April 23, 2007

An Evolutionary Quantum Game Model of Financial Market Dynamics – Theory and Evidence

Carlos Pedro Gonçalves* Carlos Gonçalves†

Abstract

The application of mathematical physics to economics has seen a recent development in the form of quantum game theory. Quantum game theory has become an important field of research in multidisciplinary applications of mathematical physics to the study of economic phenomena.

We address the empirical findings of multifractality and turbulence in financial markets' dynamics, from the point of view of evolutionary quantum game theory, proposing a quantum game theoretical model of a financial market, that extends the behavioral framework proposed by Sornette and Zhou for the self-fulfilling Ising model of the markets.

The quantum market model works with a bosonic framework for evolutionary quantum game theory introduced here and is based on recent findings within neuroeconomics and the neurobiology of decision.

The model is tested against actual market data, where it is shown that it is able to reproduce some of the main multifractal signatures present in actual markets.

Keywords: Bosonic Evolutionary Quantum Game Theory, Multifractals, Market Turbulence, Econophysics.

JEL Classification: C73, G10, G14, G32, D83, D84, D87

*Researcher at UNIDE – ISCTE Business School, quantitative methods' research line. e-mail: cpdsg@iscte.pt, carlospedrogoncalves@yahoo.com

†ISCTE – Business School, Department of Finance. e-mail: carlos.goncalves@iscte.pt

Acknowledgements: *We thank Professor Didier Sornette for the important discussions regarding the previous artificial financial market, which much contributed with elements for the current work. We also thank Maria Odete Gonçalves for the important discussions regarding cognitive science, neurobiology and philosophy, which contributed, a great deal, for the behavioral foundations of the current model. We would also like to thank the Netlogo user community and the Netlogo team at Northwestern University, for the important support in agent-based modelling.*

I. Introduction

Ever since the identification of multifractal behavior in the financial markets' price fluctuations (Mandelbrot, 1999; Mandelbrot, *et al.*, 1997; Calvet and Fisher, 2002; Muzy *et al.*, 2001), an empirical fact, unexplained within standard financial theory, there has been a search for models that provide an explanation of this phenomenon.

The major problem with financial stochastic modelling approaches such as the *multifractal model of assets' returns* (MMAR) (Calvet and Fisher, 2002) and the *multifractal random walk* (MRW) (Muzy *et al.*, 2001), is that multifractal behavior is introduced through a multiplicative cascade, without a fundamental theoretical grounding on agents' behavior. Such a modelling approach is useful in terms of the immediate end result, which is the reproduction of the price patterns, but they lack an agent-level behavioral basis for explaining these patterns.

The development of models that generate multifractal patterns in a bottom-up fashion is necessary, both for financial theorists and financial agents. The presence of multifractal patterns makes the markets more risky than predicted by standard financial models, which means that financial agents need to have models that are able to provide for guiding tools in asset and risk management.

Within standard financial theory the multifractal behavior remains unexplained. The development of a model capable of explaining the presence of multifractal signatures in the markets would be a first step towards a financial theory of market dynamics. However, such a model, and such a theory cannot be found in the framework of standard finance.

It has become necessary to review the microscopic assumptions that form the basis for classical finance, where, by microscopic assumptions, we mean the individual agent's behavior and the interaction rules between agents.

Mathematical physics, and econophysics have provided for competing fields of research where it is possible to study market dynamics from the perspective of microscopic modelling. Spin glasses provide for the oldest examples of the application of physical models to solve unexplained empirical facts in market dynamics. One of the early applications of these models dates back to Vaga's (1990) *coherent market hypothesis*, that tried to relate market dynamics to different phases, in an analogy with the different phases of a spin glass.

Although spin glasses have been an example of a simple and effective modelling tool to build microscopic theories of market dynamics, one still lacked a robust model capable of generating *self-organized multifractality*, until a recent work by Sornette and Zhou (Sornette and Zhou, 2005; Zhou and Sornette 2005, 2007), in which multifractal structure is diagnosed, not only by the standard convexity of the structure functions' exponents, but also by a continuous spectrum of power law response functions to endogenous shocks.

Sornette and Zhou stressed that this seems to be the first market model in which such a clear distinction is documented quantitatively, based only upon a *bottom-up self-organization*, which is in contrast with the previous models based upon stochastic modelling techniques like the MMAR and the MRW. This is one of the first works to produce *self-organized multifractality* in the market observables, that closely matches actual market behavior, simultaneously providing for a behavioral explanation of market dynamics.

Sornette and Zhou's model is based upon a previous work by Gonçalves (2003b), which in turn, develops a previous model proposed by Johansen, Ledoit and Sornette (2002).

The novelty of these models is that multifractality emerges from a simple microscopic mechanical rule, based upon the dynamics of social sentiment contagion. The fact that the model is grounded in simple microscopic behavioral assumptions, conveys to it an explanatory capability not present in neither standard financial theory, nor in the MMAR and MRW.

However, the model only deals with one aspect of price series, it does not capture additional empirical features like multifractality in transaction volume and the connection between market activity and price jumps.

The problem is that the model is not sufficiently dynamic with respect to the number of agents, since it works with a fixed number of agents. Taking into account that, in the model, trading is assumed to be synchronous, there are no fluctuations in the transaction volumes.

However, fluctuations in market activity and transaction volumes seem to be at the core of market turbulence and of multifractal behavior, as Mandelbrot and Hudson (2004) have discussed. Indeed, in actual markets the number of traders fluctuates. Periods of higher market turbulence are usually linked to higher market activity, with simultaneous synchronization of trading behaviors.

The existence of periods of high market activity or low market activity seems to play a central role in the production of multifractal patterns, and it provides an important piece of empirical evidence that should be taken into account when modelling the markets.

In the present work we propose a quantum game theoretical model (formalized in **section 4**) that generates *self-organized multifractal patterns* (analyzed in **section 5**) and that allows for fluctuations in the number of agents, thus accounting for the above mentioned empirical facts.

The model extends the previous work by Sornette and Zhou, and by Gonçalves, to include complex social dynamics and fluctuations in the number of agents. There are, however, some major differences, namely, the model includes a more complex semantics in agents' behavior, albeit being analytically simpler, in the sense that we obtain closed-form formulas for each trading round.

The model is based upon a bosonic framework for evolutionary quantum game theory that is laid out in **section 3**. Bosonic evolutionary quantum game theory, provides for a useful modelling tool to deal with social games in which agent populations fluctuate with agents entering and leaving, and agents changing strategies.

If strategies are made to correspond to quantum states, and agents to particles, then, the language of second quantization becomes a natural language for social systems' dynamics with fluctuating numbers of agents' implementing each strategy.

The model that we propose here is not only able to generate multifractal patterns, but it is based on a different behavioral theory than that of standard financial theory and different than that of the branch of behavioral finance based upon the *noise trader hypothesis*[1].

The *noise trader hypothesis* assumes that there is a subset of irrational agents or that rational agents may sometimes become irrational and that (irrational) crowd dynamics may dominate the markets leading to phenomena like bubbles and crashes, and to multifractal patterns. Lux and Marchesi's (1999) model is an example of a noise trader-based microscopic model, that has been successful in obtaining long range dependence in absolute returns[2] (Samanidou *et al.*, 2006).

Unlike previous microscopic models, such as Lux and Marchesi's model, that worked with the *noise trader hypothesis*, our model is based upon a different behavioral framework that incorporates recent findings within neurobiology and neuroeconomics. Both standard financial theory and the *noise trader hypothesis* have problems when considered in light of the recent empirical evidence regarding how human beings make decisions.

Indeed, both standard financial theory and the *noise trader hypothesis* are based upon a notion of rationality that does not take into account the actual role of emotions and feelings in rational thinking and rational decision. In both standard financial theory and in the *noise trader hypothesis*, emotions are always assumed to be purely irrational and are considered to be sources of market inefficiencies. However, the recent evidence, from neurobiology and neuroeconomics, does not seem to support neither this interpretation of the relation between emotions and rationality, nor the behavioral assumptions that underlie both standard financial theory and the *noise trader hypothesis*.

Emotions, feelings and processes of social cognition play a central role in rational decision-making. We shall argue here, that emotions, feelings and processes of social cognition are necessary elements for market efficiency. On the other hand, these emotions, feelings and processes of social cognition are, in our model, responsible for complex market dynamics, in such a way that complex market dynamics is always present in an efficient market, being a consequence of the very mechanisms that make the market efficient.

These behavioral and financial foundations of the model, along with the criticism to standard financial theory, are addressed in **section 2**.

Thus, the present work is divided into four parts that include **sections 2** to **6**. In **section 2**, we review the model's behavioral and financial foundations. In **section 3** we introduce the framework of quantum game theory and provide for

[1] A *noise trader* is an uninformed trader whose trades are purely random and do not reflect any information on the underlying asset. According to the hypothesis, a *noise trader* should become a driving force in market dynamics through synchronization in trades produced by a dynamics of social imitation and crowd behavior.

[2] Lux and Marchesi's model has the problem that market statistics are not robust, being sensitive to the number of agents, as noticed by Lux (2006).

the formalism for an approach to a bosonic evolutionary quantum game theory. In **section 4** the market model is formalized and in **section 5** the main empirical findings are addressed. Final reflections upon the model's significance for financial theory, quantum game theory and quantum econophysics are addressed in **section 6**.

II. Behavioral and financial foundations

Standard financial theory rests upon a hypothesis known as the *efficient market hypothesis* (EMH) which states that market prices reflect the whole information regarding the value of any asset, traded in a financial market[3]. Taken in a weaker sense, one might state that financial markets are among the most efficient socioeconomic information processing systems capable of converting the intrinsic value of an asset into a fair price for that asset.

Although, financial markets obey a general principle of efficiency, in terms of their functional nature of financial information processing systems, one should stress that EMH may have basic problems when considered as a general statement for any kind of market organization. One problem is the *existence of firms*.

This matter was first raised by the economist Ronald Coase (1937) who asked the question: *why do firms exist?* If a market is the most efficient organizational form, why do people join each other to form *firms*? These questions originated the *theories of the firm*, and are still under debate in economic theory.

The questions, themselves, raise the problem of the market as an efficient organizational form. Under such an organizational form, for instance, no worker would work for a given firm, each agent that was needed for production would meet other agents in a market, and production would proceed without any given product being produced by a given entity other than the market itself.

As it turns out, according to the recent *knowledge-based theories of the firm* (Kogut and Zander, 1992; Conner and Prahalad, 1996; Spender, 1996, Nonaka *et al.*, 2000; Gonçalves, 2005), the market seems to have a basic problem of relational stability. People must work in stable working environments in order to create knowledge more efficiently. A market is not generally efficient, as an

[3] Fama (1970) defined, on this regard, three levels of efficiency: weak, semi-strong and strong.

Markets are considered to be efficient in the *weak form* if the prices reflect all the information contained in the past price series.

On the other hand, if besides the information contained in the past price series, the prices also reflect all the past publicly available information, then, the markets are said to be efficient in the *semi-strong form*.

Finally, if private information is also reflected in the prices, then, the markets are said to be efficient in the *strong form*.

These three levels of efficiency define a type of efficiency with respect to information, in the sense that no investor should be able to profit from information since all information, relevant for valuation, is already reflected in the prices.

organizational form, when knowledge creation is a key factor of competitive advantage[4].

Markets seem, however, to be an efficient organizational form, when valuation issues are concerned. As a valuation system, a market is not compromised by any one particular interest. Economic agents actively search all the information regarding the asset under valuation, and reflect the information they gather in their trading bids.

Given a sufficient time to process all the information, the price will reach a social consensus that should reflect the intrinsic value of the asset. At the equilibrium price, no-one is willing to pay more, nor less, for the asset, it is the asset's just price.

Although the EMH does not, by itself, imply any given set of behavioral assumptions, any theory that assumes its validity will have to address how the agents process the information and place a bid based upon their information processing activity.

Standard financial theory assumes that investors have access to all the information that is relevant for asset valuation. It also assumes that the fact that all investors have access to all the relevant information is common knowledge. Even if an investor did not have access to all the relevant information, or could not process all the information, the great number of investors guarantees that the information processed by the whole market system is all the information that is necessary to evaluate an asset (Schleifer, 2000).

Besides these assumptions, financial theory also assumes that investors are *rational* and, hence, evaluate the assets *rationally* (Schleifer, 2000, p.2). Any *irrational* investors are eliminated through a process of natural selection due to accumulated losses.

The *rationality* assumption is a cornerstone in standard financial theory. However, what is meant by *being rational* begs a further inspection. There are two relevant senses which make an agent rational, in order for an agent to be

[4] In economic theory, management strategy and organizational theory, other possible explanations are found for firms to be considered as having advantages over markets include:

- Transaction costs (Coase, 1937; Williamson, 1985, 1975) – if the transactions costs are higher in the market, then transactions will be internalized in a firm;
- Bounded rationality and opportunism (Williamson, 1985; Simon, 1977, [1945]) – a firm has advantages over the market because it is able to overcome the problems of bounded rationality, asset specificity, uncertainty, transaction frequency and the problem of detecting opportunism which menaces the market transaction.
- Unique resources (Penrose (1995, [1945]; Wernerfelt, 1984; Barney, 1991) – the existence of a firm is explained by the fact that it possesses a set of unique resources, difficult to immitate and that provide it with a sustainable competitive advantage over a market organization.
- Organizational routines (Nelson and Winter, 1996, [1982]) – learning, acquisition and creation of knowledge and skills, through organizational routines, form the main engine of evolution of the organization and of value creation and are the source of the firm's advantage over a market organization.

rational she must *evaluate rationally* the several alternatives, and *act rationally*. Acting rationally, in classical decision theory, is defined in terms of *choice*. A *choice* is, in this theory, considered to be *rational* if it satisfies a *criterion for rationality*. The basic *criterion for rationality* was introduced in mathematical economics by von Neumann and Morgenstern (1990, [1953]).

Drawing upon previous economic tradition, the authors considered that an individual that attempts to maximize her personal gain is said to *act rationally*, where the personal gain is mathematically expressed by some utility function which expresses numerically the level of personal satisfaction with a given choice from among a set of alternatives.

A problem with this approach was, however, already identified by von Neumann and Morgenstern. The authors noticed that, at that time, no satisfactory treatment of the notion of rational behavior existed[5]. The notion of *rational choice*, thus, had a purely operational value and should not be considered as a fundamental notion of *rationality*.

The notion of *rational choice* does not, in fact, incorporate any theories regarding the nature of cognition, which makes it compatible with different underlying cognitive theories.

However, the recent developments within neurobiology have contributed to a new understanding of how human beings process information and how they make decisions. More specifically, new light has been shed on what it means to *evaluate rationally* the information available and to *choose rationally*.

Indeed, neurobiological evidence has shown that every act of consciousness has an emotional substractum (Damásio, 1994). The cerebral systems that are responsible for emotions and for the feelings of those emotions are fundamentally linked to the systems that are necessary for rational behavior (Damásio, 1994, p.251).

These neural structures are linked with the regulatory mechanisms of the human body. Indeed, a feeling, defined by Damásio as a private mental experience of an emotion, seems to depend on a complex system of multiple components which is linked to biological regulation, and rational thought seems to depend on cerebral systems, some of which process feelings (Damásio, 1994, p.251).

The evidence seems to point out towards the existence of an anatomic and functional linking element between reason and feelings and of these to the body (Damásio, 1994, p.251). Every act of consciousness is accompanied by a feeling.

Neurobiology shows us that there is a deep connection between feelings and emotions. Every emotion originates a feeling, if one is in a state of aware-

[5] This can be seen in the following passage (von Neumann and Morgenstern, 1990, [1953], p.9):

> "(...) it may safely be stated that there exists, at present, no satisfactory treatment of the question of rational behavior. There may, for example, exist several ways by which to reach the optimum position; they may depend upon the knowledge and understanding which the individual has and upon the paths of action open to him (...)"

ness. While an emotion is produced as a set of changes in the body's state, the essence of the feeling of an emotion is the experience of those changes in juxtaposition with the mental images that activated the emotional cycle (Damásio, 1994, p.159). The substractum of a feeling completes itself with the changes in the cognitive processes that are induced simultaneously by neurochemical substances[6] (Damásio, 1994, p.159).

Recent experiments seem to show that if the emotional basis is somehow affected, then, the ability to make rational decisions diminishes, even though individuals are capable of logical reasoning (Damásio, 1994, p.225), this constitutes a fundamental problem for traditional financial models, that have considered a purely logical, emotionless decision, as being the best basis for making *rational decisions*, that is to say, decisions which are advantageous to an individual's personal interest.

It appears to be the case that the criterion of rationality that underlies classical decision theory and classical game theory finds its validity, for human agents, only if those agents have a normally functioning emotional system. That is, an individual must have a normally functioning emotional system in order to *choose rationally* and to *evaluate rationally* the different available strategies.

Taking into account these findings, in order to have a model of an efficient market, one must take these behavioral issues into account, for, otherwise, the model would not have the sufficient explanatory power to account for market dynamics, for it would ignore how human agents actually decide.

The main problem that arises in the mainstream financial theory of market efficiency, is that it has incorporated in its theoretical basis the behavioral notion that individuals compute the economic and financial information in the same way, form homogenous expectations, fully agreeing on the implications of available information for both current prices and probability distributions on future prices of individual investment assets (Fama and Miller, 1972, p.335).

In terms of a formal representation from the point of view of agent-based modelling, the model of financial agents, conceived within this theory, is such that agents correspond to automata that are exact copies of each other and that compute the information without any emotional substractum. Such a simplification was useful mathematically, since it allowed the construction of closed-form formulas for financial economics, but it goes against the recent neurobiological

[6]We should stress that we are using Damásio's terminology associated with feelings and emotions. It is not a normalized terminology, as Damásio himself admits it.

For instance, one might not use the term *feeling* at all, and divide the term *emotion* in two components, one of which would correspond to what Damásio calls the *feeling* of the emotion.

We use Damásio's terminology for two reasons. The first, is that the neurobiological evidence seems to support a distinction between emotions and feelings, in such a way that the terminology becomes useful for dealing with the neurological data.

The second reason is that Damásio's work provided elements that support a new view of rational decision theory and a basis for neuroeconomics.

In order for a market behavioral theory to have an empirical support, expressions like *rational behavior* and *rational choice* must be considered in light of the new neurobiological basis. And, on this regard, Damásio's work provides for a useful basis for the development of new market theories.

evidence and makes the microscopic model inadequate to deal with problems that arise from deviations of actual human rational behavior from the idealized normative rationality assumed by the mainstream financial theory.

In order for individuals to possess the degree of homogeneity that the mainstream financial theory assumes, it would be necessary for them to all share the same body, have the same life experiences, be the same individual. It follows from Damásio's work on decision, that no two individuals will process the information in the same way, since different working memories, different knowledge bases, different mental imagery evoked by the information and the context of the decision itself, will necessarily produce different computations for different people.

Agreement as to the expectations stem from social consensus formation due to social information processing, and, even then, there should exist different expectations (Sornette, 2004; Gonçalves, 2003b).

The assumption of homogeneous expectations, besides assuming an emotionless neurobiological substractum for decision, also involves the assumption that all investors agree in regards to the price probability distributions, and these distributions all coincide with the actual distributions. That is, given the information set available at time t, $I(t)$, if the price $P(t')$, for $t' > t$, has the conditional probability $\mathbb{P}_t[P(t')] = \mathbb{P}[P(t')\,|I(t)]$, then, the assumption of homogeneous expectations means, in this case, that, for each agent j, $\mathbb{P}_t^j[P(t')] = \mathbb{P}_t[S(t')]$. Since all agents have the same information set, and they form homogenous expectations, the rationality criterion here becomes that each conditional probability distribution for each agent coincides with the actual probability distribution for the price.

This is useful in defining equilibrium models, but has the fundamental problem that, for actual markets, $\mathbb{P}_t$ emerges from the microscopic behavior of financial agents. Thus, taking $\mathbb{P}_t$ to be a given and assuming that the behavior of each agent is guided by $\mathbb{P}_t$ not only prevents the theory from having an explanatory model of market dynamics, but it invalidates the natural causal connection between microscopic agent-level behavior and macroscopic market dynamics, it assumes that agents are driven by some higher-order probability field with which they must agree in order to be rational.

In terms of evolutionary theory, the mainstream financial theory makes the behavior of the agents deterministically driven by the whole. The theory is close to Rupert Sheldrake's proposal for evolutionary theory in which there is a probabilistic field $-$ the *morphic field* $-$ that determines the microscopic organization in a process of formative causality, determined by the system's past (Sheldrake, 2000). In the mainstream financial theory, the conditional probability $\mathbb{P}_t$ is the financial analogue of a *morphic field* which determines the agents' behavior through a process of *formative causality*, by which the microscopic expectations are made to agree with the whole. This process of financial *formative causality* proceeds instantaneously from the hypothesis of homogeneous expectations.

The problem with this approach is that, not only, does it go against standard evolutionary theory and the way in which a complex adaptive system functions, but, also, any deviations of actual markets' from the theory automatically become unpredicted and unexplained phenomena, that are labelled as irrational for the lack of a better explanation.

The fact that $\mathbb{P}_t$ emerges from the agents' behavior is a relevant element for the construction of an efficient markets' theory. In this case, we must distinguish between two kinds of rationality: (1) *agent-level rationality* which should be grounded on some behavioral theory, which should take into account the recent findings within neurobiology and neuroeconomics; (2) *market-level rationality*.

A *rational expectations hypothesis* should hold as an emergent property of the agents' aggregate behavior, indicating an emergent *market-level rationality*.

The *market-level rationality* condition should be such that, in equilibrium, the market reflects in the expected price, the fair price for the asset, in accordance with the *rational expectations hypothesis* (Muth, 1961). The fair price corresponds to the correct valuation of the asset, reflecting the prospects for future value in the asset.

More specifically, in the case of the stock market, the fair price reflects the market's valuation of the firm's ability to generate future *free cash flows*[7], the firm's economic value is, then, an expression of a firm's current ability to generate future *free cash flows*. In the branch of financial theory that studies financial valuation, there are various approaches to assess the fair value, the *discounted free cash flow* approach has, however, become dominant within the theory of financial valuation, in this case, the firm's value is equal to the expected value of the discounted of the (random) future *free cash flows*.

Assuming that the firm goes on forever, and considering a discrete time sequence of *free cash flows*, with time intervals of some period τ, the economic and market conditions provided by the information set at time t, $I(t)$, may be assumed to provide for a specific stochastic process for the future *free cash flows* such that, on average, the *cash flows* grow at a rate $\bar{g}$, this implies that, the average present value of the infinite sequence of random *free cash flows*, is obtained by discounting the infinite sequence of *cash flows* growing on average at the rate $\bar{g}$, giving for the firm's value, at time t (Gregory, 1992, p.162):

$$\tilde{V}(t) = \frac{FCF(t)}{\kappa_c - \bar{g}} \tag{1}$$

where κ_c is the firm's cost of capital, $FCF(t)$ is the (known) *free cash flow* at time t.[8]

Fluctuations in value due to changes in the information set $I(t)$ imply that $\tilde{V}(t)$ must be changed at each time in which new information arrives, which

[7] *Free cash flow* is a notion of corporate finance, it corresponds to the *cash flow* that the firm generates from the operational and investment activities and before financial commitments. Three *operating value drivers* determine the *free cash flow* − sales growth, operating profit margin, and investment − and one *value determinant*, cash tax rate, determine the *free cash flow* (Rappaport and Maboussin, 2001, p.22)..

[8] The growth rate must satisfy the inequalities $\kappa_c > \bar{g} \geq 0$.

carries not only information regarding the current ability of the firm to generate future cash flows, but also elements that allow the prediction of the evolution of economic conditions affecting changes in this ability, that is the information set also contains elements that allow the definition of expectations regarding the evolution of economic conditions affecting the future evolution of value $\tilde{V}(t)$. A common assumption, in standard financial theory, is that the value $\tilde{V}(t)$ changes in accordance with:

$$\mathbb{E}\left[\tilde{V}(t+\tau)\middle| I(t)\right] = V(t)\exp(\bar{\mu}\tau) \tag{2}$$

Letting $v(t+\tau, t)$ be the logarithmic rate of return on value, we have that:

$$\mathbb{E}\left[v(t+\tau, t)\,|I(t)\right] = \bar{\mu}\tau \tag{3}$$

This means that although the conditions affecting a firm's ability to generate value may fluctuate, there is an average growth in value between two times t and $t+\tau$ equal to $\bar{\mu}\tau$.

If, on average, the value's logarithmic returns are equal to $\bar{\mu}\tau$, then, the condition of *rational expectations equilibrium*[9] (Muth, 1961) for the market's returns, dictates that the market should, on average, reflect correctly the average logarithmic return on value in the form of an equal average price logarithmic return, that is, for the price's logarithmic returns $R(t+\tau, t)$ we must have that:

$$\mathbb{E}\left[R(t+\tau, t)\,|I(t)\right] = \mathbb{E}\left[v(t+\tau, t)\,|I(t)\right] = \bar{\mu}\tau \tag{4}$$

when $\bar{\mu} = 0$ the conditions influencing value do not lead neither to an expected rise nor to a fall, that is, the *rational expectations equilibrium* would be:

$$\mathbb{E}\left[R(t+\tau, t)\,|I(t)\right] = \mathbb{E}\left[v(t+\tau, t)\,|I(t)\right] = 0 \tag{5}$$

The argument for rationality here is a game theoretical one, the financial market should, on average, compensate the holders of shares by an amount equal to the average returns on value. Alternatively, in terms of valuation, we may state that, the market's price should, on average, grow at the same rate than value thus reflecting the average increase in value[10].

[9] The expected value condition for the *rational expectations hypothesis*, was proposed by Muth (1961). The expected value condition also allows the definition of a *rational expectations equilibrium* in terms of the agreement between the *expected value* for the market variable and the *theoretical expected value* for that same variable.

Although this equilibrium may be a macroscopic equilibrium not necessarily brought forth by a microscopic homogeneity in expectations. Muth assumed microscopic homogeneity in expectations.

We shall see that Bachelier (1900) proposed a model of an efficient financial market in which heterogeneity in expectations was a pre-condition for a macroscopic *rational expectations equilibrium*.

[10] This argument and the equations (4) and (5) are independent of the chosen valuation method.

This criterion for rationality is macroscopic. It should emerge from microscopic agent-level behavior. Bachelier (1900) was the first to propose a mathematical model of an efficient financial market that also incorporated a microscopic behavioral reasoning, from which emerged a macroscopic rationality in the form of the global *rational expectations equilibrium* of the form of (5).

Bachelier's model for the market price was a random walk, with the Brownian motion as a limit, this model had a flaw in the fact that it could lead to negative prices, and was later on replaced by the geometric random walk, with the geometric Brownian motion as a continuous limit (Black and Scholes, 1976; Zhang, 1998). Nonetheless, by replacing the price by the price logarithm, Bachelier's main behavioral assumptions carry over directly to the geometric model.

Despite the geometric random walk and the geometric Brownian motion being used in standard financial theory, both of these models have a behavioral origin that differs from the mainstream financial theory, in terms of the main behavioral assumptions, namely these models, under Bachelier's theoretical arguments, depend largely upon heterogeneous expectations.

Bachelier assumed that the influences that determine the stock market movements are innumerable including past, current and foreseeable future events. On the other hand, besides these external drivers of market behavior, Bachelier also assumed that the stock market was also influenced by its own behavior, namely, the stock market is its own object of strategic intentionality, this means that it is also influenced by its past behavior and by its own evolutionary position within the economic system.

Taking into account the scope of factors that determine the stock market, Bachelier concluded that it is subordinated to an infinite number of factors and that it is impossible to expect a mathematical prevision of market behavior[11]. Instead, given the same information, it is possible to form one of two contradictory opinions regarding the future behavior of prices, more specifically, it is consistent with the information set to believe in a rise or in a fall. On this point, Bachelier's market theory differs from what eventually became the mainstream theory, because, expectations are, in Bachelier's model, necessarily heterogeneous.

