Multifractal analysis: a method to investigate non-stationary properties of geophysical processes

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Abstract: - Multifractal analysis is one of the new mathematical methods, which approved itself in solving many applied problems. Multifractal approach is now widely used for geophysical data processing to investigate non-stationary properties of different geophysical processes. So it is necessary to develop a common methodology for teaching of basic and applied aspects of multifractal analysis in the appropriate educational institutions. Here we present the most elaborated technique of multifractal analysis – the so-called method of moments. Modification of this method for analysis of alternating-sign functions is considered. We describe the cover-thickness capacity approach, which allows evaluating singularity spectrum both for negative and positive values of exponent q. All the multifractal spectrum calculation steps are demonstrated herein and *FractAn* software package used for learning process in St.-Petersburg State University is presented. As an example of practical application of the multifractal approach we consider analysis of geomagnetic field variations in a seismically active region in order to study the dynamics of multifractal characteristics at the successive stages of strong earthquake preparation.

Key-Words: – multifractal analysis, method of moments, singularity spectrum, geomagnetic field, earthquakes

1 Introduction

Nowadays digital methods of geophysical field registration bring huge volume of information related to the studied objects. Processing of these data demands employment of statistical approaches. Among the models of data giving best fit to real geophysical records, considerable attention has been drawn to fractal models for the last years. These models describe adequately many properties of geophysical processes, such as non-stationarity, long-term correlations, inherent hierarchy of singularities not allowing approximation of the process by a smooth function without losing the information. In many cases these properties make difficult the use of traditional methods of statistical analysis that demand stationarity and smoothness of analyzed functions for effective employment.

Numerous methods to evaluate fractal dimension of signals and data sets are already well elaborated. Fractal approach implies strict geometrical self-similarity of investigated object, which can be characterized by a single parameter: the fractal dimension. In our studies of the Earth we more often deal with the distribution of some physical property at geometrical support than with pure geometrical object. In the case of temporal geophysical set of data one can imagine the support as a sequence of cells containing values of measured parameter, cell size is equal to the time step of measurements. For analysis of similar constructions Benoit Mandelbrot invented the term multifractal, whereas mathematical concept describing the above mentioned distributions was called multifractal approach or multifractal formalism (an introduction to multifractal approach can be found in Mandelbrot's review [1]). Within the framework of multifractal approach the process or data set is characterized by a spectrum of dimensions called multifractal spectrum or singularity spectrum, because peculiarities (singularities) of different scale are supported by sets of cells with different fractal dimensions.

The theory of multifractal analysis of processes and data sets has been thoroughly elaborated in the works of Riedy, Jaffard, Lévi Véhel and others [2, 3, 4 and references therein]. During last decades we can find examples of multifractal analysis of geophysical data sets in the works of Goltz (seismic activity), Schmitt et al. (data of deep drilling of Greenland ice sheet), Hongre et al. (records of geomagnetic field), Muller and Saucier (well-log data) [5, 6, 7, 8]. In all of these works the method of moments has been used for the calculation of multifractal spectra. Here we describe this method and give an example of its application to real geophysical data.

2 Calculation of multifractal spectra for functions: method of moments

Multifractal approach was initially invented for the analysis of distribution of *measure* on geometrical support. One can give non-strict definition of measure as a "positive and additive quantity". If we consider sign-alternating temporal (or spatial) dependence $Y(t)$, it is usually referred to as *function*.

Multifractal spectra can be calculated in different ways, and method of moments is the most elaborated of them. It is based on the Legendre transform of the so-called mass exponent function or scaling function $\tau(q)$ [1-4, 9]. Details of the calculation procedure slightly differ when we analyze measures and functions. Here we consider only the analysis of functions because there are a lot of publications considering measures [1, 5, 8, 9 and other].

In classic variant of the method of moments, function $\tau(q)$ is calculated through the assessment of scaling properties of partition function or structure function of the process. Partition function $S_q(\delta)$ is the basic function of multifractal analysis and also can be defined in different ways. Method of moments for functions is based on the following definition of $S_q(\delta)$ [3]:

(1)

(Here m denotes dimension of space, where function is defined; δ is the time lag; q is an exponent called moment order – it gives the name to the method).

Then, assuming that $S_q(\delta) \sim \delta^{\tau(q)}$ when $\delta \to 0$, we can calculate the singularity spectrum

, (2)

where (3)

is scaling function, H – Hoelder exponent. In practice scaling function $\tau(q)$ is usually evaluated for every q by means of linear regression of the dependence lnon ln δ in some finite scaling interval $\delta_{min}<\delta<\delta_{max}$. Strictly speaking, if analyzed function possesses multifractal properties, then $\tau(q)$ must be a non-linear function, $\tau'(q) \geq 0$ and $\tau''(q) \leq 0$. If these conditions are valid, Legendre transform can be carried out with the use of less time-consuming formulae

; , (4)

The width of spectrum $f(H)$ depends on the range of H values presented in the considered realization of the process, i.e. characterizes the degree of data non-uniformity. The value $f(H)$ can be attributed as the dimension of a multitude of points where Hoelder exponent is equal to H. The left branch of the spectrum corresponds to large q values and reflects the contribution of anomalously high surges of measured parameter Y whereas the right branch of the spectrum (corresponding to small q values) reflects the influence of noisy, small-amplitude component of the process. This allows us to investigate thoroughly the variability of the data.

