Stochastic Finance 2004
Autumn School & International Conference

Pathwise Identification of the Memory Function of a Multifractional Market Model

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Abstract. We extend and adapt a class of estimators of the parameter H of fractional Brownian motion in order to estimate the (time-dependent) memory function of a multifractional process. We provide: (a) the estimator's distribution when H belongs to the interval $(0, \frac{3}{4})$; (b) the confidence interval under the null hypothesis $H = \frac{1}{2}$; (c) a scaling law, independent on the value of H, discriminating between fractional and multifractional processes. Furthermore, assuming as a model for the price process the multifractional Brownian motion, empirical evidence is suggested to conciliate the inconclusive and apparently contradictory results achieved in estimating the intensity of dependence in financial time series. Finally, we discuss the consistency of the memory function estimated for the S&P500 with the market mechanism.

Keywords: (multi)fractional Brownian motion; LRD estimators; financial markets

1. Introduction. In recent years, many empirical evidence have been offered that the geometric brownian motion cannot describe the behavior of financial assets. There are two main reasons for this: the actual returns seem to show some form of dependence and the empirical distributions of the log-price variations are far from gaussianity. As a model for dependence the fractional brownian motion (fBm), introduced in a celebrated paper by Mandelbrot et al. (1968), is earning a growing interest. The fBm is characterized by a slowly decaying autocorrelation function depending on the parameter $H \in (0, 1]$, named *Hurst* (or *Hölder*) exponent, and admits the following moving average representation (see, e.g., Coeurjolly (2001))

$$\begin{aligned} B_{H,K}(t) &= K \cdot V_H^{1/2} \int_{\mathbb{R}} f_t(s) dB(s) \qquad (1) \\ \text{with} \quad f_t(s) &= \frac{1}{\Gamma(H+1/2)} \{ |t-s|^{H-1/2} \mathbb{I}_{]-\infty,t]}(s) - |s|^{H-1/2} \mathbb{I}_{]-\infty,0]}(s) \} \end{aligned}$$

where $V_H = \Gamma(2H+1)\sin(\pi H)$ is a normalizing factor such that $\mathbb{E}(B_{H,K}(1) - B_{H,K}(0))^2 = K^2 \in \mathbb{R}^+$, K is a scale factor (if $K = 1$ the motion is said standard), H is the Hurst exponent and $B(\cdot)$ stands for the ordinary brownian motion. More generally, it is well known that $\mathbb{E}(B_{H,K}(t) - B_{H,K}(s))^2 = K^2 |t-s|^{2H}$ and that the fBm's increments form a zero mean, stationary, self-similar[1] and gaussian sequence with autocovariance $\varrho(h) = \frac{K^2}{2}\{(h+1)^{2H} - 2h^{2H} + |h-1|^{2H}\}$, $h \geq 0$ being the lag. In particular, when $H \in (\frac{1}{2}, 1]$ the motion displays paths more and more persistent as H increases to 1; when $H = \frac{1}{2}$, it reduces to the ordinary brownian motion and, finally, when $H \in (0, \frac{1}{2})$ the sequences show an antipersistent (or mean-reverting) behavior, the more evident the more H decreases to 0. From a geometrical point of view, H determines the (constant) roughness of the sample paths of the fBm and is linked to the fractal dimension D of the graph by the simple relation $D = 2 - H$.

Since the intensity of the long range dependence (LRD) depends on both H and h, once fixed the lag, the autocorrelation only depends on the Hurst parameter. Hence, the most immediate generalisation of the fBm can be obtained by allowing H to vary over time. This extension − referred to as *multifractional brownian motion* (mBm) (see Péltier et al. (1995), Lé vy Véhel (1995), Ayache et al. (2000a)) − is receiving an increasing attention mainly in the field of signal processing. The stochastic integral becomes in this case

[1]The stochastic process $\{X(t),\, t \in T\}$ is said *self-similar* with parameter H if for any $h > 0$ $\{X(ht)\} \stackrel{d}{=} \{h^H X(t)\}$, where the equality holds for the finite-dimensional distributions of the process [see e.g. Samorodnitsky et al. (1994)].

$$M_{H(t),K(t)}(t) = K(t) \cdot V_{H(t)}^{1/2} \int_{\mathbb{R}} f_t(s) dB(s) \tag{2}$$
$$\text{with } f_t(s) = \frac{1}{\Gamma(H(t)+1/2)} \{ |t-s|^{H(t)-1/2} \mathbb{I}_{]-\infty,t]}(s) - |s|^{H(t)-1/2} \mathbb{I}_{]-\infty,0]}(s) \}$$

where $H : [0, \infty) \to (0, 1]$ is a Hölder function[2] and the other quantities are the "dynamic" counterparts of those defined above. If $K(t) = 1$ for all t the motion is called standard (or reduced).