The main agents in Bachelier's model are speculators, for a speculator the main objective is to take advantage from the price fluctuations in order to buy low and sell high, this means that the speculator plays a minority game with the market, selling if they believe that the market will rise and buying if they believe that the market will fall. The object of strategic intentionality of each speculator is the market itself, where the market is, in this case, defined as the set of other speculators.

Each speculator will form an expectation regarding the future price movements. It might appear, at first glance, that this behavior is not a rational one,

[11] On this regard Bachelier defined a notion of informational efficiency. If the market is truly unpredictable, it is impossible to take advantage of information to predict the price movements.

since the price fluctuation is unpredictable, but such a statement reverts the terms of causality, the price fluctuation is the result of the buying and selling, it is only unpredictable because the reasons to buy balance the reasons to sell in such a way that it becomes unpredictable which speculators will form an expectation that leads them to buy and which will form an expectation that leads them to sell.

These expectations differ from the *macroscopic rational expectations*, the *macroscopic rational expectations' equilibrium* is, in this case, only attained if the microscopic expectations are not of the form of (5), that is, if each agent either expects a rise or a fall in price.

In regard to the expectation's formation process, Bachelier distinguished two notions of probability, the *mathematical probability* that can be defined mathematically, as a measure determined from some theory[12], and the probability that depends from facts to come, being impossible to predict in a mathematical fashion. It is this last notion of probability that speculators try to predict. In this sense, each speculator analyzes the reasons that might influence a rise or a fall and the amplitudes of such movements. The process is a fully personal inductive process, and has as a necessary counter-party the inverse opinion.

This behavioral theory is compatible with current neurobiology and neuroeconomics, where the induction is based on a process of evaluation of information in which emotions play a fundamental role. The emotional basis of an expectation can be included to make Bachelier's argument consistent with the recent findings in the neurobiology of decision, unlike the theory that later on developed within finance, around the *homogeneous expectations' hypothesis*, which could not be made compatible with neurobiology.

Going from the microscopic to the macroscopic, Bachelier defined that the market (the whole system) does not believe in neither a rise nor a fall in the price. This means that, while the microscopic beliefs may be biased to a rise or fall, the global expectation forms an average of zero. In this sense, the game becomes symmetric with respect to the winnings and losses of each strategy (buying strategy or selling strategy), that is $\mathbb{E}\left[R\left(t+\tau,t\right)\left|I\left(t\right)\right]=0$, which defines a balanced game.

The symmetry for each strategy, means that the game is also symmetric for a buyer and a seller, in what regards the expected logarithmic returns; they both expect to win on average a return of zero.

Bachelier's macroscopic assumption that the mathematical expectation of the speculator is null is consistent with the microscopic assumption that the personal expectations of each speculator are not null, but that they balance themselves, given the variety of information, and of the reasons to buy or sell.

Besides Bachelier's theory there is another market theory, that developed around the same period, and that also assumed market efficiency. In this theory, emotions seem to be an important element in understanding a financial agent's

[12] Bachelier considered, in this case, the classical notion of probability, as defining the underlying theoretical probability.

decisions. This market theory was formulated by Charles Dow, who was the first editor of the Wall Street Journal. The theory itself was the result of the Wall Street Journal's editorials written by Dow.

A main divergence point between Dow's theory and Bachelier's theory is that, in Dow's theory, the market trends signify an economic meaning. There is a sense to be made of the presence of positive trends, negative trends and correction movements. In Bachelier's theory, and in the geometric Brownian motion any trends are without a variational sense other than the product of randomness, trends are the result of a purely random process, they do not signify anything other than the action of aleatorial forces.

In Dow's theory, trends in prices are linked to trends in value. The market is thought of more as a valuation game, than as a speculators' game. The main difference between the two is that although a valuation may stem from a speculators' game, the object of strategic intentionality in a speculators' game is not the companies' value. A valuation game is played by value investors, that is, investors that are interested in benefiting from the companies' ability to generate future *cash flows*, they also trade in order to take advantages from over-valuation and under-valuation, which means that they also act as arbitrageurs, guaranteeing, through their buying and selling actions that the market price reflects the companies' fundamental value.

In a valuation game the trends must signify a fundamental economic meaning, in order for the market to be efficient, that is, in order for that market's fluctuations to reflect fundamental economic information, so that a price rise must reflect a rise in value and a price fall, must reflect a fall in value, a trend in price implies a trend in value, and there is a structure in the trend linked to the companies' strategy and the general economic evolution.

If the market's valuation is reflected in prices, then, for instance, sequences of rises should reflect a trend in value which means that the companies' ability to generate future *free cash flows* has been increasing, this carries a significance beyond the pure noise theory.

In Dow's market theory trends are consistent with market efficiency, indeed, the market is assumed to quickly reflect the new information, as soon as it becomes available to potential investors. Once the news are released, stock prices will change in order to reflect this new information.

The consistency of meaningful trends with the type of predictability expressed by (4) or (5) is a major fact unrecognized in the mathematical formulation of the geometric Brownian motion and the geometric random walk, the same being true for the multifractal versions of these two processes.

The fact that prices change on information implies, necessarily, that a sequence of price rises must signify an economic meaning, the market valuation process reflects expectations about the future value in the current price, therefore, a sequence of price rises must signify an economic meaning as was stated above. According to the model we build here, the presence of nonlinearities in the markets is a consequence of the very mechanisms that are responsible for market efficiency, and trends must signify an economic meaning and be reflected in the trading behavior in order for the market to be efficient, and for the relations' (4) or (5) to hold.

Although Dow's theory does not provide for a formalization of a stochastic model of market dynamics, it already recognizes basic mechanisms that play a role in market dynamics, one example is the statement that trends are confirmed by volume, periods of high volume usually correspond to clear economic signals and well-formed expectations, while periods of low volume correspond to periods of wait-and-see, in which there is some uncertainty regarding the direction of value.

In the model we propose here, this relation between volume and market dynamics is also present, that is, we have an efficient market where trends, and nonlinearities, signify an economic meaning and play a role in the individuals' decision to buy or sell.

The model proposed here deepens the bridge between Dow's theory and the EMH, by incorporating some elements present in Dow's theory and, by doing so, it is able to reproduce some of the basic unexplained patterns in actual markets. This fact is significant for financial theory, since it joins together two lines that have developed separately within finance.

Although Dow's theory is consistent with EMH, its main elements have been incorporated mostly in technical analysis' trading technologies, and these technologies do not recognize the validity of EMH (Malkiel, 1996). Such trading technologies do not form a consistent theoretical body. Modern technical analysis is mainly a set of trading rules used by investors to form expectations regarding how the market will behave. According to the standard financial theory, technical analysis should destroy any pockets of predictability present in financial time series. The bulk of modern technical analysis cannot be considered a scientific theory, being a part of actual trading behavior.

Technical analysts, try to trade on market inefficiencies linked with mean price predictability. Some proponents of the mainstream approach to financial theory, therefore, considered that Dow's theory explicitly assumed that prices were predictable on average (Malkiel, 1996), however, Dow himself considered the markets to be efficient, describing the market as a *rational barometer*, quickly reflecting the news in the price movement (Dow, 1906b). As in standard financial theory, Dow's theory also assumes, therefore, that the market works as a social information processing system transforming economic and financial information into asset prices.

We can, therefore, consider market activity as a financial valuation, where the price is the output of the valuation process, then the market is being considered in terms of a computational nature. The market's computation is a *valuation process* since the final output of the market's computation is a price.

The problem, then, is that of knowing how the market produces a particular *output*, that is, a particular price for the asset. Since the price is the result of the buying and selling of financial agents, it follows that the information is processed in a distributed way by the social system of potential investors[13].

Instead of a fixed investor set, as in Gonçalves' and Sornette and Zhou's work (Gonçalves, 2003b; Sornette and Zhou, 2005; Zhou and Sornette, 2005), it is more realistic to consider that society listens to the news that affect the markets and people form expectations, that make them want to buy or sell. This includes, as potential buyers, all kinds of investor profiles, ranging from small investors to professional traders.

The fact that economic and financial information is usually accompanied by an analysis of its meaning, allows for more people, even without training in finance, to have, at least, a basic educated opinion as to what the future prospects, for value, are.

Buying and selling is based on expectations. In Dow's theory one distinguishes two kinds of expectations, the *bullish* (optimistic) and the *bearish* (pessimistic) (Dow, 1906c). This terminology is used mainly in technical analysis, in order to identify what is called the market sentiment, if the prices rise then the dominant sentiment should be *bullish*, while that, if the prices fall, then the dominant sentiment should be *bearish*[14].

The most salient characteristic in the *bullish* and *bearish* terminology is that it implies a recognition of an emotional basis in the markets' information processing, and we can identify, in this terminology, the two basic emotional elements that underlie traders' expectations.

The actual formation of a market sentiment differs from the solipsistic information processing proposed by mainstream finance, that assumes that investors learn and act independently. In the potential investors' social system, people interact with one another, so that potential investors socially interpret the information. Neurobiology and neuroeconomics become important in providing for an explanation of how human beings decide on a social context.

The formation of an expectations' consensus by a group of people involves the formation of shared feelings regarding the information and the future prospects for the market. Therefore, we must take into account a process of contagion of market sentiment.

In the neural activity that is present during social interaction scenarios[15] mirror neurons play an important role. The knowledge of the activity of these neurons is essential for the development of a grounded financial theory, since this class of neurons processes context-dependency in sensorial stimuli, intentionality in action, and other people's emotions (Rizzolatti *et al.*, 2006, p.30). Indeed,

[13] The market computation, that constitutes the valuation process, takes the form of an evolutionary game.

The market can be considered in this computational activity, as a computer, whose computation is the result of the evolutionary game. The social learning process behind the evolutionary game, defines the nature of the game's information processing activity.

In our present case, the social learning takes the form of an evolutionary quantum computation.

[14] This largely explains the usage of spin glass models.

[15] Even those that include listening to the financial news or reading a financial paper, or visiting a site.

mirror neurons allow people to understand the intentions of others (Rizzolatti *et al.*, 2006, p.34) and identify intentionality in actions (Rizzolatti *et al.*, 2006, p.34), they are also responsible for contagion of emotions.

Mirror neurons facilitate the process of emotions' contagion by creating an emotional response in our body that mimics the emotional response of others (Goleman, 2006, p.69). The social contagion of emotions is, in turn, responsible for organized crowd dynamics, in which the crowd acts as a single entity. However, in the case of financial markets, the behavior is not as simple as that of a crowd[16]. We have to consider a dynamics of strategic thinking in the context of social cognition.

The development of shared market sentiments, which produce shared expectations, along with the dynamics of strategic thinking, shape the nature of the social and economic forces that affect market mechanics.

The basic emotions that have been categorized into *bullish* and *bearish*, form the basis for the emergence of shared expectations, since, the basic emotions that characterize the investors correspond to two kinds of expectations about value − *optimistic* and *pessimistic*. Therefore, we have a process of expectation formation, on the basis of information that arrives to the market.

However, besides expectation formation, we have to take into account a strategic deliberation that is made on the basis of these expectations, that is, the trading does not proceed directly from the expectations in the sense that *optimism* implies a *buy* and *pessimism* implies a *sale*.

The experiments developed in London University College, by the group led by Nathaniel Daw (Buron, 2007, p.48), showed that when a human being is faced with a decision there seem to be present two phases, that may alternate, an exploration phase and an exploitation phase (Buron, 2007; Daw, 2006a).

In the exploration phase, individuals collect information in order to be able to decide (Buron, 2007, p.48). The regions of the brain responsible for logical thinking become active in the information processing. In the financial markets' case the exploration phase corresponds to the phase in which potential investors interpret the meaning of the new information that arrives to the market.

In the exploitation phase, investors implement the decision and ripe the gains from the decision taken. In this phase the elements of pleasure and reward enter into play.

Individuals seem to take pleasure in making good decisions. There seems to exist a feedback loop in the form of dopamine release that signals a good decision which reinforces the memory of that decision (Buron, 2007, p.49). The memory of a good or bad decision plays a role when an individual is faced with similar decision situations. From a game theoretical point of view, this is especially important when taking into account repeated games.

[16] The purely mechanical crowd behavior was considered to be the explanation of turbulent periods in the market, by the proponents of the noise trader hypothesis. However, this hypothesis is still flawed since it uses the same criteria for rationality used in mainstream finance, which is inconsistent with neurobiological evidence.

There is also a risk sensitivity present in a decision context. Uncertainty in information along with *bullish* or *bearish* market sentiment, may guide the way in which an individual decides.

In order to have a behavioral explanation of market dynamics we must take into account these issues in model building. In the model we propose here, there is, for each trading round, an exploration phase and an exploitation phase, which means that these recent neurobiological findings are essential in establishing a bridge between our proposal and previous theories.

In particular, it should be stressed that the model we propose is not in a rupture with the EMH, we still assume that investors are rational and that behaviors satisfy von Neumann and Morgenstern's rationality criteria, however, unlike standard financial theory, we base the model on the behavioral foundations for social psychology that are now laid out by neuroeconomics and the neurobiology of decision.

The important insight gained into market behavior is that, once one introduces an emotional basis for decision, market turbulence is a by-product of efficiency. Furthermore, the patterns of transaction volume are produced by the simultaneous action of information processing and market self-regulation mechanisms introduced by agent's behavior.

Thus, for instance, if one can predict an increase in pessimism, there are agents that take advantage of this pessimism by anticipating the possibility of an undervaluation, this leads to a spike in market activity, the fall or rise in price will depend on the amount of investors that buy and the investors that sell. This is an example of a correction mechanism that prevents undervaluation. Financial theory predicts also the existence of mechanisms that prevent over-valuation. In our model, both of these mechanisms will be present. The dynamics of information processing, along with this self-regulation mechanisms, is what makes a market an efficient information processing system, however, it is the very information processing and self-regulation mechanisms that will produce turbulence and multifractality. That is, it seems, that to function properly a market must be turbulent.

Having reviewed the behavioral foundations of the model we now proceed to introduce the approach to evolutionary quantum game theory which will be used in defining the model.

III. Bosonic approach to evolutionary quantum game theory

Quantum econophysics is a recent field of research[17], that has seen an increasing number of applications, examples of several research lines include:

1. Financial markets' modelling (Schaden, 2002, 2003a; Pakuła *et al.*, 2005; Piotrowski and Sładkowski, 2002; Singh and Prabakaran, 2006; Choustova, 2006);

2. Quantum game theory (Meyer, 1999; Vaidman, 1999; Eisert *et al.*, 1999; Iqbal, 1995; Iqbal and Toor, 2001; Piotrowski and Sładkowski, 2002, 2003a, b, 2005; Flitney and Hollenberg, 2006; Melo de Sousa *et al.*, 2006; Faber *et al.*, 2006);

3. Evolutionary quantum game theory (Iqbal, 1995; Iqbal and Toor, 2001; Hidalgo, 2005, 2006a, 2006b, 2006c, 2006d);

4. Asset valuation models (Chen, 2001; Schaden, 2003b; Montagna *et al.*, 2002a, b; Bormetti *et al.*, 2006).

The pertinence of the development of a quantum econophysics can be evaluated on two levels, a theoretical level and an empirical level. If the models and economic theories developed within quantum econophysics increase the explanatory power of economic theories, then, there is both a theoretical and empirical pertinence to quantum econophysics.

Problems, still under discussion, within neurobiology, such as the possible role of quantum mechanics in consciousness[18], leaves us with the still open hypothesis that quantum mechanics and quantum computation may play a role in neurological processes involved in decision-making, however, the models that are under development within quantum econophysics do not necessarily depend upon quantum neuroscience.

An example of this is Khrennikov's work (Khrennikov, 2003, 2006, 2007). Khrenikov argues that quantum mechanics' formalism can be applied to social systems and to cognitive science. However, in Khrenikov's formalization the quantum nature of these systems is of the same kind as that of quantum mechanics, but these systems are not reduced to an underlying quantum mechanical substractum. Khrenikov proposes a quantum-like mental model in which cognitive processes are not reduced to quantum mechanical processes in the microworld, even though they show a quantum-like nature.

[17] The beginning of the field can be dated back to Wiesner's notion of *quantum money* (Weisner, 1983), which corresponds to one of the first examples of a game against a quantum nature. Although Weisner's work was not directly aimed at economic applications but rather at the discussion of the foundations of quantum mechanics.

Well-known thought experiments, such as Schrödinger's cat, albeit not being *econophysical*, can still be considered, in light of game theory's terminology, as games against nature (nature being quantum mechanical, in this case, rather than classical).

[18] See, for instance, Penrose and Hameroff (1996) and Eccles (1995).

Schaden (2003a) has successfully shown that, within a quantum theoretical description of a financial market, one is able to account for several empirical features of the markets' probabilistic behavior, which provides for some initial assurance as to the empirical value of such an approach, from a purely economic point of view.

However, the general arguments for quantum econophysics largely depend upon the interpretation of quantum theory followed by the authors.

Khrenikov's modelling of *quantum-likeness*, for instance, is based on a pure mathematical argument such that, whenever one is unable to know the conditions which determine the systems' behavior, a (mathematical) manipulation of the total probability formula leads to the *quantum formula of total probability* (Khrenikov, 2003). The result is, in this case, independent of the system's size and nature. The argument entails an equivalence between systems (physical or not) that are described by quantum mathematical formalism. However, the argument is based upon a specific interpretation of quantum probabilities, which denies the collapse postulate, and assumes that we do not have the whole knowledge of the variables affecting the system, making the interpretation close to the hidden-variables interpretation (Khrenikov, 2006, 2007).

Choustova's work (Choustova, 2006) is an example of another work in which the hidden variables' interpretation is applied within quantum econophysics.

It is also possible, on the other hand, to follow a many-worlds interpretation (Piotrowski and Sładkowski, 2003c). Within the relative state interpretation (Everett, 1957) and its development, the many-worlds interpretation (DeWitt, 1973), quantum theory applies to the entire universe, classicality emerging as a consequence of the absence of interference effects between worlds due to decoherence[19] (Zeh, 2003, 2006; Paz and Zurek, 2002; Blume-Kouhout and Zurek, 2006) or within coarse-grained descriptions[20] (Griffiths, 1984, 1993, 1994; Omnès, 1988, 1992, Gell-Mann and Hartle, 1993, 1994, 1996, 1998).

It is noticeable that, under this interpretation, even if social games follow a *quantum-like behavior*, Everett's branching of perceptions (Everett, 1973), or, in DeWitt's interpretation, of worlds (DeWitt, 1973), must occur in the same way that it is assumed to occur for quantum physical systems[21]. At each agent's decision juncture, following the many-worlds interpretation, the universe must split, since different agents' decisions will actually lead to a different world. For

[19] It should be stressed that decoherence, and, more specifically, *environmentally-induced decoherence* (Paz and Zurek, 2002) is compatible with any interpretation of quantum theory as stressed by Zeh (2006).

[20] Hartle (2005) gives, as an example, the probabilities of different alternative orbits of the Earth around the Sun.

[21] The main difference between Everett and DeWitt is that Everett's theory is single-world theory of many perceptions. That is, for the same universe, all quantum alternatives occur but the observer's consciousness splits in different states of awareness. In each state of awareness the observer sees a different outcome.

The observer's partial memory is, by assumption, unable to know that another branch possesses a different memory of the outcome.

DeWitt follows an ontological theory. In which the universe splits into different branch-worlds, which constitute separate, parallel realities, one for each alternative outcome.

instance, a world in which a financial crash or the death of a person due to a combat situation occurs is a different world than that in which the crash did not occur and the person did not die. The decision has physical effects, due to the fact that decisions made by actual physical game players have actual physical effects, therefore, the branching universes are physically different, and, so, according to Everett and DeWitt's theory, there should be a branching at each decision juncture.

For the proponents of the relative state and the many-worlds interpretations, the extension of quantum theory, to include quantum-like behavior that occur in quantum games, is not only valid, it is, in fact, necessary, in order for the interpretation to hold[22].

However, these two interpretations are not the only ones in which a quantum description of social and economic systems is valid. Indeed, Stapp (1995) argued that the Heisenberg and von Neumann's views were also consistent with such a description.

In Heisenberg's interpretation of the quantum state, the quantum formalism also finds a direct corresponding physical theory for social and economic systems. Indeed, the main elements of Heisenberg's interpretation were first advanced by Aristotle in the Book II of the *Organon*, Heisenberg recovered some of Aristotle's philosophical notions and applied them to quantum systems.

Specifically, Heisenberg held that the probability waves of Born, Kramers and Slater could be interpreted as a quantitative formulation of the Aristotelian notion of *dynamis*, or, in a later Latin version, *potentia* (Heisenberg, 1961). For Aristotle, future events are not necessarily pre-determined, the categories of possibility and contingency can be applied to future events, in the sense that they may or may not take place.

Therefore, of a future event, according to Aristotle, we cannot say that it will be realized or not. Contingency does not allow us to talk about the actual reality of an event until that event takes place. For a contingent event which has not taken place yet we only have a set of potentialities, ways in which the event may take place and ways in which the event may not take place, the actualization corresponds to the moment in which the actual reality takes place, and, in this taking place, the event either occurs in a certain mode of its occurring or not.

The Aristotelian modal logic of events, is not a binary logic. Before the event's actualization, it is neither true, nor false that the event will occur. Only after the occurrence can we talk about an either true or false occurrence. The actualization of the potentialities correspond to the collapse of the uncertainty to certainty, and the period before the actualization/collapse corresponds to the state of uncertainty described, in quantum mechanics, by the wave function.

Aristotele's modal logic as it applies to events fits well with the quantum logic that underlies the Copenhagen interpretation. This relation between quantum logic and Aristotelian modal logic was recognized by Heisenberg. According to

[22] A conclusion that is reinforced by the fact that in both these interpretations it is assumed that everything in the universe, including conscious observers are described by the Hilbert space formalism and quantum mechanics.

Heisenberg, the notion of *dynamis* assumes a new form, in the sense that it is formulated quantitatively as a probability, and subject to the mathematically expressed laws of nature (Heisenberg, 1961).

The (quantum) laws of nature no longer determine the phenomena themselves, they only determine the potentiality for the occurrence of a given phenomenon – the probability that something will occur (Heisenberg, 1961). The probability, in Heisenberg's interpretation, corresponds therefore to the *tendency* of something to be *actualized* (to acquire an *actuality*). This tendency, can be likened to a *force*, where by *force* it is meant the Aristotelian notion of *dynamis*. Popper (1992a, [1982]; 1992b, [1982]) expressed this notion of probability as a *propensity* of *something* to be *realized* with *repetition*. Since one may conceive the possibility that some events could occur only once, especially in social science[23], and the notion of *repetition* could approximate us to the ensemble interpretation, we prefer the notion that stems from Heisenberg's interpretation, of a probability as a *dynamis*, in the form of a *physical propensity* for something to be *actualized*.

In the present work, we follow this interpretation. Thus, for instance if the quantum strategic state associated with an agent, is in a superposition of strategic alternatives, this means that the agent tends to follow each of these alternatives with a given propensity.

Quantum game theory within a purely economic setting is, therefore, justifiable, when an agent is considered not to have chosen a given strategy with certainty but to have some propensity to choose each of the several pure alternative strategies available.

This economic interpretation can be extended to the evolutionary setting, in which the propensities are *physical propensities* in the social system, and the forces that affect the unitary evolution of the game correspond to social and economic forces that are formalized by the unitary operator.

Evolutionary game theory is a natural stage for dealing with social systems where there is a great number of agents, since agents may freely enter and leave the game, or change strategy. The evolution takes place at the social level, and there is a process of strategy's social selection.

Classical evolutionary game theory was developed to deal with biological systems, where natural selection operates. In this case, the rationality of the choice is imposed by the (blind) mechanism of Darwinian natural selection. The origin of this branch of game theory dates back to 1973, when Maynard Smith and G. R. Price wrote an article in *Nature* showing how game theory applies to the behavior of animals. Later on, Maynard Smith (1982) wrote a book on the subject.

The core of the theory involves an evolutionary perspective on the following notions of classical game theory (Gintis, 2000, p.178):

1. The notion of strategy – in traditional game theory players have strategy

[23] The probability is there, as a *physical propensity* present in the social and economic system, even if the event happens only once.

sets from which they choose particular strategies, in biology, *species* have strategy sets (genotypic variants), of which *individuals* inherit one or other variant (perhaps mutated);

2. Equilibrium – in place of the notion of equilibrium of non-cooperative game theory (the Nash equilibrium), Maynard Smith and Price introduced the notion of *evolutionarily stable strategy.* A strategy is evolutionarily stable if a whole population using that strategy cannot be invaded by a small group with a mutant genotype;

3. Player interactions – in evolutionary game theory we always have repeated games, where agents are randomly paired.

Evolutionary stability is usually implemented through the notion of a *replicator*, which allows the definition of a set of differential equations[24] – the *replicator equations* – that define the *replicator dynamics.* The *replicator dynamics* is an important tool that allows the extension of evolutionary game theory to social and economic problems.

A *replicator* is an entity that has some means of making approximate copies of itself. In evolutionary game theory applied to social sciences, a *replicator* can be a strategy, a technique, a convention, a cultural trait, or a more general institutional or cultural form, it takes the form of what Dawkins (1989) calls a *meme.* The *stable fixed points* of the *replicator equations* correspond to the *evolutionarily stable strategies.*

There is an advantage in using these equations, in that they allow the *fitness landscape*[25] to incorporate the distribution of the population types rather than setting constant the *fitness* of a particular type.

Hidalgo (2005, 2006a,b,c,d) has shown that there is a close relation between the differential equations that define the *replicator dynamics* and von Neumann's equation for the evolution of the quantum density operator. *Quantum evolutionary stability* has also been researched by Iqbal (1995) and by Iqbal and Toor (2002), along with frameworks for quantum repeated games.

We consider here that the language of second quantization provides for a natural quantum theoretical framework for *quantum evolutionary game theory* (QEGT), since it simultaneously allows for a quantum framework for classical evolutionary game theory, and an evolutionary framework for quantum game theory.

In standard quantum game theory one may consider a set of *player types*, $\Pi = \{p_1, ..., p_n\}$, for each *player type* there is a set of alternative pure strategies

[24] It is also possible to introduce difference equations for the *replicator dynamics.*

[25] In evolutionary biology, *fitness landscapes* or *adaptive landscapes* are used to visualize the relationship between genotypes (or phenotypes) and reproductive success.

The *fitness* of a *genetic trait* is proportional to the frequency of individuals in a species that possess that *trait.*

The set of all possible genotypes, their degree of similarity, and their related fitness values is called a *fitness landscape.*

that are represented as basis vectors in a basis[26] for the *player type*'s Hilbert space, that is, if $\mathbb{S}(p_j)$ is the set of pure strategies, then we take[27] $\mathbb{S}(p_j) = \{|s_i(p_j)\rangle : i = 1, ..., m_j\}$, where m_j is the number of alternative strategies for the *player type* p_j, and $|s_i(p_j)\rangle$ corresponds to the pure strategy s_i for the *player type* p_j.

To each player there corresponds a quantum state that can be pure or mixed. Apart from the basis states, any other pure state is a state in which there is uncertainty regarding which pure strategy the player chooses. For instance, if we have the following state, for a player corresponding of the type p_j:

$$|p_j\rangle = \sum_i c_i |s_i(p_j)\rangle \tag{6}$$

then, $\langle s_i(p_j)|p_j\rangle = c_i$ is the probability amplitude that the player plays the strategy $|s_i\rangle$. We call $|p_j\rangle$ the player's *quantum strategic state.*

We can describe the evolution in the *quantum strategic state* in terms of a unitary operator, therefore, between an initial time, t_i, and a final time, t_f, where the play must take place, we have:

$$|p_j, t_f\rangle = U(t_f, t_i)|p_j, t_i\rangle \tag{7}$$

The use of unitary operators and pure quantum states is a distinctive feature of quantum game theory[28], because it allows for interference effects (Piotrowski and Sładkowksi, 2003a).

Since, in our present case, we wish to deal with a dynamics of social learning and with a population of players, it is useful to shift the attention from the individual player to the sets of players that find themselves in a game situation against other players playing some strategy.