When we analyze one-dimensional (m=1) alternating-sign set of data, i.e. the sequence of discrete values $Y(j)$, j=1,2…N, there is a need to modify the expression for partition function by replacing the integration to summation. For example, function $S_q(\delta)$ can be defined as the average value of the function under the sign of integral [6]:

, (5)

where brackets denote the averaging over all possible values of i. It has however been shown that formula (5) works steady only for positive q, whereas for negative q scaling cannot be revealed in most cases. Several definitions of structure function have been proposed, allowing calculation of multifractal spectrum in the whole range of q values. We should only keep in mind that each estimate of singularity spectrum will inevitably differ from the others made on the base of different structure functions.

In the work of L.Hongre et al., the term *cover-thickness capacity* was introduced meaning the difference between the maximal and minimal values of the function in the interval δ [7]:

. (6)

We can define structure function on the base of $\mu_i(\delta)$:

. (7)

Scaling interval of this function sometimes differs for negative and positive q, however model calculations have shown that we can almost always find the interval of δ to get "proper" scaling function and smooth singularity spectrum. The criterion for choosing scaling interval could be the value of $\tau(q)$ least-square error. Thus, cover-thickness capacity can be recommended as an appropriate "measure" for evaluation of scaling properties of alternating-sign functions. The main steps of multifractal spectrum calculation can be summarized as follows:

- building structure function of data set in accordance with (6), (7); The range of q values should be wide enough to reach asymptotes of scaling function;
- inspection of scaling properties for various q to choose an appropriate scaling interval $\delta_{min}<\delta<\delta_{max}$;
- evaluation scaling function $\tau(q)$;
- calculation $f(H)$ spectrum according to (4).

Despite the principal possibility to automate the process of calculations (in monitoring systems, for example), visual control of resulting functions is recommended.

Analysis of processes with known scaling properties have shown, that the spectrum of a short realization can considerably differ from the true one, and the result strongly depends on the length of data set. Realizations of fractional Brownian motion (fBm) have been used to test the calculation technique. It is known, that fBm is characterized by a single value of the Hoelder exponent and its spectrum consists of single point with coordinates (H, 1). The algorithm of successive random addition has been used for creation of model data sets [9]. In the Fig.1a, b the examples of fBm realizations of 1024 points length with different values of Hoelder exponent and corresponding spectra $f(H)$ are given. It is seen, that calculated spectra are much wider than expected ones, however for each spectrum the abscissa of the maximum closely corresponds to the true value of H. Model calculations with realizations of different length have proved that spectrum tends to the point (H, 1) with increasing realization length. Nevertheless, the tendency is rather slow and theoretical value has not been reached for the maximal realization length of 2^{16} points (Fig.1c). This property should be taken into account during the analysis of real signals. For obtaining valid results in the assessment of multifractal properties of different signals, data sets have to be of the same length, while multifractal spectra must be calculated by the same technique.

Fig.1 a) Model realizations of the fractional Brownian motion (fBm) with different Hoelder exponents; b) Corresponding singularity spectra; c) Singularity spectra for fBm realizations of different length: 1– 2^{10} ; 2 – 2^{13} ; 3 – 2^{16} points (H=0.5).

In some cases comes along the necessity of different signals comparison without the whole multifractal spectrum calculation. For example, the dynamics of multifractal properties changes in long data sets can be revealed with the use of "gliding window" technique. In this case the behavior of some characteristic points of multifractal spectrum can just be followed: the abscissa of maximum (the most probable value of Hoelder exponent), the abscissas and ordinates of extreme points of the spectrum (see Fig.2). The relative width of the spectrum can be evaluated with the use of non-uniformity factor Δ [5]:

, (8)

where H_{max} and H_{min} can be replaced by any values $H(q)$ and $H(-q)$ for the simplicity of calculations.

Fig.2 Characteristic points of multifractal spectrum.

It should be noted that Hoelder exponent H, estimated in accordance with (4) for q=1, is the same as Hurst exponent which is related to fractal dimension D through the well-known equality D=2-H. The D value can be evaluated by any method of fractal analysis, for example spectral method or box method [9]. At this point, the interrelation between fractal and multifractal analysis is obvious.

3 Educational software for multifractal analysis

Nowadays there are few software packages, free and commercial, that incorporate different functions of fractal and multifractal analysis (*Fraclab* of INRIA and *Benoit* of TruSoft can be mentioned). A special educational package called *FractAn* was developed in the Earth Physics Department of Saint-Petersburg State University. It can process both *measures* and *functions,* depending on the nature of analyzed signal. The novelty of the package is that gliding window analysis with the possibility to control local scaling properties is included. *FractAn* is a Windows oriented application with a friendly interface.