Even if no longer stationary nor self-similar, compared to the fBm the mBm has the advantage to be very flexible since the function $H(t)$ can model phenomena whose sample paths display a time changing regularity.

Several techniques have been developed to estimate H but in financial applications inconclusive and apparently contradictory results have been reached.

Using the rescaled range analysis, Green et al. (1977) found significant LRD in many series of securities listed on the New York Stock Exchange. Peters (1991, 1994) confirms positive LRD for the Dow Jones and the S&P500; the same result is achieved by Cheung (1993) for some time series of exchange rates; for their part, Booth et al. (1982) find positive LRD in foreign exchange rates during the flexible regime and negative dependence in the fixed period. LRD in the daily stock returns is also uncovered by Lo et al. (1988) but, re-examining the estimation procedure, Lo (1991) himself rejects the LRD for the same data set. Using the modified R/S analysis introduced by Lo (1991), Jacobsen (1996) does not find evidence of LRD in some european and japanese stock indexes return series. Little evidence against the random walk is also offered by Pan et al. (1997) for some currency futures prices. The same result had been reached by Cha (1993) for some financial indicators referred to Korea and Sweden. Also Hiemstra et al. (1997), analyzing 1,952 common stocks, conclude that long memory is not a widespread characteristic, whereas Willinger et al. (1999) find LRD, but claim that the evidence is not conclusive since the values of H are very close to the threshold of independence ($H = 0.5$).

A very parsimonious explanation for this variety of contradictory results is the motivation of this paper and can be formulated in the light of model (2): dependence and independence could be both present in financial time series, depending on the local behavior of $H(t)$ in the time span ones looks at. Differently, assuming (and trying to estimate) a *constant* index H seems unreasonable, for it would mean asserting that investors always look at the same *past time window* to take their trading decisions through time. If so, in estimating $H(t)$ a different approach is needed which by definition cannot rely upon estimators with slow rates of convergence. This is why in this paper a "dynamical" approach is suggested to analyse the behavior of the function $H(t)$. We adapt the estimators of H introduced by Péltier et al. (1994) and provide empirical evidence that, under the assumption of *locally* log-normal distribution of the δ-dimensional returns, H does change over time.

The work is organized as follows. In Section 2 we illustrate the estimator introduced by Péltier and Lévy Véhel and properly modified; the distribution of the estimator as well as a 'parameter H-free' scaling law able to discriminate between H-constant and H-variable processes is deduced in Section 3. Section 4 is devoted to the empirical analysis of the U.S. stock index Standard & Poor's 500. Section 5 concludes.

2. Pathwise identification of $H(t)$. Assume $\{X_{i,n}\}_{i=1,\dots,n-1}$ to be a discretized version of a mBm that locally, i.e. within a window of proper lenght δ, behaves like a fBm of given Hölder exponent. With this assumption we can write

$$X_{j+q,n} - X_{j,n} \sim N\left(0, K^2 \left(\frac{q}{n-1}\right)^{2H(i)}\right) \tag{3}$$
$$\text{for } j = i-\delta, \dots, i-q; \; i = \delta+1, \dots, n; \; q = 1, .., \delta$$

[2]Given the two metric spaces (X, d_X) and (Y, d_Y), the function $f : X \to Y$ is called a Hölder function with exponent $\alpha > 0$ if, for each $x, y \in X$ such that $d_X(x, y) < 1$ there exists a constant k satisfying the condition
$$d_Y(f(x), f(y)) \le k d_X(x, y)^\alpha.$$

for some $K \in \mathbb{R}^+$ and $H(\cdot) \in (0,1]$. Given $\{X_{i,n}\}$ our aim is to estimate the function $H(i)$. In order to achieve this result we follow the guidelines of the work by Pèltier et al. (1994). We first recall that, if Y is normally distributed with mean 0 and variance σ^2, then

$$\mathbb{E}\left(|Y|^k\right) = \frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sigma^k \tag{4}$$

Define now the quantity

$$\mathbb{S}^k_{\delta,q,n}(i) = \frac{1}{(\delta - q + 1)} \sum_{j=i-\delta}^{i-q} |X_{j+q,n} - X_{j,n}|^k \qquad (i = \delta + 1, ..., n) \tag{5}$$