For a population of *n-player types*, the set of alternative pure strategy configurations, which gives the game's *strategic basis states*, is given by the tensor product of the pure strategy basis states for each player, this defines a basis

[26] Usually, this basis is defined as the basis for some game payoff observable.

[27] We use Dirac's *bra-ket* notation (Dirac, 2004, [1958]). In this case, if $\mathcal{H}$ is a Hilbert space, then a vector in that space is denoted by $|v\rangle$, which corresponds to what Dirac called a *ket vector*. The conjugate imaginary of $|v\rangle$ is the *bra vector* $\langle v|$ (Dirac, 2004, [1958], p.21)) so that the inner product in $\mathcal{H}$ is given by $(|u\rangle, |v\rangle) = \langle u|v\rangle$, and the outer product $|u\rangle\langle v|$, finally, the tensor product $|u\rangle \otimes |v\rangle$ with $|u\rangle \in \mathcal{H}$ and $|v\rangle \in \mathcal{H}'$, leads to a *ket vector* in the space $\mathcal{H} \otimes \mathcal{H}'$.

[28] In classical game theory the only probabilistic notions were expressed in terms of the mixed strategies that corresponded to mixed states of the form:

$$\sum_i w_i |p_j; s_i\rangle \langle p_j; s_i| \tag{8}$$

where $\sum_i w_i = 1$. That is, classical game theory worked either with pure quantum states in the strategic basis (the notion of pure strategy), or with mixed quantum states that are diagonal in that basis (the notion of mixed strategy).

The definition of the values for the weights w_i was defined, in classical game theory, in terms of mixed strategy Nash equilibria.

$\{|s_{i_1}...s_{i_n}\rangle = |s_{i_1}(p_1)\rangle \otimes |s_{i_2}(p_2)\rangle \otimes ... \otimes |s_{i_n}(p_n)\rangle\}$ for the game's Hilbert space $\mathcal{H}_{\mathcal{G}}$. The game's general pure state is, then, defined by:

$$|\mathcal{G}\rangle = \sum_{i_1...i_n} c_{i_1...i_n} |s_{i_1}...s_{i_n}\rangle \tag{9}$$

The connection with evolutionary game theory, comes from considering, a population of N games, that is, N sets of n players playing the game. Assuming that no entanglement occurs between games, the quantum state for such a *game population*, can be given by:

$$|\mathcal{G}_1...\mathcal{G}_N\rangle = |\mathcal{G}_1\rangle \otimes |\mathcal{G}_2\rangle \otimes ... \otimes |\mathcal{G}_N\rangle \tag{10}$$

where $|\mathcal{G}_k\rangle$ is the k-th game's quantum state.

The problem, that we now face, is that of deciding whether the state should be *symmetrical* or *antisymmetrical*, which would lead to either a bosonic or a fermionic theory. If we worked with an *antysimmetrical* state (a *fermionic game theory*), then, no more than one set of n players can choose the same strategy combination, that is, no more than one game can be in the same configuration (the same quantum state). Since in social game theory, for different sets of n players, playing independent versions of the same game, more than one player, can play the same strategy, the quantum state for the population must be taken as *symmetrical* (a *bosonic game theory*).

It should be stressed that we are dealing with the population of players only from the point of view of the strategy they play, that is, for all practical purposes, all the relevant information for the game theoretic description of each player is contained in the *quantum strategic state*. The *quantum strategic state* is all there is to be known, it completely identifies each player.

This is an important restriction to the application of this theory within social contexts. The *quantum strategic state* must contain all the relevant information, in such a way that, individual players become indistinguishable if they have the same state. This extends to the games themselves, which means that two independent games with the same quantum state in $\mathcal{H}_{\mathcal{G}}$ are indiscernible[29].

Thus, we are dealing with the population of N games as if they were an assembly of bosons[30] (Dirac, 2004, [1958]). Letting $m = \prod_j m_j$ be the number of alternative game configurations[31], then the dimensionality of the space $\mathcal{H}_{\mathcal{G}}$ is equal to m and we can introduce the numbers $n_1, n_2, ..., n_m$ as the numbers of games in the game population in which each alternative strategy combination is chosen. We can, therefore, deal with the general state of a population of games in terms of occupation number states, thus $|n_1, ..., n_m\rangle$ corresponds to a state in which n_k games are played with the k-th strategy combination, that is, n_k games are in the *k-th* quantum state, with $k = 1, ..., m$.

[29] Particle indescernibility, therefore, carries over directly to the quantum theoretic description of a game.

[30] This results, as stated above, from the fact that we work with a symmetrical quantum state for the population of games.

[31] That is, the number of pure strategies' combinations for the n-players.

We can introduce the bosonic creation and annihilation operators:

$$a_k^\dagger \left|..., n_k, ..., n_{m_j}\right\rangle = \sqrt{n_k+1}\left|..., n_k+1, ..., n_{m_j}\right\rangle \tag{11}$$

$$a_k \left|..., n_k, ..., n_{m_j}\right\rangle = \sqrt{n_k}\left|..., n_k-1, ..., n_{m_j}\right\rangle \tag{12}$$

$$[a_k, a_l] = \left[a_k^\dagger, a_l^\dagger\right] = 0 \tag{13}$$

$$\left[a_k, a_l^\dagger\right] = \delta_{kl} \tag{14}$$

Although we shall work here with a finite set of configurations, it is possible to consider an infinite number of configurations, letting $m \longrightarrow \infty$.

We can now define $|\phi\rangle$ as the quantum state for a population of games, that belongs to the Hilbert space with basis given by $\{|n_1, ..., n_m\rangle : n_k = 0, 1, ...\}$. The evolutionary quantum game is, therefore, completely defined by:

1. A set of *player types* $\Pi = \{p_1, ..., p_n\}$;
2. The strategy set for each *player type* $\mathbb{S}(p_j) = \left\{|s_1(p_j)\rangle, ..., \left|s_{m_j}\right\rangle\right\}$;
3. An initial configuration for the occupation number states $|\phi(t_i)\rangle$;
4. A unitary evolution operator $U(t_f, t_i)$ for the occupation numbers that represents an adaptive dynamics;

Given $|\phi(t_i)\rangle$ and $U(t_f, t_i)$ we can determine the probability amplitudes for each final occupation numbers as follows:

$$\langle n_1, ..., n_m | U(t_f, t_i) | \phi(t_i)\rangle \tag{15}$$

the above result gives us the probability amplitude for the transition from the initial state $|\phi(t_i)\rangle$ to the final state $|n_1, ..., n_m\rangle$.

The key element here is the evolution operator, since it is this element that incorporates the evolutionary dynamics. In the case of human social games, its form is determined by the underlying theories that define the social learning and strategic choice. The evolution $U(t_f, t_i)|\phi(t_i)\rangle$ corresponds, in this formalism, to a quantum replicator dynamics, however, the type of replicator dynamics need not be a quantum generalization of the classical form of evolutionary game theory's replicator dynamics.

The action of $U(t_f, t_i)$ on the initial game configuration $|\phi(t_i)\rangle$, may incorporate social, economic and cognitive mechanisms. Drawing upon the evidence regarding the two phases of decision (the exploration phase and the exploitation phase), we have that, for social systems, the unitary evolution operator expresses an exploration phase in which the agents learn about the alternatives, while the exploitation phase corresponds to the final moment in which agents distribute themselves between the different strategies[32]. Therefore, the form

[32] Corresponds to the moment of collapse of the state into one of the basis states in the occupation number basis.

of $U(t_f, t_i)$ will depend upon the social, economic and cognitive mechanisms involved in learning.

This framework is substantially different from classical evolutionary game theory, since in the phase of unitary evolution, we do not have generations of players being born and dying, or entering and leaving the game, the unitary evolution expresses a transition between two occupation number states, the (probabilistic) strategic choice occurs only at the end of the period, with the transition amplitudes given by (15). This matter begs, however, a deeper inspection.

An important issue, when faced with a quantum game is the problem of knowing when one can consistently assign probabilities that obey Kolmogorov's axiomatic. In our case, this issue is raised, in particular, for the final state $\left|\phi_j(t_f)\right\rangle$, where the decision takes place. Since we are dealing with closed game models, we are assuming that all the relevant elements for the game's quantum dynamics are already accounted for in the game's description, therefore, we need a criterion for consistently assigning probabilities in a closed game model[33].

We may notice that, for a general quantum system with initial state $|\psi_i\rangle$, that can be expanded in the basis $\{|O_n\rangle\}$ of some observable $\mathcal{O}$, we can introduce the complete set of projection operators $\left\{\hat{P}_n\right\}$, where $\hat{P}_n = |O_n\rangle\langle O_n|$, then, under the unitary evolution $U(t_f, t_i)$:

$$\left|\psi_f\right\rangle = U(t_f, t_i)|\psi_i\rangle \tag{16}$$

Given an operator $\hat{B}$ on the system's Hilbert space $\mathcal{H}$, we can define $\mathcal{A}$ such that:

$$\begin{aligned} \mathcal{A}\left(\hat{B}; n, m\right) &= \left\langle \psi_i \left| \hat{C}_m^{\dagger}\hat{B}\hat{C}_n \right| \psi_i \right\rangle = \langle\psi_f|O_m\rangle\langle O_n|\psi_f\rangle \left\langle O_m \left| \hat{B} \right| O_n \right\rangle & (17) \\ \hat{C}_j &= \hat{P}_j U(t_f, t_i) & (18) \end{aligned}$$

we, then, have, at the end of the unitary evolution period:

$$\left\langle \hat{B} \right\rangle = \sum_{n,m} \mathcal{A}\left(\hat{B}; n, m\right) \tag{19}$$

[33] Such a closed-game model only holds if the game is not affected by other factors unnacounted for in $U\left(t_f, t_i\right)$, such as interactions with other systems that change the game's strategic quantum state.

As we argue in the conclusion to this work, such an assumption only holds as an approximation, in which the game's strategic quantum state becomes sufficiently separable from the universe's quantum state, that allows us to assign a quantum state for the game without reference to the universe's quantum state.

Following Hartle (2005), we may consider this as a coarse-grained description of the actual system, and work with the isolated system formalism. This simplifies the game dynamics to its bare essencials without significantly changing it, which proves useful in the market model we propose here.

Furthermore, game theory usually works with closed-games, therefore, it is useful to have a link with classical closed-games.

However, as we stressed, by choosing the closed-game formalism, a consistency criterion for assigning probabilities must be provided.

If the operator $\hat{B}$ has a diagonal matrix representation in the basis $\{|O_n\rangle\}$, then, the terms $\left\langle O_m \left| \hat{B} \right| O_n \right\rangle$, with $m \neq n$ are all equal to zero, the matrix $\mathcal{A}_M$, with entries $\mathcal{A}\left(\hat{B}; n, m\right)$, reduces to a diagonal matrix. When the matrix $\mathcal{A}_M$ is diagonal the interference effects in the pure state do not affect the average of $\hat{B}$, so that, $\hat{B}$ cannot distinguish between a pure state and the mixed state with a diagonal density matrix. We, then, state that $\left|\psi_f\right\rangle$ is *decoherent relative* to $\hat{B}$. This criterion for *relative decoherence* is similar to Gell-Mann and Hartle's notion of *medium decoherence* (Gell-Mann and Hartle, 1998), where $\mathcal{A}$ plays a role similar to the decoherence functional[34].

If we define the chain for the logical disjunction between alternatives $\hat{C}_{n\vee m} = \hat{C}_n + \hat{C}_m$ we have:

$$\begin{aligned}
\mathcal{A}\left(\hat{B}; n\vee m, n\vee m\right) &= \left\langle \psi_i \left| \left(\hat{C}_n^{\dagger} + \hat{C}_m^{\dagger}\right) \hat{B} \left(\hat{C}_n + \hat{C}_m\right) \right| \psi_i \right\rangle = \\
&= p\left(O_m\right) \left\langle O_m \left| \hat{B} \right| O_m \right\rangle + p\left(O_n\right) \left\langle O_n \left| \hat{B} \right| O_n \right\rangle + \\
&+ \left\langle \psi_f | O_m \right\rangle \left\langle O_n | \psi_f \right\rangle \left\langle O_m \left| \hat{B} \right| O_n \right\rangle + \left\langle \psi_f | O_n \right\rangle \left\langle O_m | \psi_f \right\rangle \left\langle O_n \left| \hat{B} \right| O_m \right\rangle
\end{aligned}$$

If $\left|\psi_f\right\rangle$ is *decoherent relative* to $\hat{B}$, the last two interference terms vanish, and $\mathcal{A}$ reduces to the sum

$$\mathcal{A}\left(\hat{B}; n\vee m, n\vee m\right) = p\left(O_m\right) \left\langle O_m \left| \hat{B} \right| O_m \right\rangle + p\left(O_n\right) \left\langle O_n \left| \hat{B} \right| O_n \right\rangle \tag{20}$$

This result, in classical game theoretical analysis, corresponds to a player's *anticipation*[35] regarding the alternative results $|O_n\rangle$ and $|O_m\rangle$.

In this case, we have a classical probabilistic behavior, since the alternatives for $\hat{B}$ have the representation of a classical mixture of alternatives (the classical *anticipation*).

For a quantum evolutionary game, if $\hat{B}$ is an operator on the Hilbert space of the occupation number states, then, interference only affects the course of the game if $|\phi(t_f)\rangle$ is not *decoherent relative* to $\hat{B}$. When $|\phi(t_f)\rangle$ is *decoherent relative* to $\hat{B}$, the entries in the matrix representation of $\hat{B}$ only assign values to each branch defined by the projection operators $\hat{P}_{n_1 \ldots n_m}$ and the probabilities

[34] The notion of *relative decoherence*, addressed above, should not be confused with the notion of *relative decoherence* proposed by Herbut (2001), although it is shares some aspects in common with Herbut's notion.

Indeed, Herbut (2001) considered the possibility of the same quantum system being simultaneously decoherent and coherent, depending on the observer. According to Herbut's approach an observer may not 'see' the coherences, while another observer may.

Although, in our case, the same state may be 'seen' as decoherent or as coherent depending on the observable, we do not assume the existence of a classically behaving hidden-variable like Herbut (2001) does.

Our notion is not new, it is a straightforward result of the fact that if a quantum state is expanded in the basis of an observable, then, the relative phases are unobservable for every operator with a diagonal representation in the observable's basis.

[35] This term is used by Nash (1950) when discussing an individual's utility theory. We keep the term here.

for each alternative satisfy Kolmogorov's axiomatic, which means that we can consistently assign probabilities to the different alternative values of the matrix entries of $\hat{B}$.

The Hermitian operators for which $|\phi(t_f)\rangle$ satisfies the condition for *relative decoherence*, correspond to the game population's observables. In the case of financial markets there will be some important observables $-$ the logarithmic returns and the market activity. Since in social games, the game's observables form the relevant quantities, it follows that, at the final decision time, no interference effects are detected with respect to these observables' averages. The averages and the probability rules satisfy the classical rules.

The probabilities, in the context of a social game, can be interpreted as *physical propensities* in society for different strategic choices and, thus, different social distributions of the quantities represented by the game observables. these *propensities* are determined by economic, social and cognitive mechanisms.

Having introduced the general physical and mathematical formalism we are now ready to discuss the model itself.

IV. The model

The model we propose, in this section, is a single-asset model that combines Sornette and Zhou's work with the previous section's formalism. The model introduces a single *player type*, known as a value investor. Value investors reflect, in their bids, the information regarding the prospects for the value of the financial asset. This is in a direct continuity with standard financial theory, since, in order for markets to be efficient, value investors must be the dominant force behind market activity, transforming information about an asset's intrinsic value into a market price.

Information is the basic polarizing field for market sentiment. In standard finance, the simplest model for randomness in information is the binomial model (Cox *et al.*, 1979). Towards a continuity of discourse with standard financial theory, we also consider a binomial process for information. Therefore, we model the information flow in terms of classical stochastic process where a news signal $\mathcal{N}(t)$ arrives at discrete times and such that $\mathcal{N}(t) = e^{\sigma(t)}$ where $\sigma(t) = W$ with a probability p, or $\sigma(t) = -W$ with probability $1-p$, for a time-independent positive real W. In the first case, the news are good, while in the second case the news are bad[36].

We can go further on, with this continuity with standard financial theory, and assume that the information signal is a reflection of a stochastic behavior in the intrinsic value of the asset under market valuation.

Now, there are two strategies for each investor, a buying strategy and a selling strategy, which leads to the occupation number states $|n_0, n_1\rangle$ where n_0 corresponds to the number of investors that are selling and n_1 corresponds

[36] This combines standard financial theory with Gonçalves (2003b) and Sornette and Zhou's (2005) modelling of the external information signal.

to the number of investors that are buying. In this case, each value investor plays a game with information as her partner, in which, she tries, either to sell or buy in accordance with her expectations regarding future value. Each game, is therefore played by a single person, it corresponds to what is called, in game theory, a game against nature, in these cases each game's *quantum strategic state* coincides with each player's *quantum strategic state*, so that the occupation number state for the games' population is also an occupation number state for the players' population.

We assume that trading is temporally synchronous and occurs at the end of each trading round. Each trading round corresponds to an evolutionary game, in which the trading strategies are defined in accordance with the social information processing of the news signal, for the round.

Since, before evaluating the new information that has arrived, the traders' sentiment for that round is neutral, the initial state for each game is the vacuum state $|0,0\rangle$, which means that, at the beginning of each game no sentiment regarding the new information, and therefore no intention as to the strategy to follow, has yet been formed. This is consistent with the EMH, otherwise the market would not be efficient.

The occupation numbers for each trading strategies will depend on the information processing that occurs during the trading round. Each round has a duration of τ_P which defines a time horizon for the game. The temporal scale, for which τ_P is defined, will be determined empirically[37]. We also assume τ_P to be constant[38].

Since each game lasts for a time interval of τ_P, if t_k is the time of the k-th trading round, the next trading round will end at $t_k + \tau_P$ that is $t_{k+1} = t_k + \tau_P$, letting $t_0 = 0$ we have that $t_k = k\tau_P$.

At the end of each round there is a game that ends, with the trading occurring, and new information arrives, which initiates a new game.

As information is processed, physical propensities in the social system form themselves, making the system tend to buy or sell, these physical propensities are expressed mathematically by the probability amplitudes. At the end of each game, the strategic choices take place, with the probability amplitudes taken from the general rule (15). Letting t_k be the beginning of the k-th game we,

[37] As we shall see in the following section, if we take $\tau_P = 0.1$ then, the data shows that 100 trading rounds are approximately equal to one trading day. This result is based on the fact that the volatility autocorrelation function's decay approximates that of the actual markets, for a sampling scale of 100 trading rounds.

If $\tau_P = 0.1$ is equivalent to $1/100$ trading days, and taking into account, for instance, that a trading day in the US stock exchange opens at 9:30 a.m. and closes at 4:00 p.m., then, τ_P corresponds to 3.9 minutes.

[38] It would be possible to adapt the model to include effects from economic cycles, and changing trading horizons by allowing for a trading round-dependent τ_P, this would, however, introduce an artificial temporal distortion, or could even produce a multifractal pattern.

Since we want to model an emergent multifractality, it follows that we should make τ_P constant, for a first approach.

then, have the transition amplitudes for that game:

$$\Psi\left(n_0, n_1; 0, 0\right) = \langle n_0, n_1 \left| U_k\left(t_{k+1}, t_k\right) \right| 0, 0 \rangle \tag{21}$$

where $U_k\left(t_{k+1}, t_k\right)$ is the unitary operator for the *k-th round* (which corresponds to the *k-th game*).

The market behavioral theory determines the evolution operator. We let U_k vary from game to game, in order to adapt to changing market conditions. This means that each trading round can be thought analogously to a repeated quantum experiment, where the experiment's parameters are changed in accordance with the previous experiment's results.

Although U_k is assumed to be of some general form, it must change from game (*trading round*) to game (*trading round*), in accordance with some parameter that is defined in terms of some process of social cognition. Since we must consider a process of social cognition, we must also consider that the market is influenced by the new information that arrives at each round and the way in which this information is interpreted by potential investors. We leave U_k undefined, for now, and address the other assumptions made.

In terms of the total number of agents, we do not impose a fixed number of potential investors, since it would unnecessarily restrict the results to a bounded population number. Instead, we use the same simplifying assumptions that are made in queue management theory, in which one does not impose a bound on the queue.

It would be possible to extend the model to include societies with a maximum number of potential investors and to allow this number to fluctuate with time, but this would unnecessarily increase the complexity of the game setup, and for societies in which the number of individuals is very large, the assumption of an unbounded number of potential investors provides for a good approximation.

The price formation is implemented by a *market maker*, which enters in our model, in the same way as in the previous ones by Gonçalves (2003b) and by Sornette and Zhou (Sornette and Zhou, 2005; Zhou and Sornette, 2005; Zhou and Sornette, 2007), that is, solely through the price formation rule. Thus, the *market maker* is treated solely in terms of the market clearing mechanism that she induces and is not considered in terms of any strategies.

The *market maker* guarantees that the market clears by taking up the excess demand or supply. Therefore, the orders are filled by the *market maker* at a price that depends upon the net market order.

In terms of the quantum model the *market maker* enters, not as an agent, but as an automated mechanism that is formalized in terms of an observable, the *logarithmic returns' observable*, introduced in the model.

One might be inclined to assign the *market maker* the role of a measurement device, becoming entangled with the agents' decision, however, this is not the case, indeed, the *market maker* already receives the trading orders after the decisions take place, which correspond to the collapsed state at τ_P. The order placed is a collapsed orders at τ_P.

The *market maker* can be defined in terms of the *market impact function* f_M, which is the mechanism that the *market maker* uses to set the prices, and that leads to a market clearing mechanism that can be expressed in terms of a price formation rule relating the net order to the new price.

At each trading round, we take the price to be a function of the net order $S(k)$ for the k-th trading round. The net order is defined a difference between the demand and the supply, therefore, the *market maker* can be introduced solely in terms of the following algorithm to compute the price:

$$P(k) = f[P(k), S(k)] \tag{22}$$

Following Farmer (2000), we assume that f is of the form

$$f[P(k), S(k)] = P(k) f_M [S(k)] \tag{23}$$

where the market impact function f_M is taken to be an increasing function with $f_M(0) = 1$, leads to the log-linear market impact function (Farmer, 2000). In this case, following Farmer's rule, the logarithmic returns $R(k)$ are given by:

$$R(k) = \ln P(k) - \ln P(k-1) \approx \frac{S(k)}{\lambda} \tag{24}$$

where λ is a normalizing liquidity parameter.

If, following Gonçalves (2003b), Sornette and Zhou (2005), we take $S(k)$ to be the difference between the number of buyers and sellers, then the parameter λ translates $S(k)$ into units of price logarithm.

The market order can be determined in terms of the *market order observable* $\hat{S}$:

$$\hat{S} = a_1^\dagger a_1 - a_0^\dagger a_0 \tag{25}$$

therefore:

$$\hat{S} |n_0, n_1\rangle = (n_1 - n_0) |n_0, n_1\rangle \tag{26}$$

Defining the two number operators $\hat{N}_0 = a_0^\dagger a_0$ and $\hat{N}_1 = a_1^\dagger a$, the logarithmic returns' observable is, then:

$$\hat{R} = \frac{1}{\lambda}\left(\hat{N}_1 - \hat{N}_0\right) \tag{27}$$

$$\hat{R} |n_0, n_1\rangle = \frac{1}{\lambda}(n_1 - n_0) |n_0, n_1\rangle \tag{28}$$

the *market maker* therefore, enters in our model only in terms of the observable $\hat{R}$ and the price logarithm $\ln P(k)$ defined by $\ln P(k-1) + \frac{1}{\lambda}(n_1 - n_0)$.

Another important observable is the total number of investors $\hat{N} = a_0^\dagger a_0 + a_1^\dagger a_1$, which corresponds to a measure of market activity, that, in this case, coincides with transaction volume[39].

In order to have the market game completely specified, we must now define the unitary operators $U_k(t_{k+1}, t_k)$. We begin by formalizing these operators and, afterwards, address their behavioral contents and their economic and financial significance.

We define $U_k(t_{k+1}, t_k)$ as follows:

$$U_k(t_{k+1}, t_k) = \exp\left(-i(t_{k+1} - t_k)\sum_{j=0}^{1}\left(\mu_j(k)\, a_j^{\dagger} - \mu_j(k)\, a_j\right)\right) \tag{29}$$

where $\mu_j(k)$ is a game-dependent real number that will incorporate the cognitive dynamics. Since $t_{k+1} - t_k = \tau_P$, if we define $\xi_j(k, \tau_P) = -i\tau_P \mu_j(k)$ then we can re-write U_k as:

$$U_k(t_{k+1}, t_k) = \exp\left(\sum_{j=0}^{1}\left(\xi_j(k, \tau_P)\, a_j^{\dagger} - \xi_j^{*}(k, \tau_P)\, a_j\right)\right) \tag{30}$$

therefore, U_k can be expressed as a product of displacement operators D for the annihilation operators a_j:

$$D\left(\xi_j(k, \tau_P)\right) = \exp\left(\xi_j(k, \tau_P)\, a_j^{\dagger} - \xi_j^{*}(k, \tau_P)\, a_j\right) \tag{31}$$

Under the action of U_k, the vacuum state, for the beginning of each trading round, is transformed into the coherent state $|\xi_0(k, \tau_P), \xi_1(k, \tau_P)\rangle$:

$$\begin{aligned} U_k(t_{k+1}, t_k)\,|0,0\rangle &= D(\xi_0(k, \tau_P))\, D(\xi_1(k, \tau_P))\,|0,0\rangle \\ &= |\xi_0(k, \tau_P), \xi_1(k, \tau_P)\rangle \end{aligned} \tag{32}$$

where $|\xi_0(k, \tau_P), \xi_1(k, \tau_P)\rangle$ is given by:

$$|\xi_0(k, \tau_P), \xi_1(k, \tau_P)\rangle = e^{-\frac{|\xi_0(k,\tau_P)|^2 + |\xi_1(k,\tau_P)|^2}{2}} \sum_{n_0, n_1} \frac{\xi_0^{n_0}(k, \tau_P)\, \xi_1^{n_1}(k, \tau_P)}{\sqrt{n_0! n_1!}}\, |n_0, n_1\rangle \tag{33}$$

[39] This coincidence of the number of investors trading and the transaction volume is an approximation, already present in Gonçalves' model and in Sornette and Zhou's model (Sornette and Zhou, 2005; Zhou and Sornette, 2005; 2007).

The approximation is that each investor buys or sells the same (fixed) integer number of units of the financial asset.

Although this is a simplifying assumption, it provides an advantage with respect to standard financial theory, in which it was assumed that investors could buy parts of shares.

The assumption of share divisibility led to problems with respect to possible deviations from the standard financial theory, and it has been a subject of research within the field of study known as market microstructure (O'Hara, 1998).