An example of *FractAn* window with the results of analysis of fBm realization (H=0.5, length 2^{13} points) is given in the Fig.3. The user can choose for the analysis the whole realization or any part of it (zoom button). One can consider initial signal or increments or absolute increments. The window "Multifractal analysis" contains three panels. The scaling of structure function (partition sum) for any moment q can be observed in the left panel. The user can choose an appropriate interval for linear regression (L_{min}, L_{max}) and use it for further calculation of scaling function $\tau(q)$ (central panel) and $f(H)$ spectrum (right panel). Characteristic points of multifractal spectrum and non-uniformity factor are evaluated. The assessment of confidence intervals of evaluated parameters is provided. This ensures the user's control under the calculations at all steps of the procedure as well as the acquisition of correct multifractal spectra. Some procedures of fractal analysis (e.g. spectral method) are also included.

Fig.3 Interface of *FractAn* package for fractal and multifractal analysis

FractAn package have been used for teaching in Saint-Petersburg State University for many years in the framework of master lecture course "Fractals in geophysics".

4 Multifractal analysis of geomagnetic field records before the Guam earthquake of August 8, 1993

As an example of application of multifractal analysis to the real geophysical data, in the current section preliminary results of multifractal analysis of geomagnetic field records before strong (M_s=8.0) Guam earthquake of August 8, 1993, are presented. The detailed description of experimental data and the results of their fractal analysis can be found in [10].

The epicenter of the earthquake was located off the coast of Guam Island at a depth of 60 km. Geomagnetic data were available from the Guam observatory which is located in a distance of 65 km from the earthquake epicenter. Three components of magnetic variations (H^*, D and Z) were recorded on a digital cassette tape, at a sampling rate of 1 sec. The chosen sampling rate allows us to investigate the properties of ULF emissions (geomagnetic pulsations) in wide frequency range from f = 0.001 Hz to f = 0.3 Hz. This frequency range is also favorable for detection of electromagnetic signals of lithospheric origin, which could be related to the earthquake preparation processes. The corresponding consideration based on the field observations as well as on the estimations of the appropriate skin depths for electromagnetic waves is contained in [10, 11, 12]. The calculations of skin depth values show the ability for ULF electromagnetic fields of space origin to penetrate to the earthquake focal areas. Also the ULF part of electromagnetic fields, which can be generated in the earthquake focus before and during the major rupture, is able to penetrate from the focus to the Earth surface. Since the earthquake preparation processes can be connected with an alteration of the medium structure in the earthquake focal area as well as with the change of their elastic and electric parameters, such processes could be manifested also in alteration of the structure of seismic and electromagnetic noise recorded near the epicenter region. Since such alteration is expected to be of long-term character, we give our attention to the long-term variation of scaling characteristics of the observed signals.

Fig.4 Detrended one-hour records of the H-component of geomagnetic field variations at the Guam observatory before the earthquake of August 8, 1993:
a) April 20, 1993; b) July 20, 1993.

For multifractal analysis the record of *H*-component of geomagnetic field have been chosen, since the variations of that component are more pronounced and significant than variations of the other components. We analyzed the 5-months period before the earthquake (April-August 1993), and we considered the dynamics of multifractal characteristics at the noon sector of local time (12-13 LT), which corresponds to the interval 2-3 UT (universal time). Each of the one-hour records, containing 3600 points, has been first detrended and then considered as a realization of a random process. In the next step, the structure function has been calculated in according with the formulae (6), (7). It has been established that the scaling interval for most records covers a range from 16 to 1024 seconds: this range has been used for evaluation of the function $\tau(q)$ and calculation of the singularity spectrum.

In the Fig.4a, b examples of detrended records of *H*-component for two days – April 20 (far enough from the main shock) and July 20 (a short time before the main shock) – are shown. One can see, that the records are visually similar to the realizations of fBn. The changes of characteristic points of singularity spectrum in April and July-August are presented in the Fig.5. There is a well-distinguished tendency of multifractal spectrum to shift toward less values of Hoelder exponent in the month before the Guam earthquake. This means that relative input of noisy low-amplitude component of the data increases. Further detailed analysis of geomagnetic field variations in seismically active regions seems to be necessary, since the change of fractal and/or multifractal characteristics of the Earth's electromagnetic field could be a manifestation of the tectonic evolution.

Fig.5 Variations of characteristic points of the singularity spectrum: a) April, 1993; b) July-August, 1993. 1 – H_{max}, 2 – H_0, 3 – H_{min}. Monthly series of H_{max}, H_0, and H_{min} are smoothed by 5-days running average. Estimates of H in the intervals April 24-27, July 6-12, 16-17 and 25-27 are not valid due to sinusoidal components dominating in the records.

4 Conclusion

Presented modification of the method of moments with the use of cover-thickness capacity for calculation of singularity spectrum is an effective instrument for investigation of scaling properties of non-stationary random processes. Being robust and steady from computational point of view, this method is worth including to the programs of engineering education. *FractAn* software package developed in the Earth Physics Department of Saint-Petersburg State University is an easy and effective tool for the acquaintance with the method and can be used in educational process for the analysis of both model and real signals.

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