By (4), it is

$$\begin{aligned} \mathbb{E}^k_{\delta,q,n}(i) \quad &: = \mathbb{E}\left(\mathbb{S}^k_{\delta,q,n}(i)\right) = \mathbb{E}\left(\frac{1}{(\delta+1-q)}\sum_{j=i-\delta}^{i-q}|X_{j+q,n} - X_{j,n}|^k\right) = \\ &= \frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}K^k\left(\frac{q}{n-1}\right)^{kH(i)} \end{aligned}$$

from which we write

$$\frac{\mathbb{S}^k_{\delta,q,n}(i)}{\mathbb{E}^k_{\delta,q,n}(i)} = \frac{\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)}{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k\left(\frac{q}{n-1}\right)^{kH(i)}}$$

Since $\frac{\mathbb{S}^k_{\delta,q,n}(i)}{\mathbb{E}^k_{\delta,q,n}(i)} \xrightarrow{P} 1$ as $\delta \longrightarrow \infty$, we can write

$$\frac{\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)}{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k} \xrightarrow{P} \left(\frac{q}{n-1}\right)^{kH(i)}$$

from which it readily follows

$$\frac{\log\left(\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)/\left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k\right)\right)}{k\log\left(\frac{q}{n-1}\right)} \xrightarrow{P} H(i) \tag{6}$$

From (6) one has the following class of estimators of $H(i)$

$$H^k_{\delta,q,n}(i) = \frac{\log\left(\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)/\left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k\right)\right)}{k\log\left(\frac{q}{n-1}\right)}. \tag{7}$$

Remark 2.1. *In empirical applications K is generally unknown and unless the process is the standard one it has to be estimated. This problem has been solved by Coeurjolly (2001) in a different estimation context at the cost of reducing the rate of convergence of the estimator. Given (7), a wrong K causes a systematic bias of $H^k_{\delta,n}(i)$ consisting in a vertical shift of the estimated sequence which overestimates (underestimates) $H(i)$ if the true K is smaller (larger) than the assumed K. The bias can be easily seen by writing $H^k_{\delta,q,n}(i)$ as*

$$H^k_{\delta,q,n}(i) = \frac{\log\left(\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)/\left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)\right)\right)}{k\log\left(\frac{q}{n-1}\right)} - \frac{\log K}{\log\left(\frac{q}{n-1}\right)} \tag{8}$$

and observing that the last addend is increasing with K. Notice that $-$ given $n-$ the value of q does not affect the estimates only if $K = 1$ and that in continuous time, for any finite K, the bias reduces as $q \to 0$.

Remark 2.2. *Observe that*

$$\frac{\mathbb{S}^k_{\delta,q,n}(i)}{\mathbb{E}^k_{\delta,q,n}(i)} = \left(\frac{q}{n-1}\right)^{k\left(H^k_{\delta,q,n}(i)-H(i)\right)} \tag{9}$$

In fact, from (7) one has

$$\log\left(\frac{q}{n-1}\right)^{kH^k_{\delta,q,n}(i)} = \log\frac{\sqrt{\pi}\mathbb{S}^k_{\delta,q,n}(i)}{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k}$$

that is

$$\left(\frac{q}{n-1}\right)^{kH^k_{\delta,q,n}(i)} = \frac{\mathbb{S}^k_{\delta,q,n}(i)}{\mathbb{E}^k_{\delta,q,n}(i)\left(\frac{n-1}{q}\right)^{kH(i)}}$$

and finally relationship (9), which will be useful in the sequel.

3. Distribution of $H^k_{\delta,q,n}(i)$.

In this paragraph the distribution of the statistic $H^k_{\delta,q,n}(i)$ will be derived and to this we will use some known results (see e.g. Doukhan et al. (2003), Péltier et al. (1994)). The analysis will be restricted to the case $H(i) \in (0, \frac{3}{4})$, which covers the interval of financial relevance since all the estimates known in literature concern values of H smaller than 0.6. When $H(i) \in (\frac{3}{4}, 1)$ the problem can be faced by applying a theorem by Dobrushin and Major (1979) at the cost of some additional technicalities. In the following $(Y_n)_{n\in\mathbb{N}}$ will denote a stationary Gaussian sequence with $\mathbb{E}(Y_n) = 0$, $\mathbb{E}(Y_n^2) = 1$ and with autocorrelation function $c_0(n) = \mathbb{E}(Y_0 Y_n)$ satisfying the asymptotic relation $c_0(n) = n^{-\alpha}L(n)$, where $0 < \alpha < 1$ and $L(n)$ is a slowly varying at infinity function.

Hereunder four facts are recalled which provide the basis to deduce the distribution of $H^k_{\delta,q,n}(i)$.