In markets with a large number of agents, where no given agent can influence the overall market behavior the assumption that each investor buys a fixed amount of shares is not a simplification as severe as the one that assumes a share divisibility.

the transition amplitudes are, thus, given by:

$$\Psi_{k,\tau_P}(n_0,n_1;0,0) = e^{-\frac{|\xi_0(k,\tau_P)|^2+|\xi_1(k,\tau_P)|^2}{2}} \frac{\xi_0^{n_0}(k,\tau_P)\,\xi_1^{n_1}(k,\tau_P)}{\sqrt{n_0!n_1!}} \tag{34}$$

which leads to the bivariate Poissonian distribution for the occupation number states:

$$\begin{aligned} P_{\xi_0,\xi_1}(n_0,n_1) &= |\Psi_{k,\tau_P}(n_0,n_1;0,0)|^2 = \\ &= \frac{\exp\left(-|\xi_0(k,\tau_P)|^2 - |\xi_1(k,\tau_P)|^2\right)|\xi_0|^{2n_0}|\xi_1|^{2n_1}}{n_0!n_1!} \end{aligned} \tag{35}$$

The expected value for a market observable $\mathcal{O}$ at the end of each round is, then, given by:

$$\langle\mathcal{O}\rangle_{k,\tau_P} = \langle\xi_0(k,\tau_P),\xi_1(k,\tau_P)|\mathcal{O}|\,\xi_0(k,\tau_P),\xi_1(k,\tau_P)\rangle \tag{36}$$

therefore, we have, for the logarithmic returns and the market activity observables[40]:

$$\left\langle\hat{R}\right\rangle_{k,\tau_P} = \frac{1}{\lambda}\left(|\xi_1(k,\tau_P)|^2 - |\xi_0(k,\tau_P)|^2\right) \tag{37}$$

$$\left\langle\hat{N}\right\rangle_{k,\tau_P} = |\xi_1(k,\tau_P)|^2 + |\xi_0(k,\tau_P)|^2 \tag{38}$$

The market's average dynamics depends upon the dynamics of the parameters $\xi_1(k,\tau_P)$ and $\xi_0(k,\tau_P)$. It is at this stage that we must consider the behavioral issues. We begin by noticing that, from the definition of $\xi_j(k,\tau_P)$, we can write:

$$\left|\xi_j(k,\tau_P)\right|^2 = \left|-i\mu_j(k)\,\tau_P\right|^2 = \mu_j(k)^2\,\tau_P^2 \tag{39}$$

Since τ_P is a constant, it follows that the real number $\mu_j(k)$ is the trading round's varying parameter that introduces the behavioral elements. The $\mu_j(k)$ are the relevant parameters that change from game to game in accordance with the market conditions. For each trading round the two parameters $\mu_0(k)$ and $\mu_1(k)$ are updated in accordance with the following rules:

$$\mu_0(k) = \mu_0(k-1) - \omega_M(k)\,S(k-1) - \omega_N\sigma(k) \tag{40}$$

$$\mu_1(k) = \mu_1(k-1) + \omega_M(k)\,S(k-1) + \omega_N\sigma(k) \tag{41}$$

where $S(k-1)$ is the final net order for the trading round $k-1$, ω_N is a fixed news sensitive parameter, and $\omega_M(k)$ is an adaptive market contagion parameter specified, later on, in this section.

The basic reasoning behind the rule is that $\mu_0(k)$ represents the perspective from the point of view of a negative market sentiment, therefore it is increased by negative values of $\sigma(k)$, and by negative values of market order $S(k-1)$

[40] See appendix A.1 for proofs.

and decreased by positive values of these two quantities. On the other hand, $\mu_1(k)$ represents the perspective from the point of view of a positive market sentiment, therefore it increases for positive values of $S(k-1)$ and $\sigma(k)$ and decreases for negative values of $\sigma(k)$ and $S(k-1)$.

We can interpret, in light of the financial practice, $S(k-1)$ as a technical indicator on the previous market sentiment, and $\sigma(k)$ as an element of information on the asset's fundamental value. Therefore, the overall market sentiment for the k-th trading round depends upon technical factors and *market fundamentals*[41]. The market valuation, in the form of a social sentiment $V(k)$, can then be taken as:

$$V(k) = \omega_M(k)\, S(k-1) + \omega_N \sigma(k) \tag{42}$$

then, $\mu_0(k)$ varies in accordance with the negative of $V(k)$, while $\mu_1(k)$ varies in accordance with $V(k)$.

The parameters $\mu_0(k)$ and $\mu_1(k)$ reflect the economic and financial information available to investors, and correspond to two perspectives of looking at that information. For $\mu_0(k)$ we have the perspective of the investor who wishes to sell, while that, for $\mu_1(k)$ we have the perspective of the investor who wishes to buy.

If we let $\mu_0(0) = \mu_1(0) = 0$, then, we have that, for each trading round $k = 1, 2, ..., \mu_0(k) = -\mu_1(k)$. The matter that we must, now, address is that of understanding the strategic aspects implied by the dynamics of these two parameters. To that end, we work with the two number operators $\hat{N}_0$ and $\hat{N}_1$. The average values for each of these operators are, for the k-th round:

$$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} = \mu_0(k)^2 \tau_P^2 \tag{43}$$

$$\left\langle \hat{N}_1 \right\rangle_{k,\tau_P} = \mu_1(k)^2 \tau_P^2 \tag{44}$$

Since $\mu_0(k) = -\mu_1(k)$ it follows that:

$$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} = \left\langle \hat{N}_1 \right\rangle_{k,\tau_P} \tag{45}$$

therefore, we have:

$$\left\langle \hat{R} \right\rangle_{k,\tau_P} = 0 \tag{46}$$

this is the first important result, in what regards the connection with standard financial theory, indeed this result coincides with the condition expressed by (15). In accordance with standard financial theory, a criterion of market efficiency, as we saw, when we discussed Bachelier's theory, is the fact that we cannot use the information available at the beginning of each trading round to

[41] The financial term, *market fundamentals* refers to the non-technical indicators, that is all those indicators that do not refer to past market behavior and that refer only to financial and economic information that is important in order to assess an asset's fundamental value.

predict the returns at the end of that trading round, more properly, the expected value of the returns, given the information available at the beginning of the trading round, must be equal to zero.

Otherwise, not all of the information would be reflected in the returns, and the market would fail to be efficient. Therefore, we have that (46) means that our model satisfies this criterion for market efficiency.

In order to understand how information is reflected in the returns, we must consider the behavioral interpretations of the variations of $\left\langle \hat{N}_0 \right\rangle_{k,\tau_P}$ and $\left\langle \hat{N}_1 \right\rangle_{k,\tau_P}$ between trading rounds.

We can introduce the following table:

	$V(k) < 0$	$V(k) > 0$
$\mu_0(k-1) < 0$	$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} < \left\langle \hat{N}_0 \right\rangle_{k-1,\tau_P}$	$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} > \left\langle \hat{N}_0 \right\rangle_{k-1,\tau_P}$
$\mu_0(k-1) > 0$	$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} > \left\langle \hat{N}_0 \right\rangle_{k-1,\tau_P}$	$\left\langle \hat{N}_0 \right\rangle_{k,\tau_P} < \left\langle \hat{N}_0 \right\rangle_{k-1,\tau_P}$

Taking into account the relation between μ_0 and μ_1, it follows that the above table contains all the information regarding alternative combinations of values. When $V(k) < 0$ the social sentiment regarding the intrinsic value is negative, which means that the social system of potential investors foresees a value loss.

In this case, from the point of view of an agent playing a selling strategy, if $\mu_0(k-1) < 0$, then the past pessimism has a negative sign, which means that there was a past optimism, therefore the negative prospects signalled by $V(k) < 0$ are counterweighted by a past optimism, this means that there is an ambiguous signal, there is not yet a clear signal whether one should buy or sell, this uncertainty leads to an average decrease in sales because those who hold stock will tend to wait and see whether new information will confirm a trend reversal, and those who do not hold stock will not risk a buy or a sale[42].

On the other hand, from the point of view of a buying strategy, if $\mu_0(k-1) < 0$ then $\mu_1(k-1) > 0$, the past optimism expressed by $\mu_1(k-1)$ is positive, therefore, once more, the negative prospects are counterweighted by a past optimism, the ambiguous signal also leads to a risk sensible strategy in which individuals prefer to wait, in order to see whether or not a trend reversal is present. The *wait and see strategy* is followed by all investors, which means that, on average, both buys and sales diminish, and, therefore, we have an average decrease in market activity.

If, for $V(k) < 0$, $\mu_0(k-1) > 0$, then, the past pessimism is confirmed by the present social sentiment regarding the intrinsic value, which means that, from the point of view of a selling strategy, this, clearly negative, signal prompts an average increase in selling.

If this was the only force on the market, then the sales could accumulate and lead to large jumps and to an undervaluation due to the market following

[42] A sale without holding the asset is called a short sale. In standard financial theory one usually assumes no limits to short selling, we also make this simplifying assumption here.

a synchronous selling dynamics, which would introduce market inefficiencies[43]. However, as predicted by standard financial theory, there should, in principle, exist market regulation mechanisms, in the form of investors who would be able to anticipate an overshoot in pessimism, and therefore enter the market anticipating a decrease in price below the asset's fair price.

This strategic reasoning is present in the above formulation. Indeed, from the point of view of a buying strategy, when $V(k) < 0$ and $\mu_1(k-1) < 0$ there is an accumulated pessimism, potential investors anticipate that an increase in sales, may lead to an undervaluation in the absence of a compensating demand, which makes this a good time to buy. This leads to an average increase in both the number of buyers and sellers, and, therefore, there is an average increase in market activity.

Let us, now, consider that $V(k) > 0$, and that $\mu_1(k-1) > 0$, in this case, there is a past optimism that is confirmed by the present social sentiment regarding value, therefore, there are good prospects for value which means that there is an average increase in buys. On the other hand, since $\mu_0(k-1) < 0$, there is also an average increase in sales, because there will be investors that anticipate the possibility of a contagion of optimism leading to an overvaluation of the asset.

Finally, if $V(k) > 0$, $\mu_0(k-1) > 0$ and $\mu_1(k-1) < 0$, there is an ambiguous signal, and potential investors will wait and see whether or not there is a trend reversal.

Taking into account this behavioral interpretation of the rules (40) and (41), we can see that the operator for each trading round U_k is modelling an efficient market in which a strategic thinking is present, where individuals not only follow strategies based on an emotional response associated with information processing, where pessimism leads to a sale and optimism to a buy, but they also anticipate the other agents' behavior and implement trading strategies to take advantage of opportunities presented by market behavior that could lead to overvaluation or undervaluation. There is also a responsiveness to uncertainty present in ambiguous stimuli, which leads the agents to *wait and see*.

We can see that we have a market that is efficient with respect to the expected value of returns, so that the EMH holds in our model, albeit with different microscopic behavioral assumptions than those of the standard financial theory.

Furthermore, our model incorporates basic dynamical elements already addressed by Dow. Indeed, the increase in transaction volume occurs only when there is a clear cognitive signal, that allows the investors to have a well defined sense of the direction in value, otherwise, volume decreases. This is in direct

[43] Indeed, in the models proposed by Gonçalves (2003b) and by Sornette and Zhou (Sornette and Zhou, 2005; Zhou and Sornette, 2005, 2007) it is this inneficient dynamics that produces jumps in prices and crashes. In the case of Gonçalves' model the coherent dynamics of sales and buys occurred too often leading to unrealistically bimodal distributions in logarithmic returns, and to predictable patterns in these returns, this led Sornette and Zhou to propose an alternative model in which social inertia prevented the market to explore the coherent dynamics too often.

agreement with Dow's findings for the actual markets. Therefore, our model expresses mathematically Dow's notion of the market as a *rational barometer*.

In order to have the model fully specified, we must, however, provide for the dynamics of $\omega_M(k)$. The two parameters $\omega_M(k)$ and ω_N control, respectively, for the importance given to the technical information and to the news signal. Since we take ω_N as fixed, the dynamics of the parameter $\omega_M(k)$ will determine the relative importance that each information source will have on the social sentiment regarding value.

The dynamics for $\omega_M(k)$ is similar the one proposed by Sornette and Zhou for the sentiment contagion. In this case, since $\sigma(k) = \pm M$, where M is a positive real, the extension of Sornette and Zhou's rule becomes:

$$\omega_M(k) = \bar{\omega} + \gamma\omega_M(k-1) + \beta r(k-1)\frac{\sigma(k)}{M} \tag{47}$$

where $\bar{\omega}$ is a base propensity for sentiment contagion.

The normalization of the news signal provided by $\frac{\sigma(k)}{M}$ is necessary in order to keep the basic rationale of the original rule, basically we are multiplying $r(k-1)$ by the sign of the news signal, so that if the news and the returns have the same sign, then there is a contribution of $+\beta|r(k-1)|$ to $\omega_M(k)$. On the other hand, if the news and the returns have opposite signs, then there is a contribution of $-\beta|r(k-1)|$ to $\omega_M(k)$.

The main reasoning behind this rule, for $\beta > 0$, is that, if the market and the news have the same sign, then, the previous market state has anticipated the news, which means that the technical indicator was a good predictor of the new information's sense (optimistic or pessimistic), this means that the sensitivity of the social system to the sentiment associated with the technical indicator is increased by a factor proportional to the absolute value of the returns[44]. On the other hand, if the returns and the news have opposite signs the technical indicator loses importance in μ_j. The term $\gamma\omega_M(k-1)$ provides for a social inertia effect.

Despite the similarity between the above rule and Sornette and Zhou's rule, there is a fundamental difference that has behavioral consequences in the way in which the parameter β is interpreted. The main difference lies in the fact that Sornette and Zhou worked with the product $r(k-1)\sigma(k-1)$ and considered that $\beta < 0$ is rational and $\beta > 0$ is irrational, in our case, $\beta > 0$ is not only rational, but necessary for market efficiency.

This difference in the sign of β is due to the fact that Sornette and Zhou considered a market whose main object of strategic intention is the market itself, therefore, investors use the news signal's information to predict market behavior. In our model investors are not trying to predict market behavior, instead, their object of strategic intention is the asset's fundamental value. The game of value investors is not an n-person game where each player plays against the collective of other agents, instead, it is what is called, in game theory, as we stressed before, a game against nature.

[44] It does not matter the pessimistic or optimistic sense, only if returns predicted the news.

Agents buy shares, because these shares give them a right over the companies' value, what interests a value investor is the companies' shareholder value, not the market behavior itself. The two sources of information about value, technical information (in the form of the market's past performance) and fundamental information (in the form of the news signal), are the two main information sources that allow potential investors to form expectations regarding value.

Since the product $r(k-1)\,\sigma(k)$ is positive, only when the previous period's returns were able to predict the present information, and is negative when the previous period's returns were unable to predict present information, then, the term $\beta r(k-1)\frac{\sigma(k)}{M}$, that gives the strength of the technical information, should increase when the past returns predict the current news and decrease when the past returns fail to predict the current news, which means that β must be made positive. It is rational to give more weight to the technical indicator if this indicator was able to provide with a better estimate of value.

In the rule proposed by Gonçalves (2003b), where $\gamma = 0$ and $\beta = 1$, one also worked with $r(k-1)\,\sigma(k-1)$, but the interpretation was closer to the one we follow here. The product with $r(k-1)\,\sigma(k-1)$ is open to two interpretations: either one considers that the market is the object of strategic intention and, in that case, investors are behaving like speculators that are trying to use the news to predict the market, in which case it is rational for $\beta < 0$; or value is considered as the object of intention, and investors are trying to predict the news signal and, thus, become more sensitive to the returns, if these returns were able to predict the news signal, therefore, in this second case, $\beta > 0$ is the rational behavior.

It is important to notice that, in the context of investors whose object of strategic intention is the value itself, if the weight given to the news becomes too high, the market may actually become less efficient, since it may become too sensitive to the immediate news, being unable to put into context the recent past. Furthermore, not all the information regarding value is reflected in the news signal. In accordance to standard financial theory, both technical information and external news should be taken into account, in order for the full information set to be reflected in the valuation.

In the model we propose here, if the past technical information is a good predictor of the current news, then, investors pay more attention to the technical indicator in (42) (the market sentiment $S(k-1)$). The technical indicator provides for a context for the news, if the news $\sigma(k)$ are consistent with the previous returns $r(k-1)$, then, this reinforces the previous market valuation, while an inconsistency, leads to a greater attention to the new external information and to a strategy of waiting to see whether there is a trend reversal in value or not.

We could consider the yet more general rule:

$$\omega_M(k) = \bar{\omega} + \gamma\omega_M(k-1) + \sum_j \beta_j r(j-1)\frac{\sigma(j)}{M} \tag{48}$$

Although this rule is more general with respect to the one we present here, we study here only the simpler version of (48).

As we show here, the simpler rule (47), is already able to produce irregularities that contain multifractal signatures close to the ones found in the actual markets. Further studies should be made for the extension expressed by (48), and other extensions including other kinds of investors such as speculators playing a minority game with the market.

The simpler model provided by rule (47), allows us, however, to establish our main point which is that multifractality is compatible with a version of EMH based on more realistic behavioral assumptions.

This completes the model's formalization. We are now ready to address the main findings that result from simulations for different parameter ranges.

V. Main results

The main parameters that determine the market's behavior are γ, β, τ_P and $\bar{\omega}$. For the purpose of comparison with the work developed by Sornette and Zhou, we study the case of $\beta = 1$. The larger the value of τ_P is, the larger will be the oscillations, the smaller the value of τ_P is, the smaller will be the oscillations. The normal size of the oscillations is controlled by the two parameters τ_P and λ.

In order to set the oscillations close to actual markets we found out that $\tau_P = 0.1$ and $\lambda = 2000$ provides for a good approximation.

Regarding the combination of the parameters γ and $\bar{\omega}$ we found out that, if we take $\bar{\omega} = 0$, the market tends to produce larger fluctuations than if $\bar{\omega} \neq 0$. For $\bar{\omega} \neq 0$, the parameter γ may have stabilizing effects if it is small, but if it becomes too large it may produce larger fluctuations than those of actual markets. In these cases we must take $\lambda > 2000$, in order for the maximum fluctuation size to match that of the markets.

For parameters in the range $\bar{\omega} \in (0, 0.1]$ and $\gamma \in (0, 0.1]$, with $\tau_P = 0.1$ and $\lambda = 2000$, we found a good agreement with market dynamics, as long as the amount $M \geq 1$. The restriction that $M \geq 1$ is necessary in general, since for values of M smaller than 1, the market will take some time to converge to the actual dynamics, since it tends to produce long periods with zero market activity followed by some jump. The smaller is M the more this behavior will hold.

We also assume that the probabilities for the Bernoulli process for $\sigma(k) = \pm M$, $p = 1 - p = 0.5$, this provides for a driftless information signal. The presence of a drift corresponds to the market expression of a long-term average growth in the economy. We regard the signal $\sigma(k)$ to be an extra information beyond the drift, this justifies the definition of a single value for M and $p = 0.5$, which makes the signal symmetric in regard to the drift.

The inclusion of a drift follows from the condition expressed by equation (4), in order for there to exist a consistency with (4), that is, for the market to obey a macroscopic condition of *rational expectations*, we should have:

$$\left\langle \hat{R} \right\rangle_{k,\tau_P} = \bar{\mu}\tau_P \tag{49}$$

where $\bar{\mu}$ corresponds to the average return for a long-term investor who holds the stock without selling. In that case, $\xi_1(k)$ would have to be changed to:

$$\xi_1(k) = -i\left[\left(\sqrt{1+\bar{\mu}\tau_P}\right)\mu_1(k)\,\tau_P\right] \tag{50}$$

The presence of the term $\sqrt{1+\bar{\mu}\tau_P}$ accounts for a natural tendency of investors to buy just because on average the companies' value increases, that is, the companies' ability to generate future *free cash flows* increases.

Since Sornette and Zhou did not work with an intrinsic drift, we take $\bar{\mu} = 0$, for the sake of comparison. Nonetheless, we have added an extra simulation with $\bar{\mu} = 0.1$. We verified that, when we add a drift, the parameter λ has to be increased in order for the returns to match the real markets' data. In the simulation with $\bar{\mu} = 0.1$, we set $\lambda = 2500$, $\bar{\omega} = 0.05$, $\gamma = 0.1$, $\beta = 1$, $\tau_P = 0.1$, $M = 1$ and the number of simulation steps was taken as 2^{18}.

In all the other simulations, analyzed here, we have set $\bar{\omega} = 0.1$, $\gamma \in \{0, 0.05, 0.1\}$, $\beta = 1$, $\tau_P = 0.1$, $\lambda = 2000$, the number of simulation steps was always taken as 10^{10}. All the simulations were performed in Netlogo, and the fractal analysis was implemented with the Matlab toolbox's Fraclab, developed by the INRIA group.

A. Multifractal properties

As stated in the introduction, there are three main multifractal patterns that we address here:

- Logarithmic price series' multifractality;
- Absolute returns' multifractality;
- Transaction volume's multifractality.

The identification of the presence of multifractal patterns in financial logarithmic price series, led Mandelbrot, Fisher and Calvet (1997) to the proposal of the MMAR. Their main proposal was to consider the price logarithm in terms of a continuous time stochastic process, such that it has stationary increments and (Calvet and Fisher, 2002, p.17):

$$|\ln P(t+dt) - \ln P(t)| \sim C_t\,(dt)^{\alpha(t)} \tag{51}$$

where $\alpha(t)$ and C_t are, respectively, the local Hölder exponent and the prefactor at t.

The local Hölder exponent quantifies the structure of the local singularities and the local scaling. In continuous unifractal processes, the Hölder exponent takes on a unique value H. In standard finance, $H = \frac{1}{2}$, and it was assumed that any variations in market volatility were due to changes in the prefactor C_t. In a multifractal process, the exponents $\alpha(t)$ change with time, in such a way that the market may become more regular or less regular in accordance with the values of $\alpha(t)$.

The local exponent's relative frequency can be represented by a renormalized density called the multifractal spectrum, which provides for statistical information on the distribution of Hölder exponents. Applying the large deviation formalism (Calvet and Fisher, 2002), we may begin by noticing that the Hölder exponents can be expressed by:

$$a(t) = \lim_{l \to 0} \sup \frac{\ln |Y(t,l)|}{\ln l} \tag{52}$$

$$Y(t,l) = \ln P(t+l) - \ln P(t) \tag{53}$$

We may, now, partition the time interval $[0,T]$ into b^k subintervals of size $\left[t_i, t_i + b^{-k}T\right]$ and determine, for each subinterval the coarse-grained Hölder exponent:

$$\alpha_k(t_i) \equiv \frac{\ln |Y(t,l)|}{\ln l} \tag{54}$$

This operation generates a set $\{\alpha_k(t_i)\}$ of b^k observations. The range of exponents can, then, be divided into small intervals of length $\triangle\alpha$. If we denote the number of coarse-grained exponents contained in $(\alpha, \alpha+\varepsilon]$ by $N_k^{\varepsilon}(\alpha)$, then, the following limit represents a renormalized probability distribution of local Hölder exponents, and is called the *large deviation multifractal spectrum* (Mandelbrot, 1989):

$$f(\alpha) = \lim_{\varepsilon \to 0} \lim_{k \to \infty} \left\{ \frac{N_k^{\varepsilon}(\alpha)}{\ln b^k} \right\} \tag{55}$$

The classical method for computing this spectrum is based upon a histogram computation and leads to numerous problems such as the choice of the partition. Fraclab's estimator is based upon a kernel method used in densities' estimation[45].

In **figures 27** and **28** we provide for the estimated spectra for the simulated price series. The convex structure of the spectra and the wide range of exponents with high values of $f(\alpha)$ is evidence favorable to the hypothesis that the price logaritm's series possesses multifractal patterns.

The ranges of Hölder exponents span from 0.3 to values larger than 0.9, with maximum of $f(\alpha)$ in values close to $\alpha_0 = 0.6$. This type of scaling profile has been observed in the actual markets[46]. In the case of the series with drift, with 2^{18} data points, we also find the same profile of exponents.

[45] Fraclab's algorithm provides for smooth kernels and seems to produce more accurate results than alternative multifractal spectra implemented in the same program, such as the *Legendre spectrum*, which is a concave approximation to the *large deviation spectrum*.

[46] Mandelbrot *et al.* (1997) observed a maximum exponent of 0.6 for the foreign exchange markets and Ohashi *et al.* (2003) observed maxima close to 0.6 for the indexes S&P500 and Nikkei. The range of the spectra are also close to the findings by Ohashi *et al.* (2003), with $\alpha_{\min}$ between 0.3 and 0.4, and $\alpha_{\max}$ around 0.9.

We also divided the simulated series with $\bar{\mu} = 0.1$ into eight intervals of equal length, and estimated the spectra for each subseries (see **figures 29** to **36**). Although the spectra change, there is an overall consistency in regards to the smallest exponents which are usually between 0.3 and 0.4, and in regards to the maximum exponents which are usually close to 0.9, furthermore the most probable Hölder exponent α_0 is usually in a range close to 0.6.

The finding of maxima for $f(\alpha)$ larger than 0.5, in actual financial price series by Mandelbrot *et al.* (1997) and by Ohashi *et al.* (2003), means that financial markets show higher persistence than predicted by standard financial theory's Wiener Brownian motion. This greater regularity is unaccounted for in models based on the multifractal random walk and in the MMAR, since, in these models, multifractality is introduced *a priori* and the multifractal spectrum is adjusted in the cascade mechanism to fit the data.

In our case, the most probable local Hölder exponent emerges from the market behavior, it is not adjusted to fit the data. The existence of an α_0 greater than 0.5, in our model, may be a consequence of the long range dependency of the $\mu_j(k)$ on the past realizations of the returns.

Another factor noticed by Calvet and Fisher (2002) was that the concavity of the spectrum, for the exchange rates series, also implied the existence of lower Hölder exponents that corresponded to more irregular instants of the price process, contributing disproportionately to volatility.

Regarding volatility, Calvet and Fisher also added that sets of instants with a particular value of α tend to be clumped together, simultaneously generating more risk and long memory in volatility.

The existence of volatility clusters can be confirmed in the returns' graphs (see **figures 3**, **4**, **9** and **10**). Each of these graphs exhibits a visual evidence of turbulence, where there are periods of low volatility (*laminarity*) followed by periods of strong volatility (*turbulence*).

If, following Sornette and Zhou (2005), we use the absolute returns as a surrogate for volatility, then the type of temporal dependence of volatility is exhibited in the autocorrelation functions. A signature of multifractality, as shown by Muzy *et al.* (2000, 2001), is an autocorrelation function that decays linearly with the logarithm of the time lag. This type of memory structure in the risk, associated with volatility, can be seen in **figures 13** and **15**, in both these plots we see that the autocorrelation functions follow a nonlinear approximation to the linear decay reported by Muzy *et al.* (2001), the same profile occuring for market activity (**figures 14** and **16**). This is due to the fact that our market is a short trading-period model. If we provide for a regular sampling of the simulation results we obtain an approximation to a linear decay in the lag logarithm as the sampling period is increased.

As an example, in **figures 17** to **19**, we provide for the autocorrelations for the absolute returns with sampling scales equal to 10, 20 and 100 trading rounds, for $\gamma = 0.1$. We can see that as the sampling scale is increased the nonlinear approximation becomes smaller and the linear decay sets in more quickly. **Figures 19** to **22**, we provide for the absolute returns' correlations for each simulated series, with a sampling scale of 100 trading rounds. We can see that all the series possess a linear decay for 100 trading rounds.

For comparison purposes, we provide for similar plots, for actual market data. **Figures 23** to **26** show the results for daily market data for the financial indexes S&P 500, Nasdaq, Nikkei and Hang Seng. The results closely match those of the actual data, in the general form of the linear decay. However, there are differences. For instance, the Nasdaq has a slower decay for 10^2 lags. For the simulations with $\bar{\mu} = 0$, there are negative values in the autocorrelations for lags close, in the logarithmic scale, to 10^2.

The case of $\bar{\mu} = 0.1$ best approximates the type of decay seen in the actual markets, since the linear decay does not reach negative values. Nonetheless, the linear decay seen in all simulations, seems to yield a good approximation of daily market sessions' data for sampling periods of 100 data points, which means that 100 rounds of our game approximate a single trading day.

Although the previous artificial financial market studied by Sornette and Zhou, showed such evidence, in our model there is the added advantage that we can see the direct connection between volatility and volume statistics. Indeed, using market activity measured by the number of buying and selling orders, we can see for the market activity's observable, a similar nonlinear approximation to a linear decay profile in the autocorrelation function plotted against the logarithm of the time lag.

Furthermore, increases in market activity seem to be accompanied by increases in volatility. This can be seen by comparing the returns graphs of **figures 3**, **4**, **9** and **10**, with the corresponding market activity graphs of **figures 5**, **6**, **11** and **12**. We can see, from these figures, that when market activity increases there is also an increase in market volatility, furthermore, the volatility clusters (the *turbulent periods*) correspond to increased market activity clusters.

