seeing initial state again

=> **Frequency estimator**, giving MC A_W

Reset strategy: reset when we reach s_F or s_0 :

e.g.:
2 runs: $s_0 \ s_1 \ s_2 \ s_1 \ s_2 \ s_1 \ s_2 \ s_F$
2 runs: $s_0 \ s_1 \ s_0$



Theorem

For $\varepsilon = \sqrt{\frac{1}{2n} \log \frac{2}{\delta}}$, and the frequentist estimator A_W , we have:

$$P(|\gamma(A, \varphi) - \gamma(A_W, \varphi)| > \varepsilon) \leq \delta$$

Sketch of proof:

- 1 Use the Okamoto bound gives that $P(|\gamma(A, \varphi) - \gamma(W, \varphi)| > \varepsilon) \leq \delta$
- 2 Show that $\gamma(W, \varphi) = \gamma(A_W, \varphi)$ (not so easy)

Result 2:

What about learning a Markov chain good for all properties?

=> find A_W , ε such that **uniformly**, for all formula φ , we have:

$$P(|\gamma(A, \varphi) - \gamma(A_W, \varphi)| > \varepsilon) \leq \delta$$

- All properties of LTL?

Not possible (ask arbitrarily high precision for arbitrarily nested formula). [Daca et al.'16]

Possible for **depth k** formulas, but $O(\exp(k))$

- All properties of PCTL?

Not possible

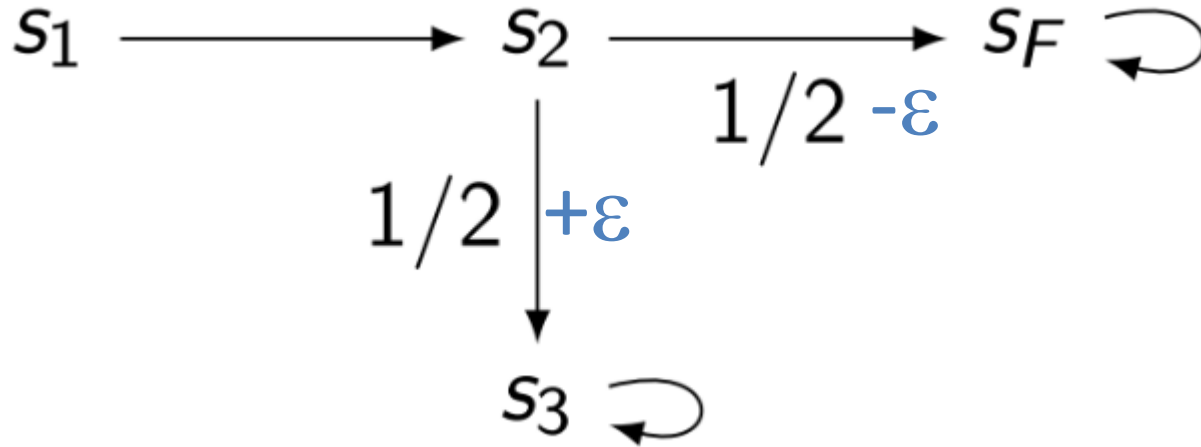
(PCTL cannot handle any statistical error)

- All properties of CTL?

Possible, and not so complex!

In particular, all reachability properties

PCTL cannot handle statistical error



PCTL Property $\varphi : (s_2 \Rightarrow \text{Proba}(X(s_F)) \geq \frac{1}{2})$
 $P^A(\varphi) = 1$ in the original system A.

If we learn A_W from observations W , we can
do a small statistical error ε and obtain

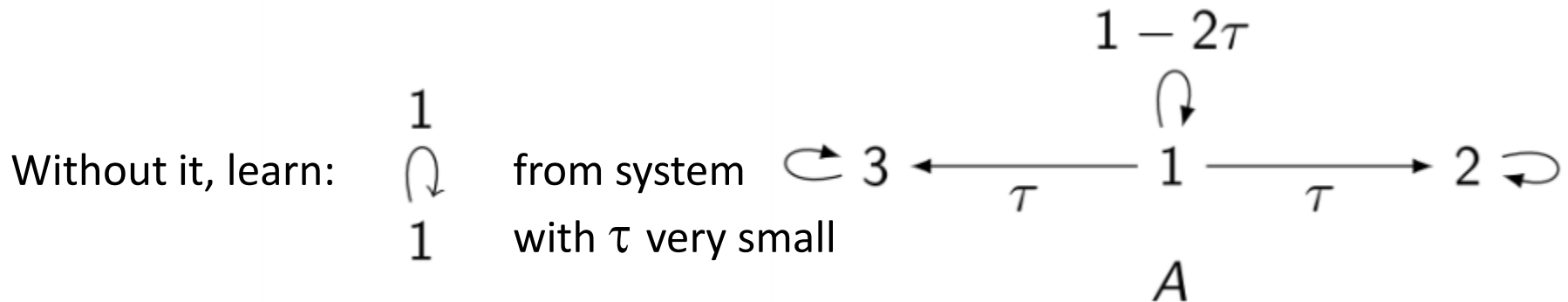
$$P^{A_W}(\varphi) = 0$$

Global PAC bounds for CTL

Error wrt Probability($X\phi \mid \phi U \psi \mid G\phi$), for all ϕ, ψ CTL formulae.