The intertwining between price logarithms, returns, and volatility risk, is not present in the MMAR nor in the MRW. In these models, the cascade process, that produces the multifractal patterns, is an independent, exogenous, process. In our model, on the contrary, there is a theoretical argument, based on market behavioral theory, that justifies the existence of a dependence of the price process on volatility and returns. There is a feedback mechanism from the volatility to the market's evaluation of the fundamental value, indeed replacing (47) in (42) and with some manipulation of the expression we obtain:

$$\begin{aligned} V(k) &= \left(\bar{\omega} + \gamma\omega_M(k-1) + \beta\frac{S(k-1)}{\lambda}\frac{\sigma(k)}{M}\right) S_{k-1} + \omega_N \sigma(k) \\ &= \lambda\left[\bar{\omega} + \gamma\omega(k-1)\right] r_{k-1} + \left(\frac{\lambda\beta}{M} r_{k-1}^2 + \omega_N\right)\sigma(k) \end{aligned} \tag{56}$$

where, in this last expression, the volatility is measured by the squared logarithmic returns r_{k-1}^2. Equation (56), determines the $\mu_j(k)$ and therefore strategies' occupation number probability distribution, which, in turn will determine the probability of each r_{k-1}, we can clearly see here the dependence of $V(k)$ and of the basic market observables' dynamics upon both the returns and the volatility.

The market returns' process participates actively in the dynamics that produces the returns themselves, and, by that means, also produces the multifractal patterns. This is a major difference with respect to the models that introduce independent exogenous cascade process, of which the MMAR and the MRW are examples.

As such there may be a direct connection between the irregularities at each point in the series with both market activity and volatility. A simple method of analysis, that is sensitive to changes in local irregularities consists in computing a generalized iterated function system (GIFS) that provides for a best approximation to the signal[47]. The difference between an iterated function system (IFS) and a GIFS is that, for a GIFS, the number of maps and the various parameters are allowed to change at each scale, this makes a GIFS more appropriate to capture multifractal patterns in data.

Once we have the GIFS that best approximates a signal[48], we can obtain, analytically, a Hölder exponent at each point, of the GIFS. In this way the signal can be approximated in terms of a GIFS with a given pointwise Hölder function. The pointwise Hölder function of the GIFS that best approximates the signal provides with an analysis of the original signal's regularities.

In **appendix B.4.1**, we provide for the Hölder functions for a sample of 65536 price logarithms[49] (**B.4.1**), taken from the simulated series analyzed in **appendix B.1**, with $\bar{\mu} = 0$, and also provide for the Hölder function for the entire simulated series with $\bar{\mu} = 0.01$. We can see that in all the cases there is a good agreement between the pointwise Hölder exponents and the multifractal spectra obtained for each price logarithm[50]. In **appendix B.4.2** we provide for the same analysis, for actual market data, for the indexes: S&P500, Nasdaq, Nikkei and Hang Seng.

In both the simulated price series and the actual market data, there are three salient features in the Hölder functions for the GIFS that best approximate the simulated price logarithms:

- The pointwise Hölder function is not smooth exhibiting strong fluctuations, this is a well documented feature of financial systems (Ayache and

[47] We shall be analyzing three signals: the price logarithm, the absolute returns and the market activity.

[48] Fraclab implements this procedure by computing a discrete wavelet transform of the signal (using, in this case, Daubechies 4 discrete wavelet transform). The parameters of the GIFS are then obtained as ratios of the wavelet coefficients.

Once the GIFS is known, the estimation of the exponent follows immediately, because there is an analytical formula which gives the exponent at each point of a GIFS as a function of its parameters. Because the formula is only valid in the limit of infinite resolution, the obtained result is a finite size approximation.

[49] The method only works for samples that are in powers of 2, therefore, we use the largest power of 2 that is smaller than 100000.

[50] There is one outlier in each of the cases with $\gamma = 0$, $\gamma = 0.05$ and $\bar{\mu} = 0.01$, this outlier corresponds to values of the Hölder exponents smaller than those predicted by the spectra, but this is an isolated case, in all the three simulations.

For all the simulated price series there is an overall good agreement between the large deviation spectra and the Hölder functions.

Lévy Véhel, 2004) and it is a feature in common with other systems with multifractal signatures (Ayache and Lévy Véhel, 2004);

- The function fluctuates around mean values close to the maxima of the estimated *large deviation spectrum*, which accounts for a consistency between the two analysis and reinforces the evidence of multifractal patterns in the logarithmic price series;

- There is a slowly varying component in the Hölder functions that accounts for slowly varying irregularities in the markets[51].

In order to see how the Hölder regularities present in the price logarithm are related to the Hölder regularities in the absolute returns and market activity we applied the same analysis to the absolute returns and the market activity. We can see that in all the cases, the pointwise Hölder function is not smooth and has slowly varying components, which is consistent with the presence of multifractal signatures in both the absolute returns and transaction volume (measured in terms of market activity).

The slowly varying components are relevant because they indicate medium to long-term variations in persistence patterns. If we determine the cross-correlations of the price's slowly-varying component with the absolute returns' and market activity's slowly varying components[52], we can see that there is a correlation with a slow oscillatory decay to zero in all the simulations. This general pattern is also present in the actual markets, as can be seen in **appendix B.4.2**.

The existence of slow oscillatory decay in the cross correlations reveals a nontrivial dependence between the slowly-varying components. The fact that the decay is slow means that the long term persistence in the price logarithm is related to the past long term persistence in the absolute returns and market activity and vice-versa, on the other hand, the presence of an oscillatory decay means that there is a nontrivial relation between the past persistence in one market observable and the present persistence in the other observable.

For instance, in the case of the market with $\gamma = 0$ we can see that there are negative correlations between the slowly-varying component of the price logarithm's Hölder function and that same component for the absolute returns for lags up to 10^4, which shows that as the persistence increases in the absolute returns, the price logarithm tends to become less persistent.

[51] In order to analyze the relation between the slow component of the pointwise Hölder functions, we have implemented a denoising technique known as wavelet pumping, available in Fraclab.

This technique consists in obtaining a wavelet transform of the original signal (the estimated Hölder functions) and multiplying the coefficients at scale j by 2^{-dj}. The wavelet we used was Daubechies 10.

As can be seen in the graphs of **figures 37** to **48**, the smoothed signal approximates well the slowly varying component in the Hölder functions.

The same smoothing procedure was also applied to the real markets' Hölder functions.

[52] Since the cross-correlation function is symmetric we only present one of the two halves of the function.

On the other hand, for lags close to 3×10^4 we find positive correlations with magnitudes comparable to those of the initial lag differences but with opposite sign. This means that, in terms of the slow-varying component in the Hölder functions, an increase in volatility's regularity tends, in a more recent past, to make the current price trajectories more irregular, while that, for a more distant past, we may find that increased past regularity in volatility tends to make the current price trajectories more regular.

These findings, along with the other analysis provided above, show that the model proposed is able to produce many of the multifractal patterns present in actual markets.

VI. Conclusions

The work developed here has two major lines of implications: (1) implications for financial theory; (2) implications for quantum game theory and for quantum econophysics.

Regarding the implications for financial theory, we were able to provide for a microscopic model of an efficient market that reproduces multifractal signatures similar to the ones present in actual markets. Although the previous models by Sornette and Zhou (Sornette and Zhou, 2005; Zhou and Sornette, 2005, 2007), that served as basis for this model, also generated multifractal signatures, this model differs from the models of Sornette and Zhou in a number of instances. The agent-level strategic reasoning and behavioral rules are more complex than the one proposed by Gonçalves (2003b), and by Sornette and Zhou (2005); both herding effects, and anticipatory correction movements are present and the model is more dynamic in the sense that it allows for permanent fluctuations in the transaction volume.

Another important finding of this model is the ability to capture multifractal patterns in the transaction volume and the relation between transaction volume and market turbulence. In our model, there is a direct correspondence between high volume (high market activity) and *turbulent periods*, and between low volume (low market activity) and *laminar periods*.

With respect to the MRW and MMAR, this model has the major advantage of endogeneizing the multifractality. The multifractality emerges from the microscopic behavior of the social system, and it is not imposed as an exogenously generated multifractal temporal distortion or as an exogenously fluctuating volatility term.

Besides these empirical issues, the model also presents a theoretical novelty in its foundations. More specifically, we showed that EMH can still hold as an hypothesis, in more realistic behavioral settings than the ones that came to dominate the mainstream finance. The EMH seemed to be threatened by the deviations to the mainstream models, and the microscopic models were usually developed to explore these deviations to the EMH. Our model, instead, shows that a macroscopic *rational expectations' equilibrium* and an efficient market is compatible with the real-life turbulent markets. Turbulence and multifractality do not necessarily follow from '*irrational behavior*' or from deviations from efficiency, instead, these phenomena may be the result of the very mechanisms that are responsible for efficiency.

In terms of the underlying financial theory, it should also be stressed that the model proposed here combines the EMH with Dow's theory, which constitutes a novel approach, since Dow's theory was mostly incorporated in the technical analysis trading technologies and was largely considered as incompatible with the EMH and mainstream financial theory. Although Dow's theory is, indeed, incompatible with the mainstream theory, it is still compatible with EMH and provides important insights into market behavior, constituting a useful basis for theory building, being easier to combine with neuroeconomics and neurobiology of decision.

Another implication of our model, for financial theory, quantum game theory and quantum econophysics, is that it is one of the first applications of quantum game theory to financial economics with an extensive empirical testing. Several puzzling market features are captured by the same quantum model, which constitutes sufficient reason to pursue further extensions of the model itself and to pursue further applications of quantum theory to social science and economics.

Regarding quantum game theory, we have provided a framework for evolutionary game theory that may prove useful in other economical applications with a fluctuating number of agents.

Although the model is able to implement a quantum description of the market, it does not contain a full quantum description of the whole process, since the news process is classically stochastic. Instead, one may assume a quantum description for the news process itself. In this case, instead of $\sigma(t)$, we could define the operator $\hat{\sigma}$ such that:

$$\hat{\sigma} = M \cdot \hat{\sigma}_z \tag{57}$$

$$\hat{\sigma}_z = |M\rangle\langle M| - |-M\rangle\langle -M| \tag{58}$$

where $\hat{\sigma}_z$ is Pauli's operator in the basis $\{|M\rangle, |-M\rangle\}$. This leads to the eigenstates:

$$\hat{\sigma}|M\rangle = M|M\rangle \tag{59}$$

$$\hat{\sigma}|-M\rangle = -M|-M\rangle \tag{60}$$

For each trading round the news state would, then, be given by the unitary evolution of the resulting signal of the previous trading round, with unitary operator defined by the Haddamard transform:

$$U_H = \frac{1}{\sqrt{2}}\left(|M\rangle\langle M| - |-M\rangle\langle -M|\right) + \frac{1}{\sqrt{2}}\left(|M\rangle\langle -M| + |-M\rangle\langle M|\right) \tag{61}$$

we, then, have, the following trading rounds' transition amplitudes:

$$\langle M\,|U_H|\,M\rangle = \frac{1}{\sqrt{2}},\ \langle -M\,|U_H|-M\rangle = -\frac{1}{\sqrt{2}} \tag{62}$$

$$\langle M\,|U_H|-M\rangle = \langle -M\,|U_H|\,M\rangle = -\frac{1}{\sqrt{2}} \tag{63}$$

Another issue that should be taken into account is that we have assumed a collapse at the end of each trading round, which is consistent with the theory that at the end of the game, the decision is accompanied by an actualization of an event (the agents' choice) and, therefore, by the collapse of the strategic state.

We can see that we may consistently assign probabilities to that collapse, since the condition for *relative decoherence* is satisfied, which means that, for each trading round (each quantum game), the market's quantum state, at the end of the social learning, is always *decoherent relative* to the market's observables. The same reasoning applies to the observable $\hat{\sigma}$, in the quantum description we have just provided for the news signal.

As a final note, regarding the nature and status of quantum econophysics, we stress that if, as our model seems to indicate, quantum econophysics provides for economic theories with greater explanatory power, then, taking into account the discussion of **section 3**, one may consider the more general consequences of such a field of research. As the name indicates the field itself seems to form a transdisciplinary whole, made of the fusion of two disciplines: economics (*econo*) and quantum physics.

Its place in a discursive network with both economics and physics must always be taken into account, and the tensional links that make econophysics an autonomous node in the network, must be relevated in any inquiry, if quantum econophysics is to grow as a well-formed theoretical body. It is important, in particular, to consider the fitting between quantum econophysics and quantum cosmology.

Hartle (2003) addressed this issue, by trying to answer Gell-Mann's question "If you know the wave function of the universe, why aren't you rich?". Taken in a generalized sense, every event that takes place in the universe should have a nonzero probability of occurring, that is, it should be predicted by the quantum state of the universe. It is in this sense that quantum cosmology, is considered, by Gell-Mann and Hartle (1996), as predicting the probabilities of alternative *decohering coarse-grained histories* of the universe.

Since the universe, in its beginning, was such that, quantum effects played a relevant role, it follows that the events that, presently, take place in our planet are taking place within a system (the universe) that began as a quantum system. Therefore, with a more advanced theory of quantum cosmology[53], along with the computational power that allowed us to perform the different calculations,

[53] After unification of general relativity with quantum theory.

it should, in principle, be possible to provide for a probabilistic description of the entire universe's evolution[54].

However, Hartle adds that, it is too much to hope that a probability like that for the FTSE to go up tomorrow by one tick, falls in the category of events in which there is a significant result using only quantum cosmology, or a theory like M theory. The argument is that since a rise tomorrow is an event in the history of the universe, a quantum mechanical probability could, in principle be calculated for it, but it is likely that, after all the work, the predicted probability for a one-tick rise would be 50%. In this sense, a quantum theory of the universe in which every event is associated with a probability, cannot provide for a market theory, only by itself.

The problem is that a quantum cosmological theory may have to compress too much information, losing explanatory power, with respect to some phenomena, as was identified by Hartle (2003). The quantum state of the universe, as it is formulated within quantum cosmology, is able to deal with some general features of the universe, mostly linked with geometry, but as the universe gains in complexity, the compressibility of systems implies limits to a fully reductionist approach. We need higher-order theories that deal with emergent relations and emergent symmetries, so that a theory for the quantum mechanics of a quantum system that is formed of smaller components, may have to involve the introduction of new observables and new degrees of freedom linked to the relations between the subsystems that compose the higher-order system. An example can be found, for instance, in the hydrogen atom's wave function.

Higherarchical complexity in the universe is usually accompanied by the need to introduce new elements in a theory that must account for relations between the parts that compose a given system.

A market quantum theory may also be considered as a higher-order quantum theory, that deals with markets. Now, if a market quantum theory predicts some price rise with a probability equal to p, then, we have several different possible histories for the universe, and the number p becomes a relevant quantum cosmological matter. But, then, one must consider the nature and meaning of that p.

As systems become more complex, new degrees of freedom may be taken into consideration. Thus, we can only speak of the hydrogen atom's wave function, and of observables for the hydrogen atom, after an hydrogen atom forms. This means that observables may emerge and quantum states for complex systems may form, upon which the probabilities for those observables may be calculated.

The quantum strategic state and the market quantum state are just other examples of this phenomenon. The market quantum state, tells us about a system composed of several complex bodies that behave with the same dynamical rules of a quantum system. The strategic observables in a quantum game, correspond to a certain symmetry property with respect to the different strategies[55].

[54] This argument is a quantum analogue of Laplace's demon (Hawking, 2001).

[55] Two strategies in a quantum game leads to the $SU(2)$ group for the strategy set, three strategies lead to the $SU(3)$ and so on.

In the same way that *isospin* is a property of a nucleon (that can be a proton or a neutron), we have, for a game, a *strategy property* that an agent may exemplify, and by which that agent can be described in terms of the corresponding strategy.

An atomic theory only applies in a universe where an atomic system forms, while that, a quantum market theory only becomes relevant in a universe where a market forms. In the same way that atomic theory allows us to calculate the relevant probabilities for an atomic system, a quantum market theory, calculates probabilities for a market system, and a quantum game theory, calculates the probabilities for a quantum game.

In principle, the probability p for a price rise, predicted by a quantum market theory, should be obtained from the local quantum state of the market. Local, in this case, with respect to the rest of the universe. If the market's quantum state is sufficiently separable we can develop a quantum market theory, without reference to the rest of the universe. This largely depends on the relation between the relevant information for valuation and other systems' observables. The FTSE does not change for particle collisions taking place in a particle accelerator on earth. This statement expresses an example of the deeper approach that isolates the main information sources for a financial market. In this case, we can see that the nature of these sources seems to be sufficiently separable from the rest of the universe, in order for one to work out a quantum market theory with a model of an isolated market.

On the other hand, such a model can only be considered as an approximation. Indeed, a more realistic approach, would be to introduce some measurement-like interactions. The model proposed here should be taken as an isolated game approximation. Usually, quantum theoretical games, like classical ones, are addressed on their own, without reference to the rest of the world, while, in fact, that is not the case. Human decision is such that the individual is aware of the decision taken, which means that, besides the game's occupation number state $|n_0, n_1\rangle$, we should also introduce a cognitive awareness state such that each individual is aware of the strategy chosen. In this case, an extra occupation number state for the awareness of following a buying or selling strategy could be introduced, leading to the entangled state for each trading round:

$$|Cog; Market\rangle = e^{-\frac{|\xi_0(k,\tau_P)|^2+|\xi_1(k,\tau_P)|^2}{2}} \sum_{n_0,n_1} \frac{\xi_0^{n_0}(k,\tau_P)\,\xi_1^{n_1}(k,\tau_P)}{\sqrt{n_0!n_1!}} |n_0, n_1; n_0, n_1\rangle \tag{64}$$

where the first pair of numbers correspond to the agents' awareness of the decision, and the second pair, to the agents' actions themselves.

Also, upon deeper inspection one may argue that the argument of separability is not completely accurate. Indeed, although we may argue, as we did above, that the market is not likely to be affected by quantum events such as the ones that occur in a particle accelerator on earth (following the above argumentative illustration), the events that occur in the market carry physical effects to the universe histories.

For instance, whenever the market price is broadcasted in the internet or through some other media, physical effects take place. The message that passes on depends on the market's events that lead to a certain market price, and this message is carried by some physical media.

The neurological state of a potential investor changes while listening to the news, electrical impulses in the brain, the areas of the brain that become active, chemical substances that are released, all these may change with a message about the current market price.

All this entails physical effects that, as can be seen from the examples above, are not even limited to the macroscopic. Decisions and events always entail physical effects. No market fluctuation leaves the universe unchanged. Therefore, besides the agent's cognitive state, and the actual action, we may include, in a very crude approximation, an *environment* that will be in a different state for each alternative, which takes into account the physical effects of the agents' decisions, this leads to the final entangled state:

$$|Cog; Market; \mathcal{E}\rangle = e^{-\frac{|\xi_0(k,\tau_P)|^2+|\xi_1(k,\tau_P)|^2}{2}} \sum_{n_0,n_1} \frac{\xi_0^{n_0}(k,\tau_P)\,\xi_1^{n_1}(k,\tau_P)}{\sqrt{n_0!n_1!}} |n_0, n_1; n_0, n_1; \mathcal{E}_{n_0,n_1}\rangle \tag{65}$$

Such entangled states, would, however, imply little change in the analysis, since, we would still have to take into account these elements as making part of the game, in the sense that we would be adding game observers. We would still have to determine probabilities for the game, and we could only do so if a condition of consistency would be met. Or, alternatively, we could trace over the environment's degrees of freedom, and obtain a mixed state for each market session, in either case, we would obtain the same probability distribution for each decision.

Such a formalization would have, unnecessarily complicated the model, leading to the same results. The *relative decoherence* condition allowed us to consistently assign probabilities to the final occupation number states for the game observables.

Nonetheless, it turns out, that, in a more general perspective, whatever the market theory, the probabilities determined from it are always approximations. In terms of principle, and within a quantum cosmological line of thought, the market quantum state can also be considered a part of the universe's quantum state, which means that past quantum events have some bearing in the market's probabilities. Therefore, an actual probability estimate would still have to be calculated from the probability for the different universe's histories that are consistent with the existence of a financial market driven by human agents. It is, however, debatable whether such an approach would provide further relevant insights for the financial problem of the modelling of market dynamics.

As it now stands, quantum econophysics needs further research, both theoretical and empirical, connecting mathematical physics and mathematical economics.

References

[1]Aristóteles, *Organon, Livro II, Periérmeneias*, Trans. Pinharanda Gomes, Guimarães Editores.

[2]Ayache, A.; Lévy Véhel, J., 2004, *On identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion*, Stochastic Processes Appl. 111, No.1, pp. 119−156.

[3]Bachelier, L., 1900, *Théorie de la spéculation*, Annales scientifiques de l'É.N.S., 3^{e} série, tome 17, pp. 21−86.

[4]Barney, Jay, 1991, *Firm Resources and Sustained Competitive Advantage*, Journal of Management, Vol. 17, No. 199-120.

[5]Black, F.; Scholes, M., 1973, *The pricing of options and corporate liabilities*, Journal of Political Economy **81**: 637−654.

[6]Blume-Kouhout; Zurek, W. H., 2006, *Quantum Darwinism: Entanglement, branches, and the emergent classicality of redundantly stored quantum information*, Phys. Rev. A 73, 062310.

[7]Bormetti, G.; Montagna, G.; Moretti, N.; Nicrosini, O., 2006, *Pricing exotic options in a path integral approach*, Quantitative Finance, 6, pp. 55−66.

[8]Buron, Valérie, 2007, *Le circuit cérebral de la décision*, Le Monde de l'Intelligence, No.8, Janvie, Février, Mars, pp. 47−49.

[9]Callen, E.; Shapero, D., 1974, *A Theory of Social Imitation.* Physics Today 27, No.7, pp.23−28.

[10]Calvet, Laurent ; Fisher, Adlai, 2002, *Multifractality in asset returns: Theory and evidence, Review of Economics and Statistics*, Vol.84, 3, August, pp. 381-406.

[11]Chen, Z., 2001, *Quantum theory for the binomial model in finance theory*, arXiv:quant-ph/0112156.

[12]Choustova, 2006, *Toward Quantum Behavioral Finances: Bohmian Approach*, arXiv:quant-ph/0109122.

[13]Coase, Ronald H., 1937, *The Nature of the Firm*, Economica, 4, pp. 386-405.

[14]Conner, K. R.; Prahalad C. K., 1996, *A Resource-Based Theory of the Firm: Knowkedge versus Opportunism*, Organization Science, Vol. 7, No. 5, 477-501.

[15]Cox, J.; Ross, S. A.; Rubinstein, M., 1979, *Option pricing: A simplified approach*, Journal of Financial Economics **7**: 229−263.

[16]Damásio, António, 1994, *O erro de Descartes: emoção, razão e cérebro humano*, Trans. Dora Vicente and Georgina Segurado, Publicações Europa-América, Lda.

[17]Damásio, António, 1995, [1994], *Descartes' error: emotion, reason and human brain*, Harper Perennial.

[18]Damásio, António, 1999, *The feeling of what happens – Body and emotion in the making of consciousness*, Harvest Book, Harcourt Inc.

[19]Damásio, António, 2003, *Looking for Spinoza – Joy, sorrow and the human brain*, Harvest Book, Harcourt Inc.

[20]Daw, Nathaniel; Doya, Kenji, 2006, *The computational neurobiology of learning and reward*, Current Opinion in Neurobiology 2006, 16:199–204.

[21]Daw, Nathaniel, 2006a, *Exploration versus exploitation in humans: neural and behavioral approaches*, invited seminar, Neuroeconomics workshop series, Stanford University, Palo Alto.

[22]Daw, Nathaniel, 2006b, *The misbehavior of value and the discipline of the will*, Neural Networks 19, pp. 1153–1160.

[23]Dawkins, Richard, 1989, *The Selfish Gene*, Oxford University Press.

[24]DeWitt, B.S., 1973, *Quantum mechanics and reality*, *in* The Many-Worlds Interpretation of Quantum Mechanics, ed.: B.S. DeWitt and N. Graham, Princeton University Press.

[25]Dirac, P. A. M., 2004, [1958], *The Principles of Quantum Mechanics*, Oxford Science Publications.

[26]Dow, Charles, 1904, *Are prices rising or falling*, The Wall Street Journal, Editorial, September 17.

[27]Dow, Charles, 1906a, *The price movement*, The Wall Street Journal, Editorial, May 19.

[28]Dow, Charles, 1906b, *The Wall Street barometer*, The Wall Street Journal, Editorial, June 29.

[29]Dow, Charles, 1906c, *Is it a bull swing?*, The Wall Street Journal, Editorial, August 21.

[30]Eccles, J. C., 1995, *How the Self Controls its Brain*, Springer.

[31]Eisert, J.; Wilkens, M.; Lewenstein, M., 1999, *Quantum games and quantum strategies*, Phys. Rev. Lett., **83**, 3077−3080.

[32]Everett, H., 1957, *Relative state formulation of quantum mechanics*, Rev. Mod. Phys. **29**, pp.454−462.

[33]Everett, H., 1973, *The theory of the universal wave function*, *in* The Many-Worlds Interpretation of Quantum Mechanics, ed.: B.S. DeWitt and N. Graham, Princeton University Press.

[34]Faber, J.; Portugal, R.; Rosa, L. P., 2006, *Quantum games in open systems using biophysic Hamiltonians*, arXiv:quant-ph/060219.

[35]Fama, Eugene, 1970, *Efficient capital markets: a review of theory and empirical work*, Journal of Finance, May, pp. 383−417.

[36]Fama, Eugene; Miller, Merton, 1972, *The Theory of Finance*, Holt, Rinehart and Winston.

[37]Farmer, Doyne, 1998, *Market, force, ecology and evolution*, Santa Fe Institute working paper 98-12-117.

[38]Farmer, Doyne, 2000, *Market, force, ecology and evolution*, arXiv:adap-org/9812005.

[39]Flitney, A. P., 2005, *Aspects of quantum game theory*, PhD thesis submitted to Department of Electrical and Electronic Engineering, Faculty of Engineering, Computer and Mathematical Sciences, The University of Adelaide, Australia.

[40]Flitney, A. P.; Hollenberg, Lloyd C. L., 2006, *Nash equilibria in quantum games with generalized two-parameter strategies*, arXiv:quant-ph/0610084.

[41]Gardiner, C.W.; Zoller, P., 2004, *Quantum Noise*, Springer-Verlag.

[42]Gintis, Herbert, 2000, *Game Theory Evolving*, Princeton University Press.

[43]Gell-Mann, Murray; Hartle, James, 1993, *Classical Equations for Quantum Systems*, Phys.Rev. D47, 3345-3382.

[44]Gell-Mann, Murray; Hartle, James, 1994, *Time Symmetry and Asymmetry in Quantum Mechanics and Quantum Cosmology*, *in* Physical Origins of Time Asymmetry, ed.: J. Halliwell, J. Perez-Mercader, and W. Zurek, Cambridge University Press, Cambridge.

[45]Gell-Mann, Murray; Hartle, James, 1996, *Equivalent Sets of Histories and Multiple Quasiclassical Realms*, gr-qc/9404013.

[46]Gell-Mann, Murray; Hartle, James, 1998, *Strong Decoherence*, *in* Proceedings of the 4th Drexel Conference on Quantum Non-Integrability: The Quantum-Classical Correspondence; ed.: D.-H. Feng and B.-L. Hu, International Press of Boston.

[47]Goleman, Daniel, 2006, *Social Intelligence*, Bantam.

[48]Gonçalves, Carlos, 2005, *Uma Teoria da Empresa Ágil Criadora de Conhecimento. Contributos*, PhD Thesis, ISCTE-Business School.

[49]Gonçalves, Carlos, 2006, *Criação de Conhecimento e Agilidade: Novos Desafios Competitivos*, Economia Global e Gestão, Vol. XI, No. 1, pp. 125-139.

[50]Gonçalves, C. Pedro, 2003a, *Herding*, http://ccl.northwestern.edu/netlogo/models/community/Herding

[51]Gonçalves, C. Pedro, 2003b, *Artificial financial market* http://ccl.northwestern.edu/netlogo/models/community/Artificial%20Financial%20Market

[52]Gonçalves, C. Pedro, 2005, *Artificial financial market II – Tail risk*, http://ccl.northwestern.edu/netlogo/models/community/Artificial%20Financial%20Market%20II%20-%20Tail%20Risk

[53]Gonçalves, C. Pedro; Curto, J.D., 2005, *Risco, Agentes e Complexidade*, paper presented at the seminar *Grupos, Tecnologia e Criatividade*, ISCTE.

[54]Gonçalves, C. P.; Ferreira, M. A., 2006, *Self-Organized Criticality in Synchronized Loss Dynamics – A Behavioral Account*, working paper presented at the seminar *Mercado Financeiro Artificial: Simulação Multiagentes e Dinâmicas de Mercado*, ISCTE, http://dfc.gestao.iscte.org/seminarios/downloads.php?item=6&tipo=file.

[55]Gregory, Alan, 1992, *Valuing Companies – Analysing Business Worth*, Woodhead-Faulkner.

[56]Griffiths, R. B., 1984, *Constistent Histories and the Interpretation of Quantum Mechanics*, J. Stat. Phys. **36**, pp. 219–272.

[57]Griffiths, R. B., 1993, *Consistent interpretation of quantum mechanics using quantum trajectories*, Phys. Rev. Lett. **70**, pp. 219–272.

[58]Griffiths, R. B., 1994, *A consistent history approach to the logic of quantum mechanics*, *in* Symposium on the Foundations of Modern Physics, ed.: K.V. Laurikainen, C. Montonen and K. Sunnarborg, Éditions Frontières.

[59]Hameroff, S.; Penrose, R., 1996, *Orchestrated reduction of quantum coherence in brain microtubules – a model for consciousness*, *in* Toward a Science of Consciousness: Contributions from th 1994 Tucson Conference, ed.: S. Hameroff, A. Kaszniak and A. Scott, Cambridge, MA: MIT Press.

[60]Hartle, James B.; Hawking, Stephen, 1987, *Wave Function of the Universe*, *in* Quantum Cosmology, ed.: L.Z. Fang and R. Ruffini, World Scientific Publishing.

[61]Hartle, James B., 2003, *Theories of Everything and Hawking's Wave Function of the Universe*, *in* The Future of Theoretical Physics and Cosmology, ed.: G. W. Gibons, E. P. S. Shellard and S. J. Rankin, Cambridge University Press, arXiv:gr-qc/0209047.

[62]Hartle, James B., 2004, Linear Positivity and Virtual Probability, Phys. Rev. A, 70, 02210, arXiv:quant-ph/0401108.

[63]Hawking, Stephen, 1987, *The Quantum State of the Universe*, *in* Quantum Cosmology, ed.: L.Z. Fang and R. Ruffini, World Scientific Publishing.

[64]Hawking, Stephen, 2001, *The Universe in a Nutshell*, Bantam.

[65]Heisenberg, W., 1961, *Planck's Discovery and the Philosophical Problems of Atomic Physics*, *in* On Modern Physics, ed.: C. N. Polter, Orion Press.

[66]Herbut, F., 2001, *Is quantum decoherence reality or appearance?*, arXiv:quant-ph/0107049.

[67]Hidalgo, E. Guevara, 2005, *Quantum replicator dynamics*, arXiv:quant-ph/0510238.

[68]Hidalgo, E. Guevara, 2006a, *Introduction to the study of entropy in quantum games*, arXiv:quant-ph/0604170.

[69]Hidalgo, E. Guevara, 2006b, *Quantum games entropy*, arXiv:quant-ph/0606045.

[70]Hidalgo, E. Guevara, 2006c, *The why to the applicability of statistical physics to economics*, arXiv:physics/0609088.

[71]Hidalgo, E. Guevara, 2006d, *Quantum econophysics*, arXiv:physics/0609245.

[72]Iqbal, A., 2004, *Studies in the Theory of Quantum Games*, PhD thesis, Quaid-i-Azam University, Department of Electronics.

[73]Iqbal, A.; Toor, A.H., 2002, *Quantum repeated games*, Physics Letters A **300**, **6**, pp. 541−546.

[74]Johansen, A.; Ledoit, O.; Sornette, D., 2002, *Crashes as Critical Points*, arXiv:cond-mat/9810071.

[75]Johansen, A., Sornette, D., 2002, *Endogenous versus exogenous crashes in financial markets*, arXiv:cond-mat/0210509.

[76]Khrennikov, A., 2003, *On the cognitive experiments to test quantum-like behaviour of mind*, arXiv:quant-ph/0205092v2.

[77]Khrennikov, A., 2006, *A formula of total probability with interference term and the Hilbert space representation of the contextual Kolmogorovian model*, arXiv:math/0609197.

[78]Khrennikov, A., 2007, *Quantum-like Probabilistic Models outside Physics*, arXiv:physics/0702250.

[79]Kindleberger, Charles P., 2000, *Manias, Panics and Crashes*, John Wiley & Sons.

[80]Kogut, Bruce; Zander, Udo, 1992, *Knowledge of the firm, combinative capabilities, and the replication of technology*, Organization Science, Vol.3 No. 3, August.

[81]Kogut, Bruce; Zander, Udo, 1996, *What Firms Do? Coordination, Identity, and Learning*, Organization Science, Vol. 7, No. 5, pp. 502-518.

[82]Lesourne, Jacques; Orléan, André; Walliser, Bernard; Bourgine, P.; Fauchart E.; Laslier, J.-F.; Marengo, L.; Moreau, F.; Umbhauer, G., 2006, [2002], *Evolutionary Microeconomics*, Springer.

[83]Lux, T.; Marchesi, M., 1999, *Scaling and criticality in a stochastic multi-agent model of a financial market*, Nature, 397, pp. 498−500.

[84]Lux, T.; Marchesi, M., 2000, *Volatility clustering in financial markets: A micro-simulation of interacting agents*, International Journal of Theoretical and Applied Finance, 3, pp. 67−702.

[85]Lux, T., 2006, *Financial power laws : empirical evidence, models, and mechanism*, http://opus.zbw-kiel.de/volltexte/2006/5159/pdf/EWP-2006-12.pdf.

[86]Malkiel, Burton G., 1996, *A Random Walk down Wall Street*, W.W. Norton & Company.

[87]Mandelbrot, Benoît, 1962a, *Sur certains prix spéculatifs: faits empiriques et modèle basé sur les processus stables additives de Paul Lévy*, Comptes Rendues, 254, pp. 3968-3970.

[88]Mandelbrot, Benoît, 1962b, *The variation of certain speculative prices*, IBM Research Report NC-87, March.

[89]Mandelbrot, Benoît, 1963, *The variation of certain speculative prices*, Journal of Business, 36, pp. 394-419.

[90]Mandelbrot, Benoît; Fisher, Adlai and Calvet, Laurent, 1997, *A multifractal model of asset returns*, Cowles Foundation Discussion Paper, 1164.

[91]Mandelbrot, Benoît, 1989, *Multifractal measures, especially for the geophysicist*, Pure and Applied Geophysics **131**, 5−42.

[92]Mandelbrot, Benoît, 1999, *Survey of multifractality in finance*, Cowles Foundation Discussion Paper 1238.

[93]Mandelbrot, Benoît; Hudson, Richard L., 2004, *The (Mis)Behavior of Markets – A Fractal View of Risk, Ruin, and Reward*, Basic Books.

[94]Melo de Sousa, P. B.; Ramos, R. V.; Costa Filho, J. T., 2006, *New models of quantum games*, arXiv:quant-ph/0608131.

[95]Meyer, David A., 1999, *Quantum strategies*, Phys. Rev. Lett., **82**, pp.1052−1055.

[96] Montagna, G.; Nicrosini, O.; Passarino, G.; Moreni, N., 2002a, *A path integral way to option pricing*, arXiv:cond-mat/0202143.

[97] Montagna, G.; Nicrosini, O.; Passarino, G.; Moreni, N., 2002b, *Pricing derivatives by path integral and neural networks*, Proc. of International Econophysics Conference, Bali, August 38−31, arXiv:cond-mat/0211260.

[98] Montagna, G.; Nicrosini, O.; Passarino, G.; Moreni, N., 2004, *Pricing exotic options in a path integral approach*, Quantitative Finance, 6, pp. 55−66, arXiv:cond-mat/0407321.

[99] Muth, John F., 1961, *Rational expectations and the theory of price movements*, Econometrica, Vol.29, No.3.

[100] Muzy, J. F., Delour, J., Bacry, E., 2000, *Modelling fluctuations of financial time series: from cascade process to stochastic volatility model*, The European physical Journal B 17, 537-548.

[101] Muzy, J. F.; Sornette, D.; Delour, J.; Arneodo, A., 2001, *Multifractal returns and hierarchical portfolio theory*, Quantitative Finance 1 (1), pp. 131−148, arXiv:cond-mat/0008069.

[102] Nelson, Richard R.; Winter, Sidney G., 1996, [1982], *An Evolutionary Theory of Economic Change*, The Belknap Press of Harvard University Press.

[103] Nonaka, I.; Toyama, R.; Nagata, Y., 2000, *A Firm as a Knowledge-creating Entity: A New Perspective on the Theory of the Firm*, Industrial and Corporate Change, Volume 9, Number I.

[104] O'Hara, Maureen, 1998, *Market Microstructure Theory*, Blackwello Publishing.

[105] Ohashi, K.; Amaral, L. A. N.; Natelson, B. H.; Yamamoto, Yoshiharu, 2003, *Asymmetrical singularities in real-world signals*, Physical Review E **68**, 065204.

[106] Omnès, R., 1988, *Logical reformulation of quantum mechanics* I−III, J. Stat. Phys. **53**, pp. 893−975.

[107] Omnès, R., 1992, *Consistent interpretation of quantum mechanics*, Rev. Mod. Phys. **64**, pp.339−382.

[108] Pakuła, I.; Piotrowski, E. W.; Sładkowski, J., 2005, *Universality of measurements in quantum markets*, arXiv:quant-ph/0504036.

[109] Paz, J. P.; Zurek, W.H., 2002, *Environment-induced decoherence and the transition from quantum to classical*, *in* Fundamentals of Quantum Information, Springer.

[110] Penrose, Edith T., 1995, [1959], *The Theory of the Growth of the Firm*, John Wiley & Sons.

[111]Pesseglione, M.; Seymour, B.; Flanding, G.; Dolan, R.; Firth, C., 2006, *Dopamine-dependent prediction errors underpin reward seeking behaviour in humans*, Nature, Aug. 31; 442(7106):1042−5.

[112]Piotrowski, E.; Sładkowski, J., 2002, *Quantum market games*, Physica A 312, 208, arXiv:quant-ph/0104006.

[113]Piotrowski, E.; Sładkowski, J., 2003a, *An invitation to quantum game theory*, Int. J. Theor. Phys. 42, 1089 arXiv:quant-ph/0211191.

[114]Piotrowski, E.; Sładkowski, J., 2003b, *Trading by quantum rules − quantum anthropic principle*, Int. J. Theor. Phys. 42, 1101, arXiv:quant-ph/0201045.

[115]Piotrowski, E.; Sładkowski, J., 2005, *Quantum games and programmable quantum systems*, arXiv:quant-ph/0504010.

[116]Popper, Karl, 1992a, [1982], *O Universo Aberto*, Tradução de Nuno Ferreira da Fonseca, Publicações Dom Quixote, Lisboa.

[117]Popper, Karl, 1992b, [1982], *A Teoria dos Quanta e o Cisma da Física*, Tradução de Nuno Ferreira da Fonseca, Publicações Dom Quixote, Lisboa.

[118]Rappaport, A.; Maboussin, M. J., 2001, *Expectations Investing*, Harvard Business School Press.

[119]Rizzolati, G.; Fogassi, L.; Gallese, V., 2006, *Mirrors in the mind*, Scientific American, vol.195, No.5, November, pp. 30−35.

[120]Samanidou, Egle; Zschischang, Elmar; Stauffer, Dietrich; Lux, Thomas, 2006, *Microscopic models of financial markets*, Economics Working Paper Series, No. 2006-15, Christian-Albrechts-Universität Kiel, Department of Economics, http://opus.zbw-kiel.de/volltexte/2006/5162/pdf/EWP-2006-15.pdf.

[121]Schaden, Martin, 2002, *Quantum finance*, Physica A 316, 511, arXiv:physics/0203006.

[122]Schaden, Martin, 2003a, *A quantum approach to stock price fluctuations*, arXiv:physics/0205053.

[123]Schaden, Martin, 2003b, *Pricing European options in realistic markets*, arXiv:physics/0210025.

[124]Sheldrake, Rupert, 2000, *The Presence of the Past: Morphic Resonance and the Habits of Nature*, Inner Traditions Bear and Company.

[125]Sheldrake, Rupert, 2002, *Seven Experiments that could Change the World − A Do-it-yourself Guide to Revolutionary Science*, Inner Traditions Bear and Company.

[126]Shleifer, Andrei, 2000, *Inneficient Markets, An Introduction to Behavioral Finance*, Oxford University Press, US.

[127]Simon, Herbert A., 1977, [1945], *Administrative Behavior*, The Free Press, A Division of Macmillan Publishing, Co., Inc..

[128]Singh, J. P.; Prabakaran, S., 2006, *A toy model of financial markets*, EJTP 3, No. 11, pp. 11−27.

[129]Smith, J. M.; Price, G. R., 1973, *The logic of animal conflict*, Nature 246: 15−18.

[130]Smith, J. M., 1982, *Evolution and the Theory of Games*, Cambridge University Press.

[131]Sornette, Didier, 2003, *Critical market crashes*, arXiv:cond-mat/0301543.

[132]Sornette, Didier; Zhou, Wei-Xing, 2003, *Predictability of large future changes in major financial indexes*, International Journal of Forecasting 22, pp. 153−168.

[133]Sornette, Didier, 2004, *Why Stock Market Crash: Critical Events in Complex Financial Systems*, Princeton University Press.

[134]Sornette, Didier; Zhou, Wei-Xing, 2005, *Importance of positive feedbacks and over-confidence in a self-fulfilling Ising model of financial markets*, arXiv:cond-mat/0503607.

[135]Sornette, Didier, 2006, *Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-organization and Disorder: Concepts and Tools*, Springer.

[136]Vaga, Tonis, 1990, *The Coherent Market Hypothesis*, Financial Analysts Journal, November/December.

[137]Vaidman, L., 1999, *Variations on the theme of the Greenberger-Horne-Zeilinger proof*, Foundations of Physics, Vol.29, No.4, pp. 615−630

[138]Voit, J., 2001, *The Statistical Mechanics of Financial Markets*, Springer.

[139]Von Neumann, John; Morgenstern, Oskar, 1990, [1953], *The Theory of Games and Economic Behavior*, Princeton University Press, US.

[140]Weisner, S., 1983, *Conjugate coding*, SIGCAT News, **15:1**, pp. 78−88.

[141]Wernerfelt, Bierger, 1984, *A Resource-based View of the Firm*, Strategic Management Journal, Vol. 5, pp. 171−180.

[142]Williamson, Oliver E., 1975, *Markets and Hierarchies: Analysis and Antitrust Implications*, The Free Press, Collier Macmillan Publishers, London.

[143]Williamson, Oliver E., 1985, *The Economic Institutions of Capitalism*, The Free Press, A Division of Simon & Shuster Inc..

[144]Zeh, H. D., 2002, *The Wave Function: It or Bit?*, *in* Science and Ultimate Reality, ed.: J. D. Barrow, P. C. W. Davies, and C. L. Harper Jr., Cambridge University Press.

[145]Zeh, H. D., 2006, *Roots and fruits of decoherence*, *in* Quantum Decoherence, ed.: B. Duplantier, J. -M. Raimond and V. Rivasseau, Birkhäuser.

[146]Zhang, Peter G., 1998, *Exotic Options*, World Scientific.

[147]Zhou, Wei-Xing; Sornette, Didier, 2005 *Self-fulfilling Ising model of financial markets*, arXiv:cond-mat/0503230.

[148]Zhou, Wei-Xing; Sornette, Didier, 2006, *Fundamental factors versus herding in the 2000-2005 US stock market and prediction*, Physica A 360, pp. 459−482.

[149]Zhou, Wei-Xing; Sornette, Didier, 2007 *Self-organizing Ising model of financial markets*, The European Physical Journal B, vol.55, no.2, pp.175-181.

Appendix A. Logarithmic returns and market activity observables

Let us consider, first, the returns observable, $\hat{R}$. Denoting $\xi_0\left(k,\tau_P\right)$ by ξ_0 and $\xi_1\left(k,\tau_P\right)$ by ξ_1, we have that:

$$\begin{aligned}\left\langle \hat{R} \right\rangle_{k,\tau_P} &= \left\langle \xi_0,\xi_1 \left| \frac{1}{\lambda}\left(\hat{N}_1 - \hat{N}_0\right) \right| \xi_0,\xi_1 \right\rangle \\ &= \frac{1}{\lambda}\left(\left\langle \xi_0,\xi_1 \left| \hat{N}_1 \right| \xi_0,\xi_1 \right\rangle - \left\langle \xi_0,\xi_1 \left| \hat{N}_0 \right| \xi_0,\xi_1 \right\rangle\right) \\ &= \frac{1}{\lambda}\left(\left|\xi_1\right|^2 - \left|\xi_0\right|^2\right)\end{aligned}$$

on the other hand, for market activity $\hat{N}$, we have:

$$\begin{aligned}\left\langle \hat{R} \right\rangle_{k,\tau_P} &= \left\langle \xi_0,\xi_1 \left| \left(\hat{N}_1 + \hat{N}_0\right) \right| \xi_0,\xi_1 \right\rangle \\ &= \left\langle \xi_0,\xi_1 \left| \hat{N}_1 \right| \xi_0,\xi_1 \right\rangle + \left\langle \xi_0,\xi_1 \left| \hat{N}_0 \right| \xi_0,\xi_1 \right\rangle \\ &= \left|\xi_1\right|^2 + \left|\xi_0\right|^2\end{aligned}$$

Appendix B

B.1 Simulated financial series

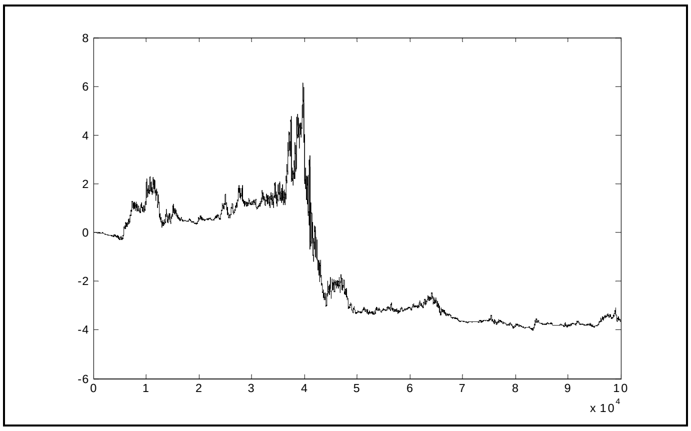


Fig.1 – Price logarithm, γ = 0.

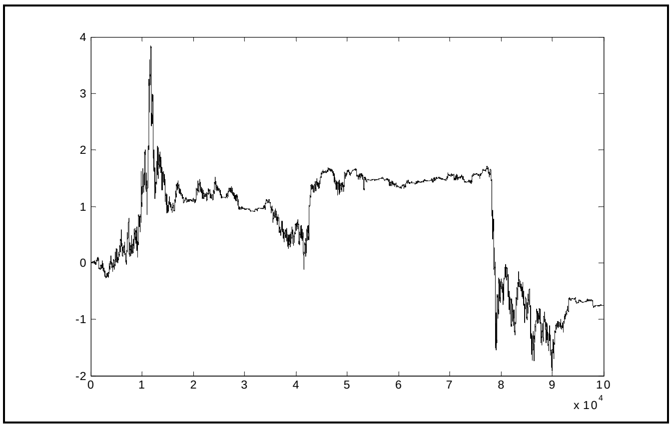


Fig.2 – Price logarithm, γ = 0.05.

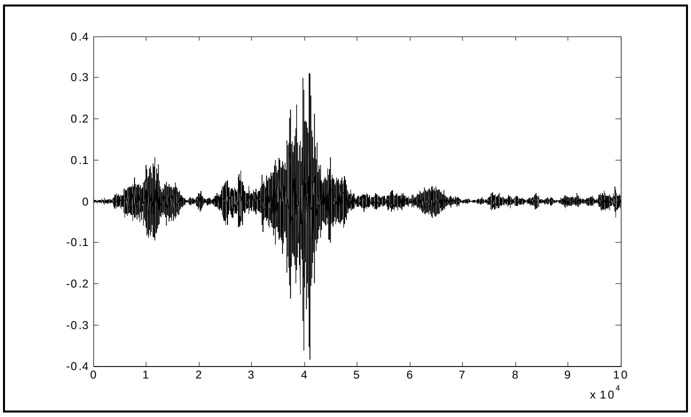


Fig.3 – Logarithmic returns for γ = 0.

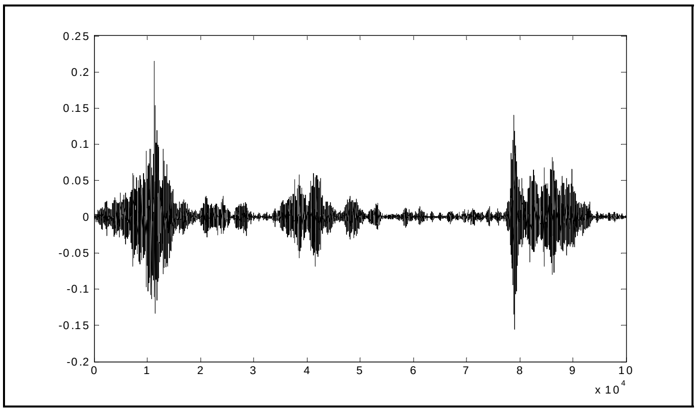


Fig.4 – Logarithmic returns for γ = 0.05.

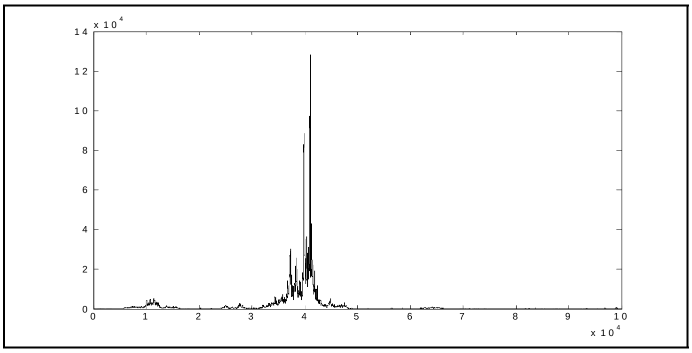


Fig.5 – Market activity for γ = 0.

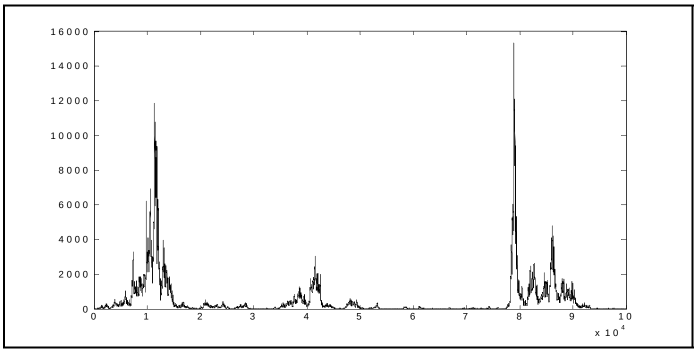


Fig.6 – Market activity for γ = 0.05.

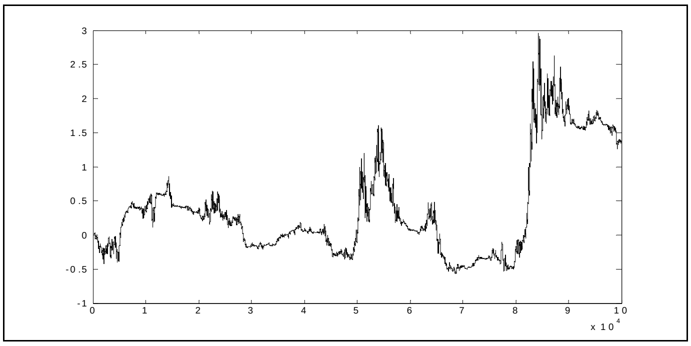


Fig. 7 – Price logarithm for $\gamma = 0.1$.

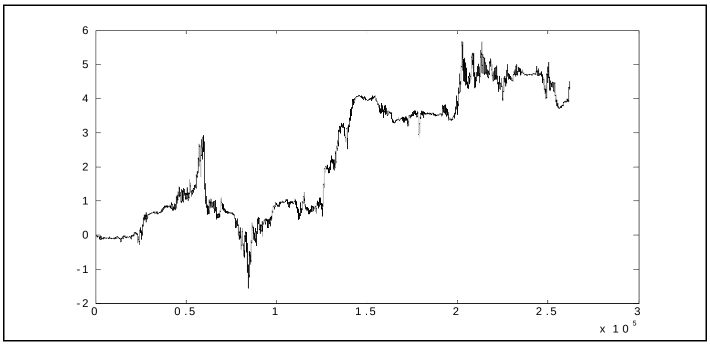


Fig. 8 – Price logarithm for $\overline{\mu} = 0.1$.

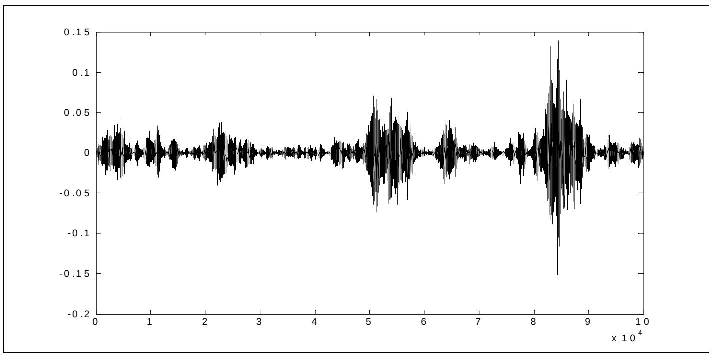


Fig. 9 – Logarithmic returns for $\gamma = 0.1$.

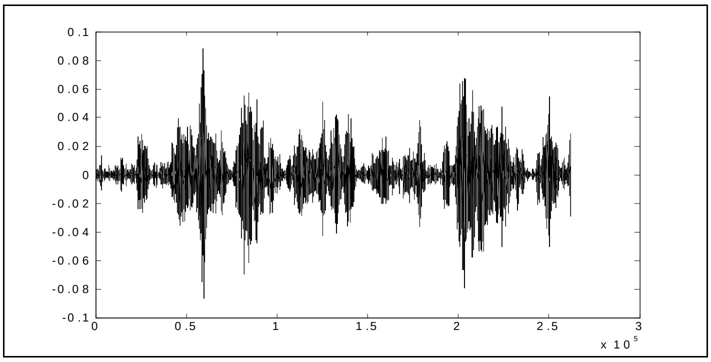


Fig. 10 – Logarithmic returns for $\overline{\mu} = 0.1$.

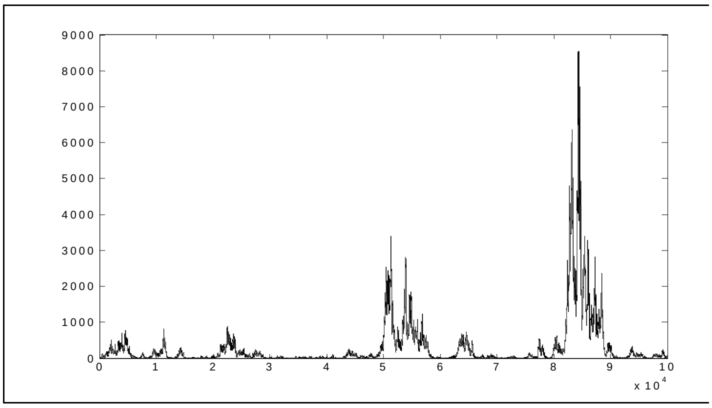


Fig.11 – Market activity for $\gamma = 0.1$.