observations $W = \{\ll u s v s \gg\}$: reset when same state seen twice

Need to know the **support** of transitions
(not the case for fixed timed before failure)



=> We use **Laplace smoothing** to ensure the probabilities are all >0 .

Global PAC bounds for CTL

Error wrt Probability($X\varphi \mid \varphi U \psi \mid G\varphi$), for all φ, ψ CTL formulae.

observations $W = \{\llbracket u \mid s \mid v \mid s \rrbracket\}$: reset when same state seen twice

Theorem

There exists A_W such that for all CTL-like formula φ ,

$$P(|\gamma(A, \varphi) - \gamma(A_W, \varphi)| > \varepsilon) \leq \delta$$

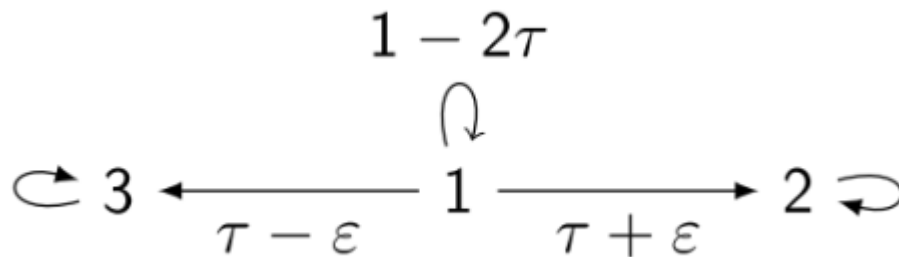
Sketch of proof

- ① Use Okamoto bound to obtain PAC bound on transition probabilities:
 $P(|A_W(i, j) - A(i, j)| < \varepsilon) \geq 1 - \delta$
- ② Use Laplace smoothing to obtain that
 $P(|\bar{A}_W(i, j) - A(i, j)| < \varepsilon) \geq 1 - 1.1\delta$
- ③ Thus, $P(\forall i, j, |\bar{A}_W(i, j) - A(i, j)| < \varepsilon) \geq 1 - 1.1 \cdot m^2 \cdot \delta$
- ④

... Next slide

Conditioning

$Cond(M) = \min_{s \in S, i \in R(s)} P_i(F_{\leq m} \cap R(s))$ and $R(s)$ is the set of states other than s that can reach s , that is, $Cond(M)$ is the minimal probability to move away from $R(s)$ in at most m steps.



$$R(2)=R(3)=\{1\} \text{ and } R(1)=\{\}$$

$$P_1(\text{Leave}_{\leq m} 1) = 1 - (1 - 2\tau)^m \approx 2 m \tau$$

$$\Rightarrow Cond(M) \approx 2 m \tau$$

Conditioning

γ wrt Probability($X\phi \mid \phi U \psi \mid G\phi$), for all ϕ, ψ CTL formulae.

Let M, M_ε such that $\forall i, j, |M(i, j) - M_\varepsilon(i, j)| < \varepsilon$ and
 $M(i, j) > 0 \Leftrightarrow M(i, j)_\varepsilon > 0$...true thanks to Laplace smoothing

Theorem

Denoting $\gamma(A)$ the probability to reach a state s in DTMC A , we have

$$|\gamma(M) - \gamma(M_\varepsilon)| < \frac{m^2 \cdot \varepsilon}{\text{Cond}(M)}$$

Experimental Results

	System 1	System 2	System 3	System 4	System 5
# states	3	3	30	64	200
# transitions	4	7	900	204	40000
# events for time-to-failure	191 (16%)	991 (10%)	2753 (7.4%)	1386 (17.9%)	18335 (7.2%)
# events for full CTL	1463 (12.9%)	4159 (11.7%)	8404 (3.8%)	1872863	79823 (1.7%)

Table 1: Average number of observed events N (and relative standard deviation) given $\epsilon = 0.1$ and $\delta = 0.05$ for a time-to-failure property and for the full CTL

Conclusion

Understand learning algorithms from AI:

- Frequentist estimator enough for one fixed property.

- β -Laplace smoothing to keep >0 transitions.

- Useful for providing bounds for CTL, rationale to fix β .

Provide **Global bounds on behaviors** of the model learnt for CTL.

Not possible for PCTL or unbounded LTL.