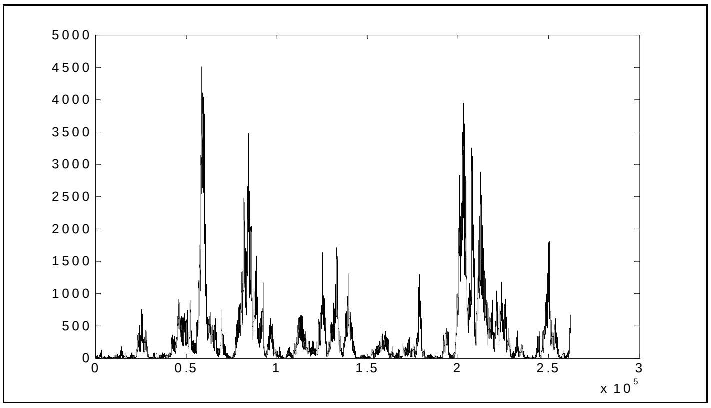


Fig.12 – Market activity for $\overline{\mu} = 0.1$.

B.2 Autocorrelation functions

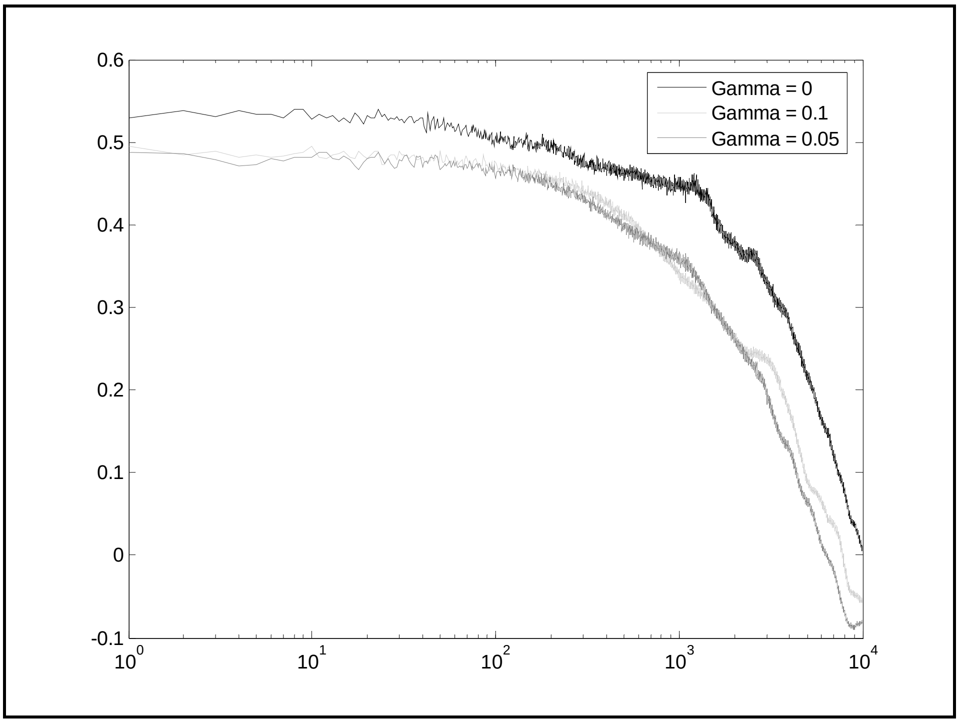


Fig.13 – Autocorrelation functions for absolute returns in the cases with $\overline{\mu} = 0$.

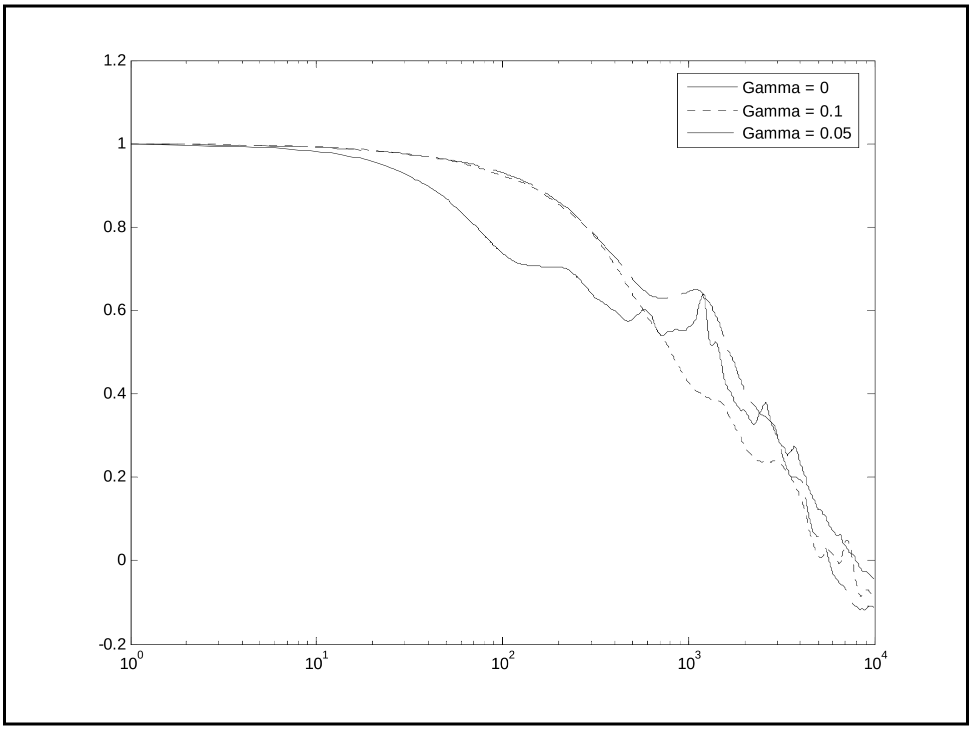


Fig.14 – Autocorrelation functions for market activity in the cases with $\overline{\mu} = 0$.

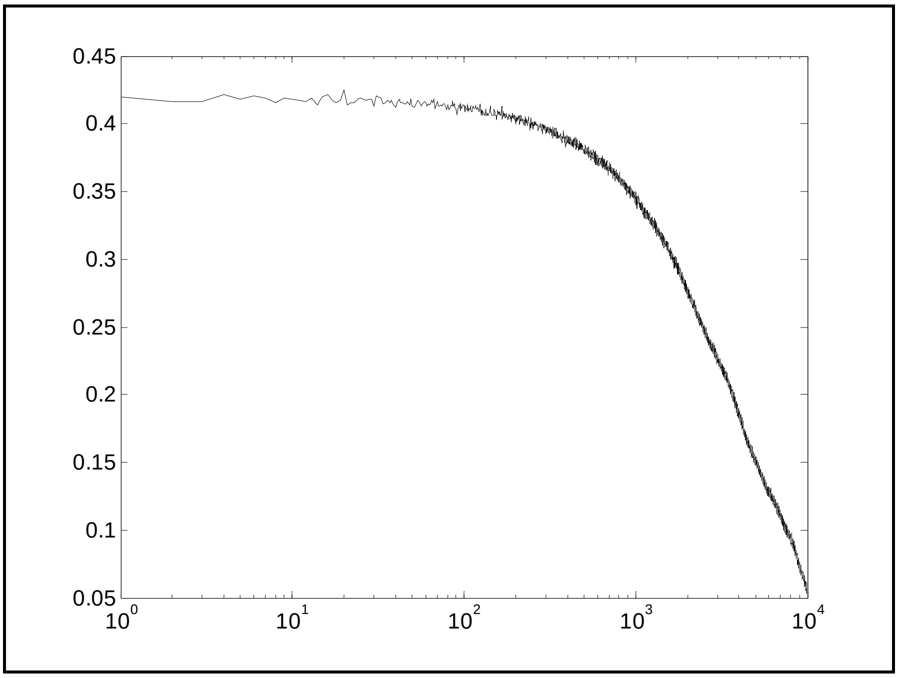


Fig.15 – Autocorrelation functions for absolute returns in the cases with $\overline{\mu} = 0.1$.

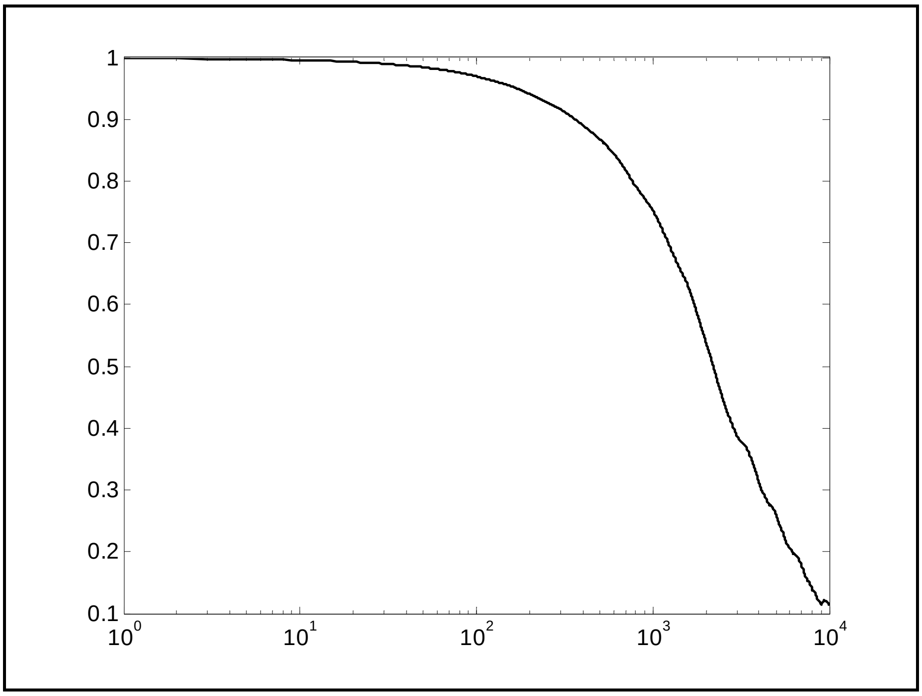


Fig.16 – Autocorrelation functions for market activity in the cases with $\overline{\mu} = 0.1$.

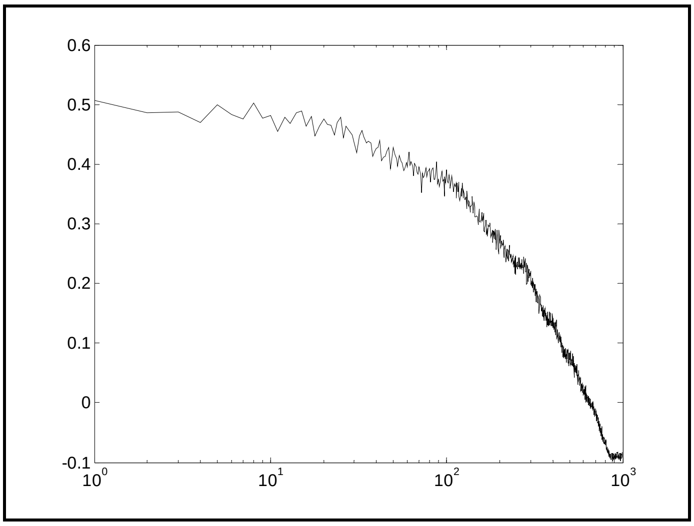


Fig.17 – Autocorrelation functions for absolute returns, $\gamma = 0$, sampling scale 10.

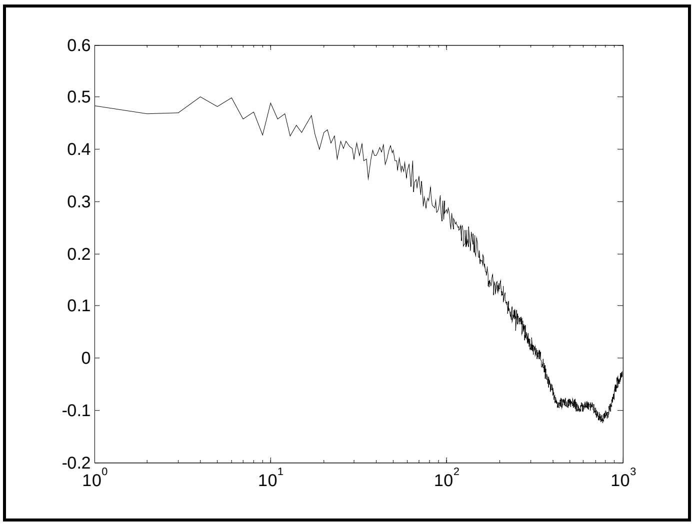


Fig.18 – Autocorrelation functions for absolute returns, $\gamma = 0$, sampling scale 20.

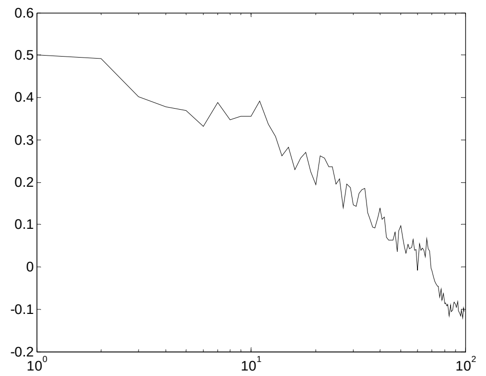


Fig.19 – Autocorrelation functions for absolute returns, γ = 0, sampling scale 100.

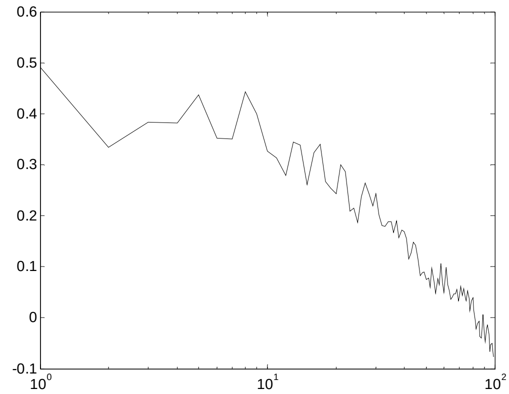


Fig.20 – Autocorrelation functions for absolute returns, γ = 0.05, sampling scale 100.

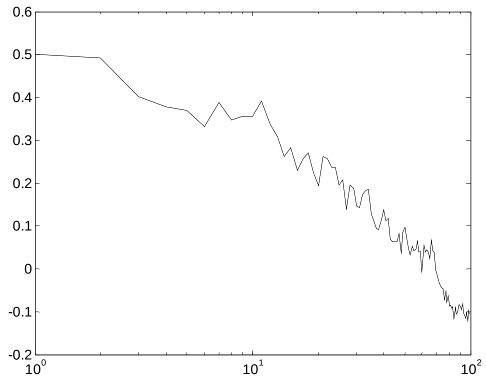


Fig.21 – Autocorrelation functions for absolute returns, γ = 0.1, sampling scale 100.

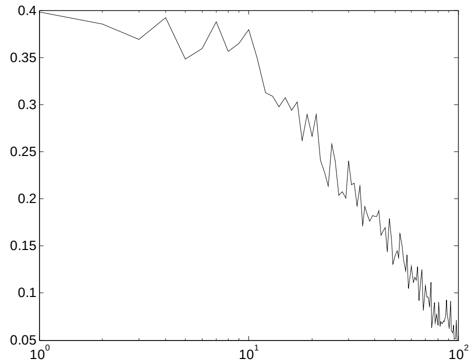


Fig.22 – Autocorrelation functions for absolute returns, $\overline{\mu} = 0.1$, sampling scale 100.

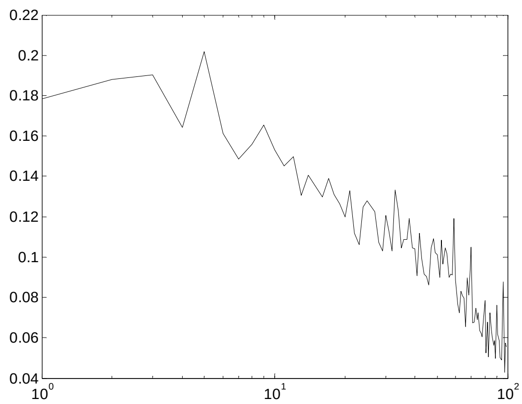


Fig.23 – Autocorrelation functions for daily absolute returns, S&P500 (05-02-1971 to 02-01-2007).

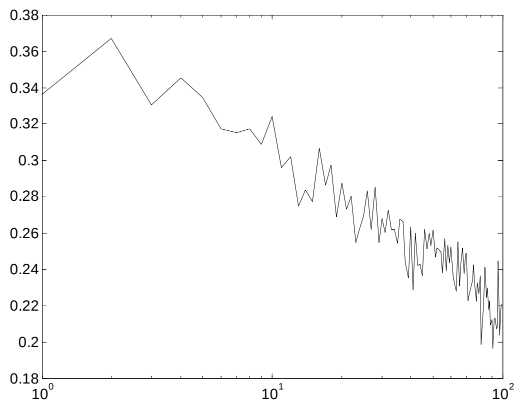
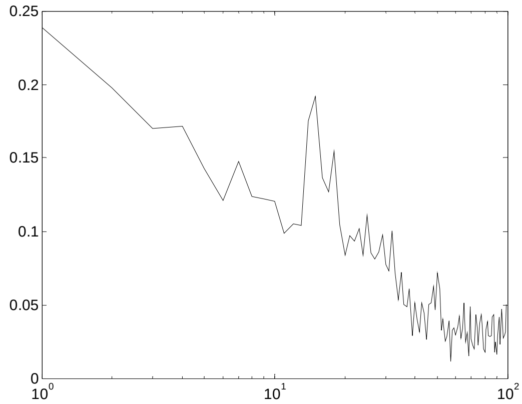
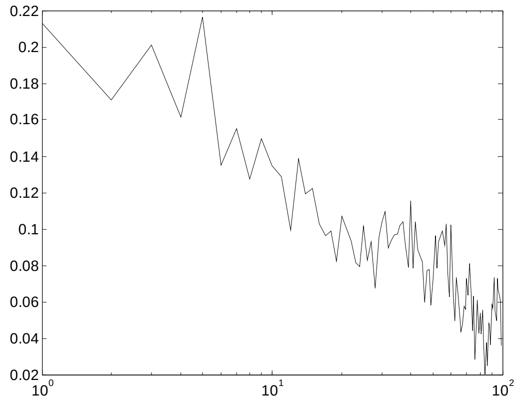


Fig.24 – Autocorrelation functions for daily absolute returns, Nasdaq (05-02-1971 to 02-01-2007).

 Fig.25 – Autocorrelation functions for daily absolute returns, Nikkei (02-04-1986 to 02-01-2007).	 Fig.26 – Autocorrelation functions for daily absolute returns, Hang Seng(02-04-1986 to 02-01-2007).

B.3 Multifractal spectra

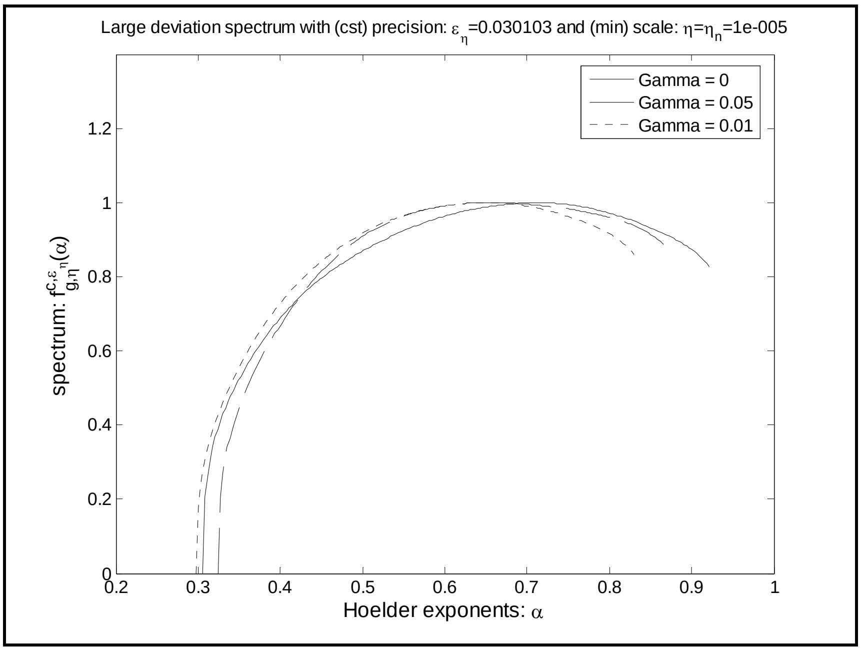


Fig.27 – Price logaritm's large deviation spectrum for the cases with $\overline{\mu} = 0$.

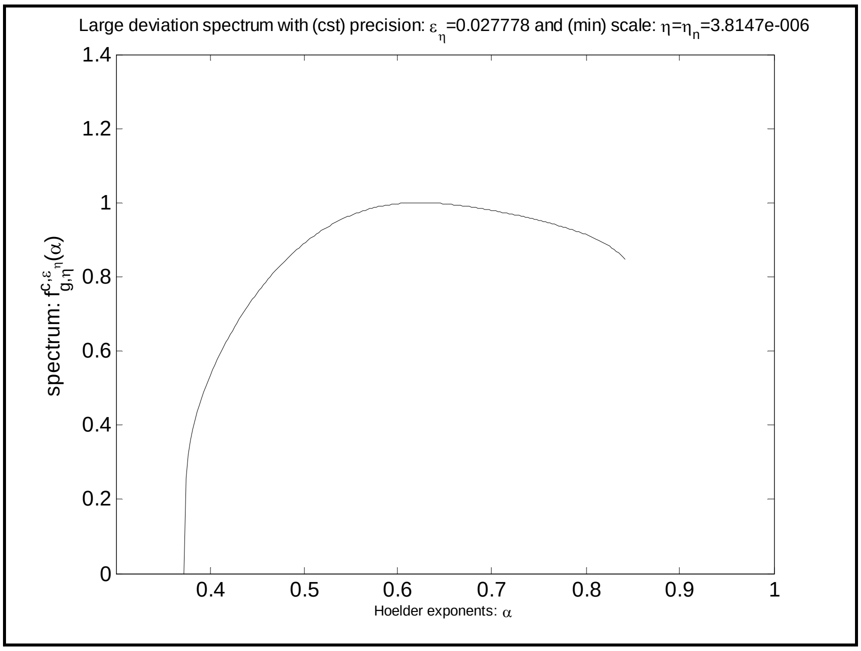


Fig.28 – Price logaritm's large deviation spectrum for $\overline{\mu} = 0.1$.

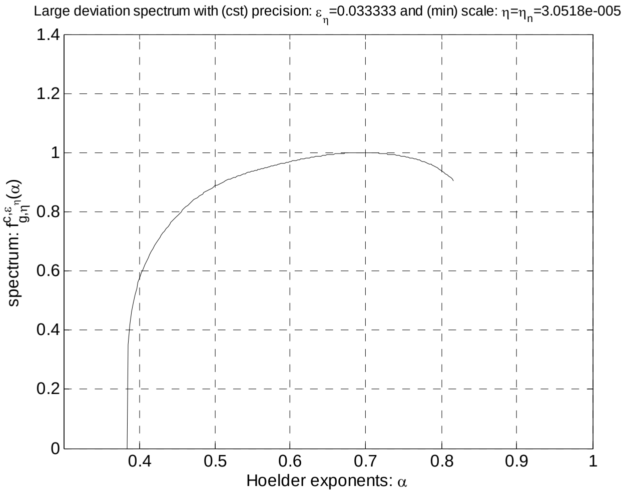


Fig.29 – Price logaritm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 1 to 32768.

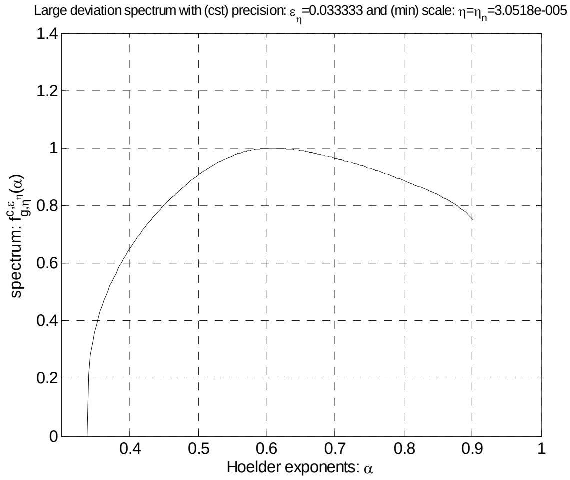


Fig.30 – Price logaritm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 32769 to 65536.

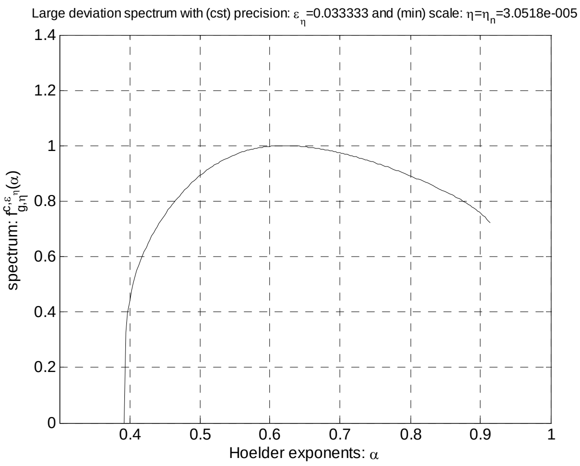


Fig.31 – Price logarithm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 65537 to 98304.

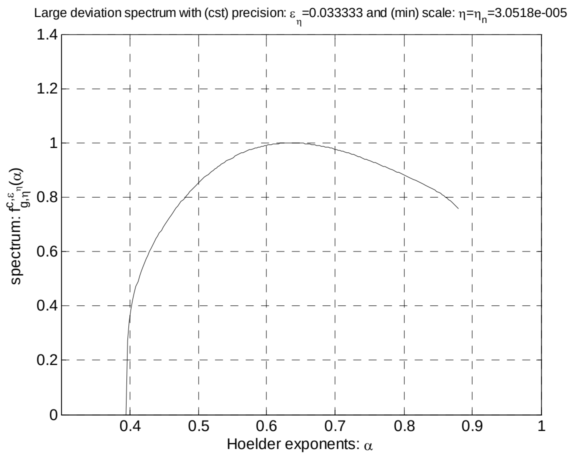


Fig.32 – Price logarithm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 98305 to 131072.

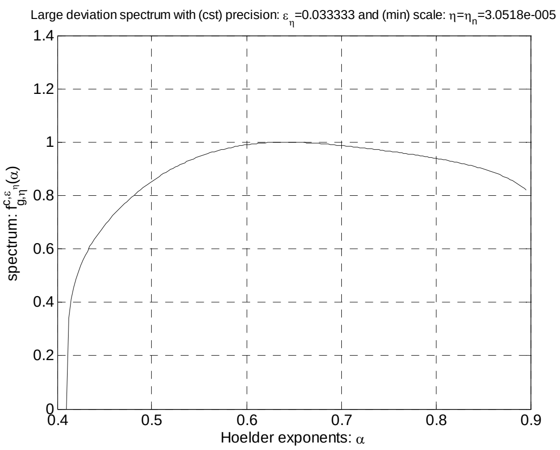


Fig.33 – Price logarithm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 131073 to 163840.

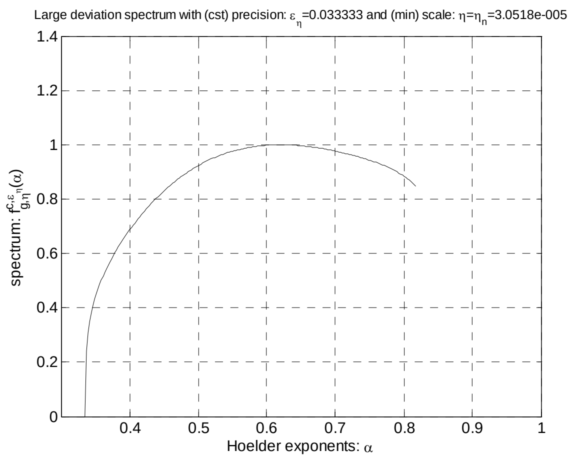


Fig.34 – Price logarithm's large deviation spectrum for $\overline{\mu} = 0.1$, sample rounds 163841 to 196608.

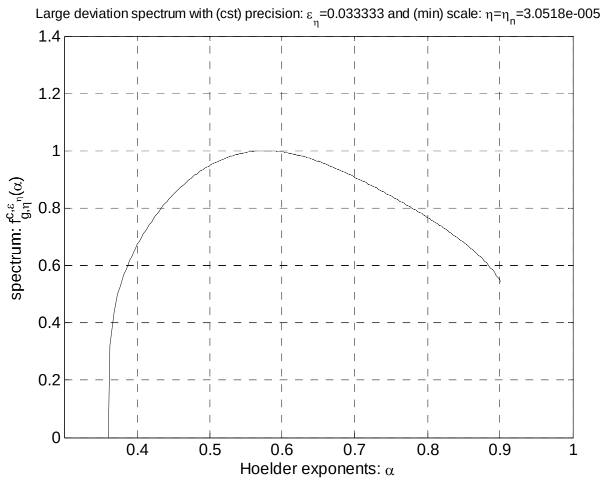


Fig.35 – Price logarithm’s large deviation spectrum for $\overline{\mu}=0.1$, sample rounds 196609 to 229376.

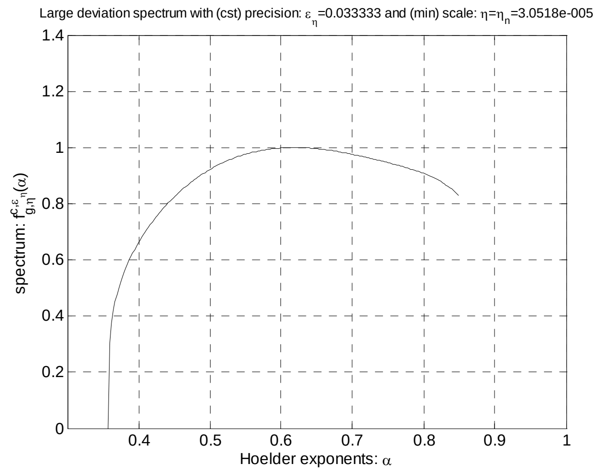


Fig.36 – Price logarithm’s large deviation spectrum for $\overline{\mu}=0.1$, sample rounds 229377 to 262144.

B.4 Hölder functions

B.4.1. Simulations’ results

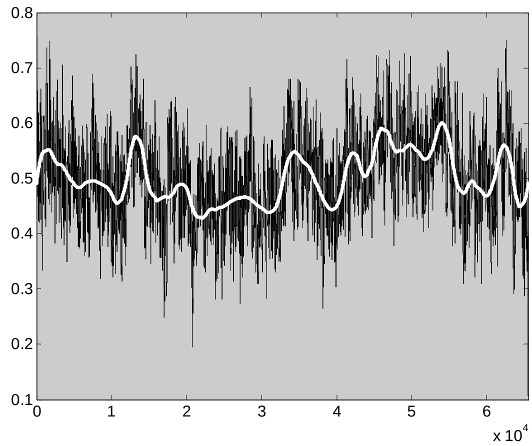


Fig.37 - Price logarithm's Hölder func., γ=0.

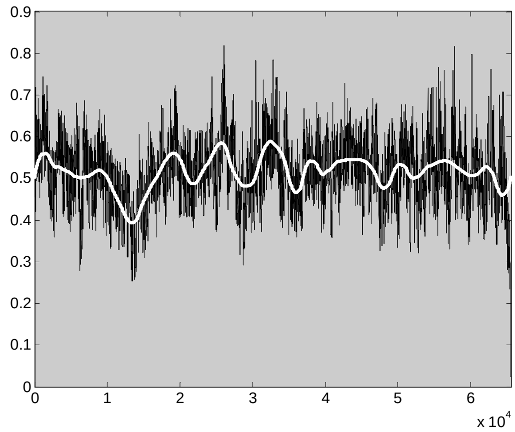


Fig.38 - Price logarithm's Hölder func., γ=0.05.

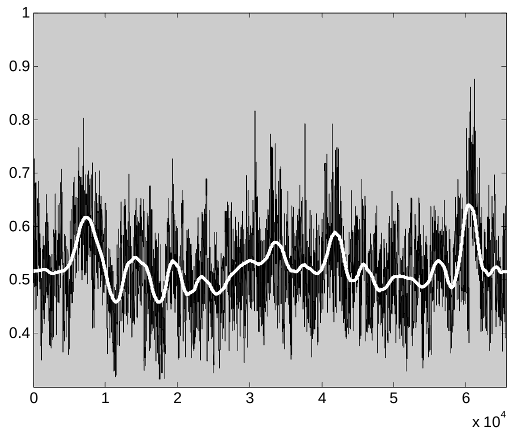


Fig.39 - Price logarithm's Hölder func., γ=0.1.

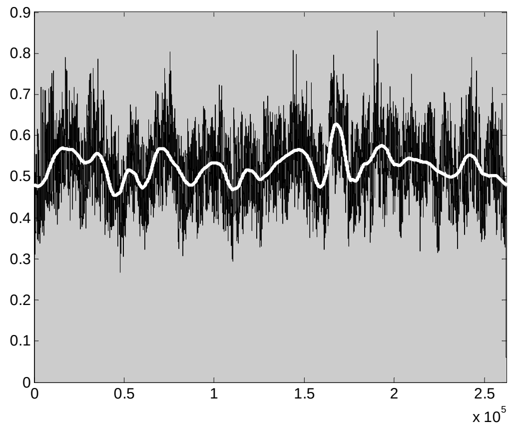


Fig.40 - Price logarithm's Hölder func., $\overline{\mu} = 0.1$.

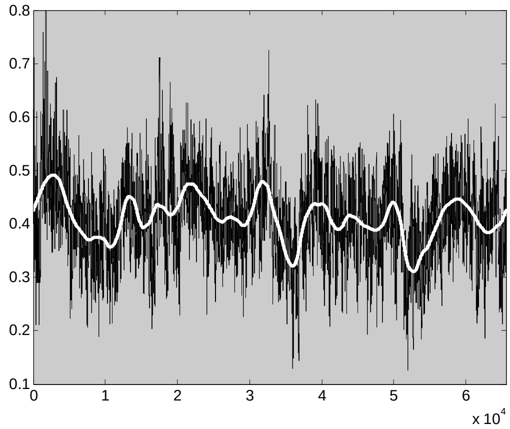


Fig.41 - Volatility's Hölder func., γ=0.

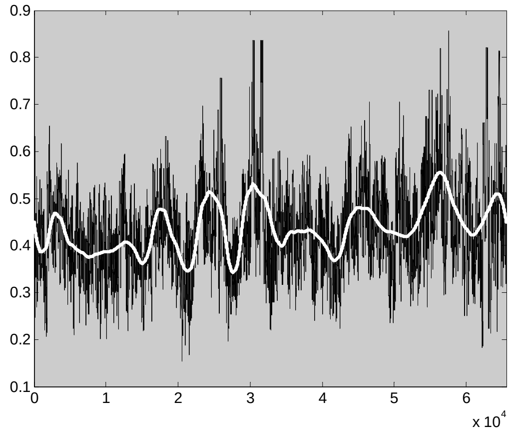


Fig.42 - Volatility's Hölder func., γ=0.05.

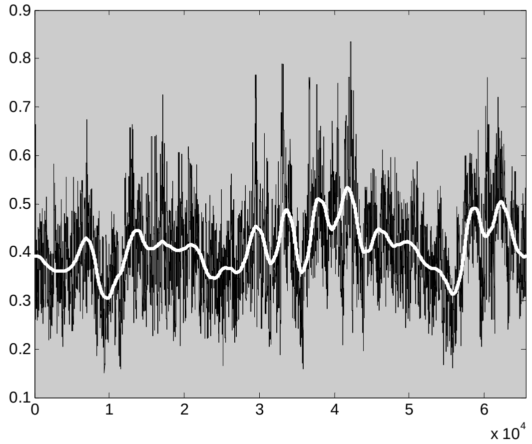


Fig.43 - Volatility's Hölder func., γ=0.1.

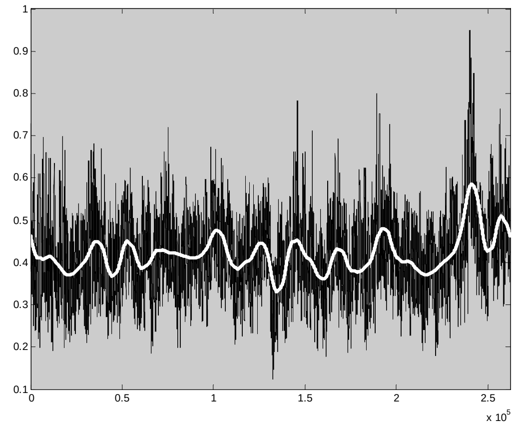


Fig.44 - Volatility's Hölder func., $\overline{\mu} = 0.1$

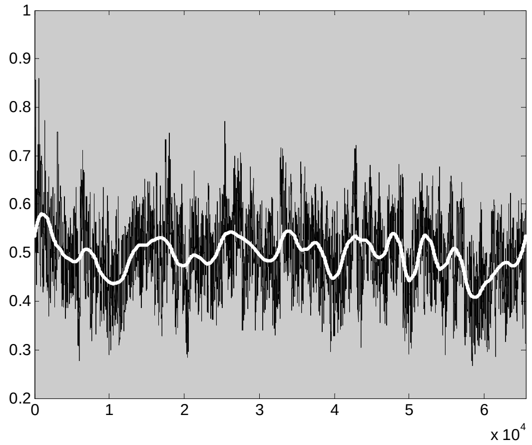


Fig.45 - Market activity's Hölder func., γ=0.

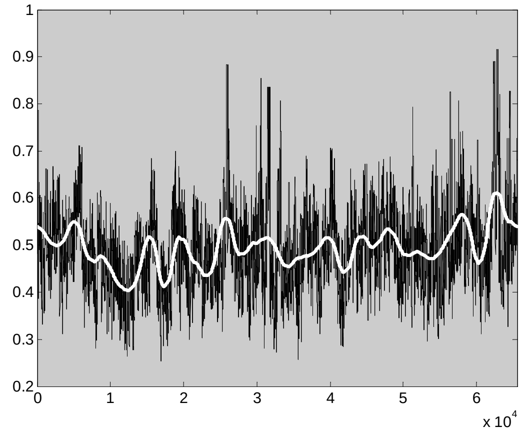


Fig.46 - Market activity's Hölder func., γ=0.05.

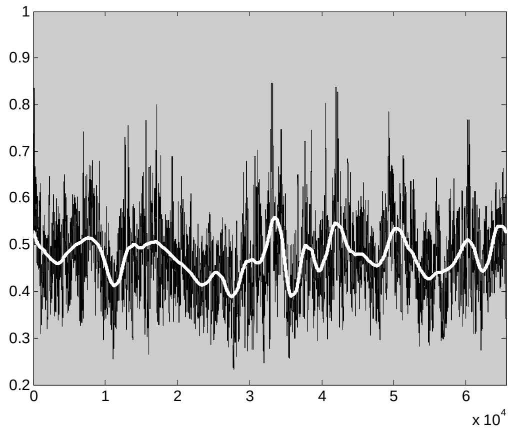


Fig.47 - Market activity's Hölder func., γ=0.1.

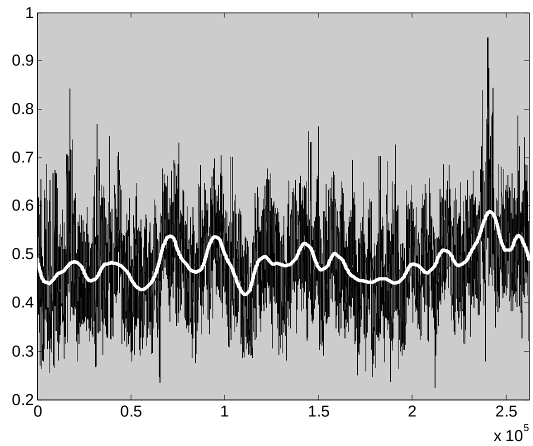


Fig.48 - Market activity's Hölder func., $\overline{\mu} = 0.1$.

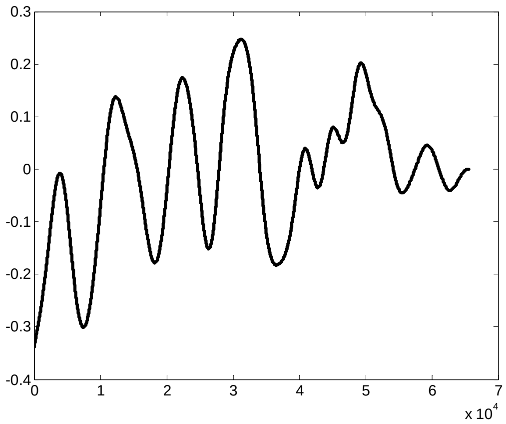


Fig.49 - Cross-correlations between price logarithm's Hölder func. and volatility's Hölder func., γ=0.

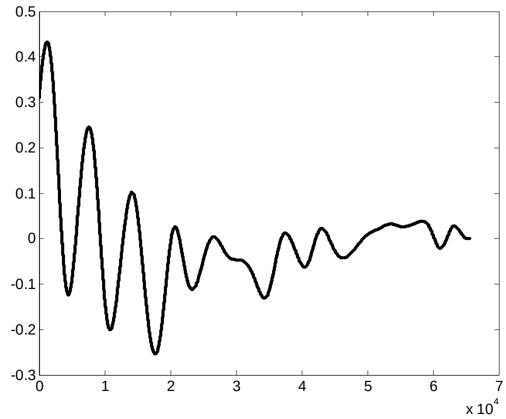


Fig.50 - Cross-correlations between price logarithm's Hölder func. and volatility's Hölder func., γ=0.05.

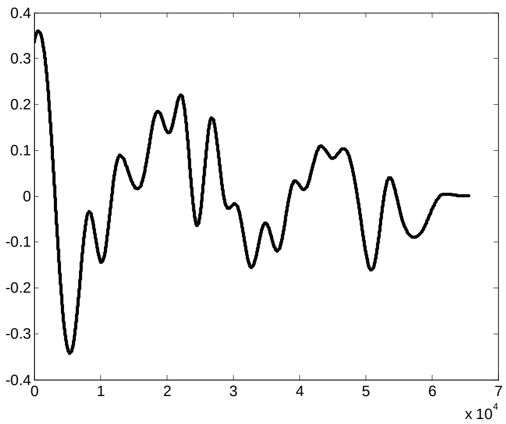


Fig.51-Cross-correlations between price logarithm's Hölder func. and volatility's Hölder func., γ=0.1.

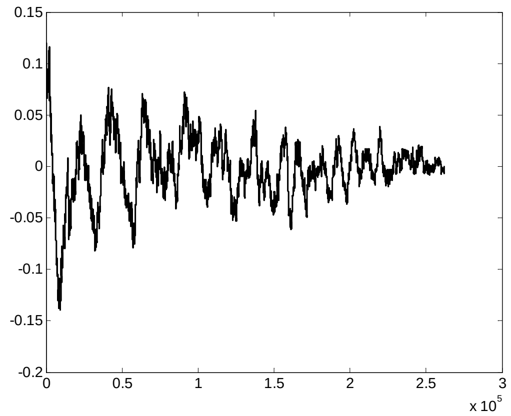


Fig.52 - Cross-correlations between price logarithm's Hölder func. and volatility's Hölder func., $\overline{\mu} = 0.1$.

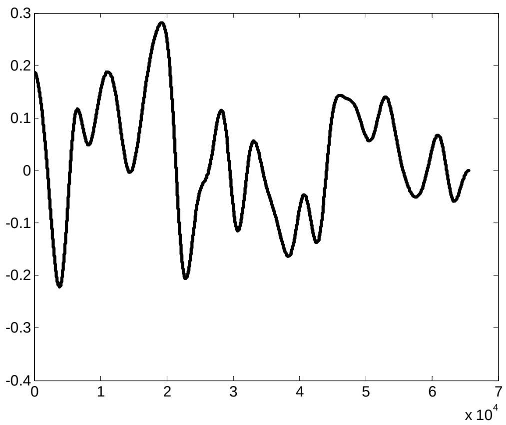


Fig.53 - Cross-corr. between price logarithm's Hölder func. and market activity's Hölder func., γ=0.

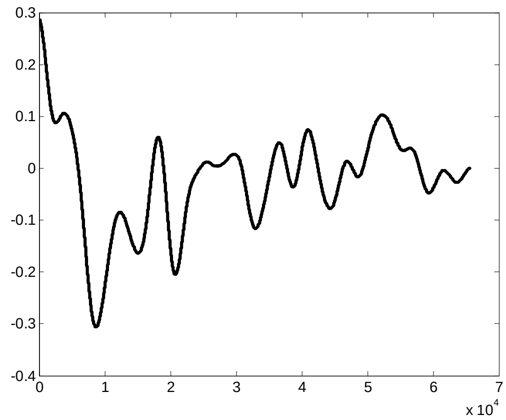


Fig.54 - Cross-corr. between price logarithm's Hölder func. and market activity's Hölder func., γ=0.05.

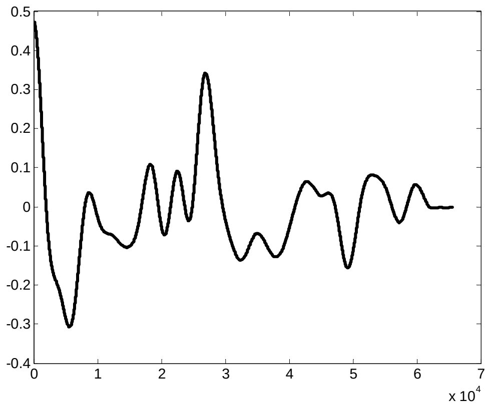


Fig.55-Cross-corr. between price logarithm's Hölder func. and market activity's Hölder func., γ=0.1.

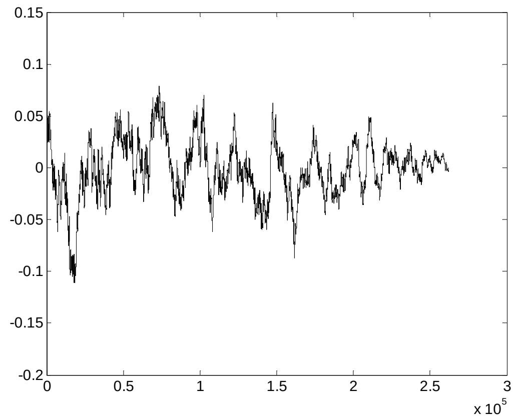


Fig.56-Cross-corr. between price logarithm's Hölder func. and market activity's Hölder func., $\overline{\mu} = 0.1$.

B.4.2. Markets' results[1]

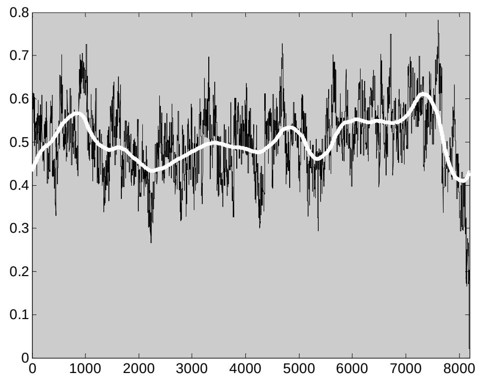


Fig.57 - S&P500's price logarithm's Hölder function.

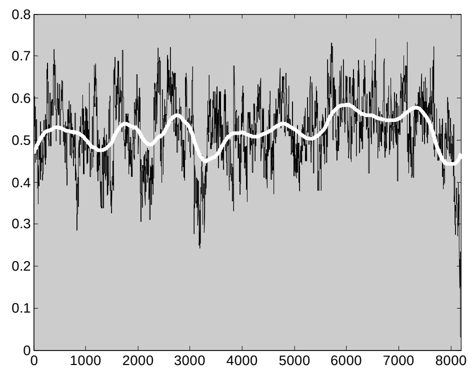


Fig.58 - NASDAQ's price logarithm's Hölder function.

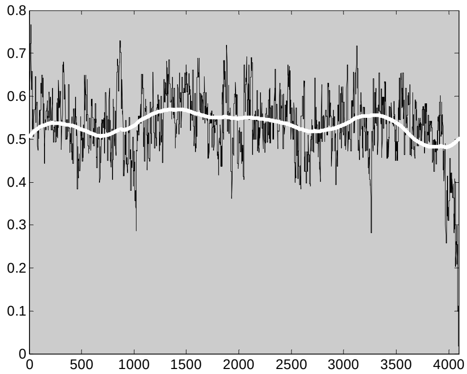


Fig.59 - Nikkei's price logarithm's Hölder function.

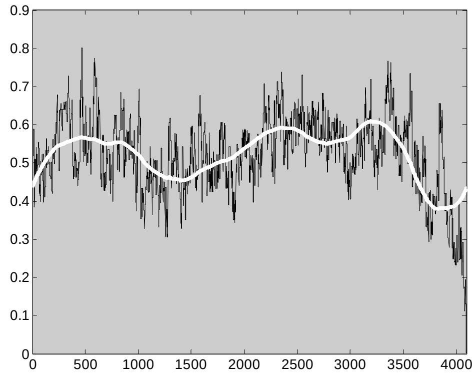


Fig.60 - Hang Seng's price logarithm's Hölder function.

[1] For this analysis had to take a subsample from the actual data to fit a power of 2, in this case the Nikkei and Hang Seng have 4096 data points which corresponds to the interval from 23-04-1991 to 2-01-2007, while the S&P500 and the Nasdaq data have 8192 data points, which corresponds to the interval from 11-08-1975 to 02-01-2007.

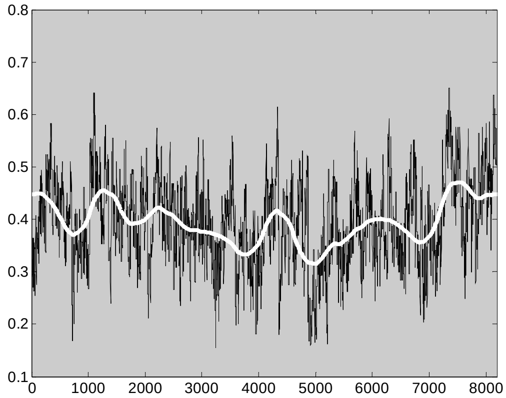


Fig.61 - S&P500's volatity's Hölder function.

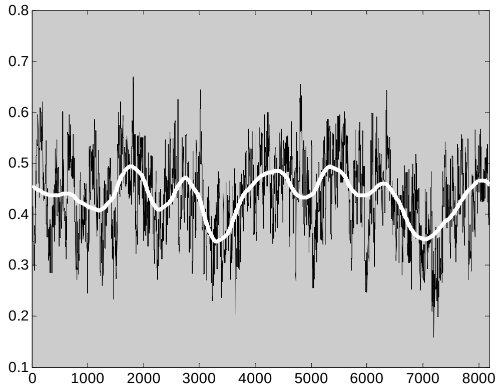


Fig.62 - Nasdaq's volatility's Hölder function.

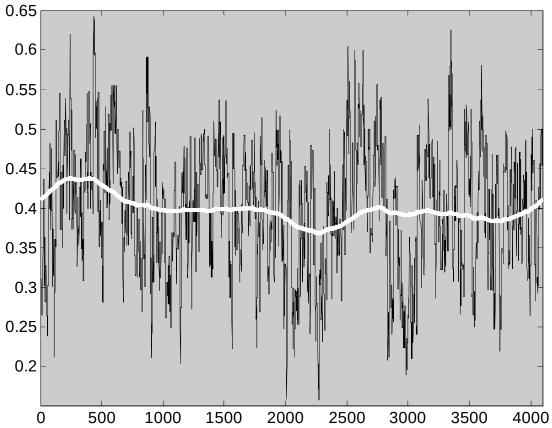


Fig.63 - Nikkei's volatility's Hölder function.

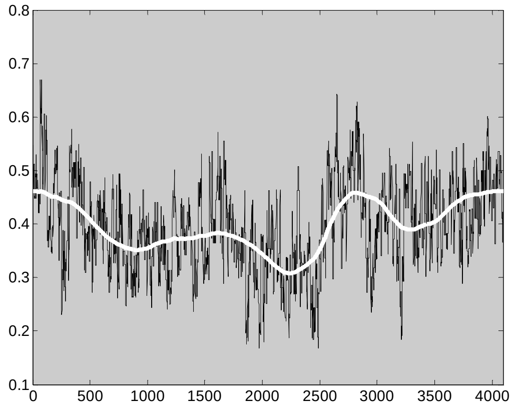


Fig.64 - Hang Seng's volatility's Hölder function.

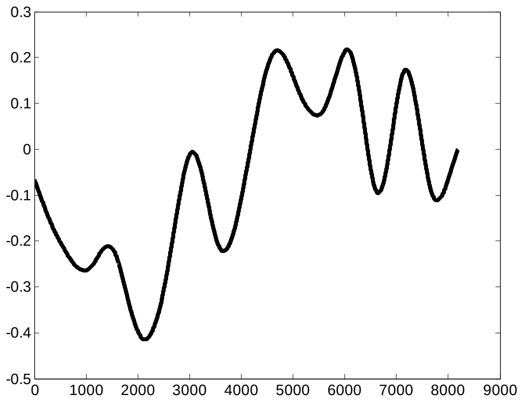


Fig.65 - Cross-corr. between price logarithm's Hölder func. and volatility's Hölder func., S&P500.

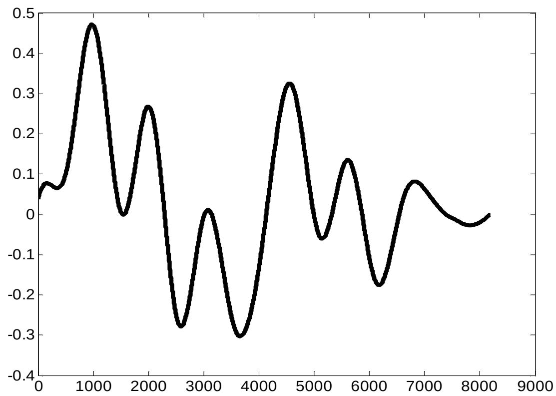


Fig.66 - Cross-corr. between price logarithm's Hölder func. and volatility's Hölder func., Nasdaq.

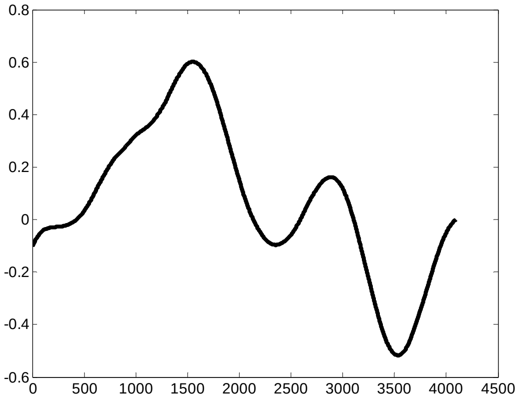


Fig.67 - Cross-corr. between price logarithm's Hölder func. and volatility's Hölder func., Nikkei.

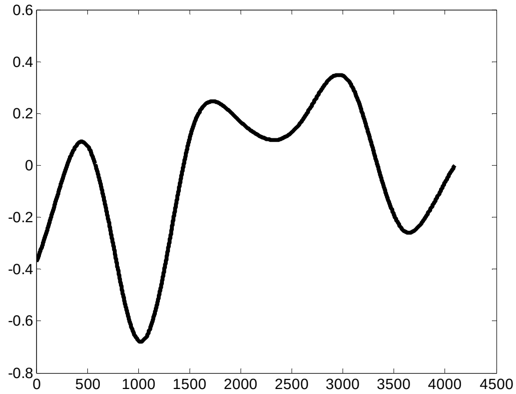


Fig.68 - Cross-corr. between price logarithm's Hölder func. and volatility's Hölder func., Hang Seng.