Fact 3.1. *Let $\mathbb{H}(y)$ be a real function satisfying the following requirements:*

(a) $\mathbb{H}(y)$ does not vanish on a set of positive measure;

(b) $\int_{\mathbb{R}} \mathbb{H}(y)\exp\left(-\frac{y^2}{2}\right)dy = 0$;

(c) $\int_{\mathbb{R}} [\mathbb{H}(y)]^2\exp\left(-\frac{y^2}{2}\right)dy < \infty$.

With these assumptions, $\mathbb{H}(y)$ admits the following representation, known as Fourier-Hermite series expansion

$$\mathbb{H}(y) = \sum_{j=1}^{\infty} a_j H_j(y) \tag{10}$$

where $\sum_{j=1}^{\infty} a_j^2 j! < \infty$ and H_j is the j-th Hermite polynomial with leading coefficient 1.

Fact 3.2. (Theorem A, by Breuer and Major (1983)) *Let $Z_n = G(Y_n)$ be a non-linear function of Y_n such that $\mathbb{E}Z_n = 0$ and $\mathbb{E}Z_n^2 < \infty$. Denoted by $m = \min\{j \geq 1 : a_j \neq 0\}$ the Hermite rank of the function G and by $T_N = N^{-1/2}\sum_{n=1}^{N} Z_n$, if the condition $\sum_{n\in\mathbb{N}} |c_o(n)|^m < \infty$ holds, then $\lim_{N\to\infty}\mathbb{E}(T_N)^2 = \sigma^2 < \infty$ and $T_N \xrightarrow{d} \sigma T$ as $N \to \infty$, T being $N(0,1)$.*

Fact 3.3. (Péltier and Lévy Véhel (1994)) *The autocorrelation function of the increments of the $B_{H,K}(t)$ can be written as $c_0(n) = \frac{H(2H-1)}{n^{2(1-H)}} + o\left(\frac{1}{n^{2(1-H)}}\right)$. Setting $\alpha = 2(1-H)$ and*[3] *$L(n) \sim H(2H-1)$ and observing that $L(n)$ is a slowly varying at infinity function, one has $c_0(n) = n^{-\alpha}L(n)$. It follows that the condition $\sum_{n\in\mathbb{N}} |c_o(n)|^m < \infty$ of Theorem A becomes $\sum_{n\in\mathbb{N}} |n^{-m\alpha}L^m(n)| < \infty$. To ensure the convergence of the series one has $\frac{1}{2m(1-H)} < 1$, namely $H < \frac{2m-1}{2m}$, so Theorem A applies to the trajectories of the process (2) under the assumption given by (3).*

[3] We write $a_n \sim b_n$ if $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

Fact 3.4. *Let $(J_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ be two sequences of random variables such that $J_n = (1+o_P(1))K_n$ as $n \to \infty$. If $J_n \xrightarrow{d} J$ then $K_n \xrightarrow{d} J$.*

Equipped with Facts 3.1.-3.4., we can now calculate the distribution of the estimator $H^k_{\delta,q,n}(i)$ through the following three steps:

Step 1. First notice that, setting $Z^k_{\delta,q,n}(i) = \frac{\mathbb{S}^k_{\delta,q,n}(i)}{\mathbb{E}^k_{\delta,q,n}(i)} - 1$, we can write

$$Z^k_{\delta,q,n}(i) = (1+o_P(1))k(H(i) - H^k_{\delta,q,n}(i))\ln\left(\frac{n-1}{q}\right) \tag{11}$$

In fact a different way of writing (9) is

$$Z^k_{\delta,q,n}(i) = \left(\frac{n-1}{q}\right)^{k(H(i)-H^k_{\delta,q,n}(i))} - 1 = e^{k(H(i)-H^k_{\delta,q,n}(i))\ln\left(\frac{n-1}{q}\right)} - 1$$

and recalling that $Z^k_{\delta,q,n}(i) \xrightarrow{P} 0$ as $\delta \longrightarrow \infty$ by virtue of the Slutsky's theorem, one gets (11).

Step 2. For each integer $k \geq 1$ set

$$\mathbb{H}^k(y) = |y|^k - \mathbb{E}_k, \tag{12}$$

where $\mathbb{E}_k = \mathbb{E}(|Y|^k)$, $Y \sim N(0,1)$. Since:

(a') $\mathbb{H}^k(y)$ does not vanish on a set of positive measure;

(b')

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{H}(y)\exp\left(-\frac{y^2}{2}\right)dy &= \int_{\mathbb{R}}\left(|y|^k - \mathbb{E}_k\right)\exp\left(-\frac{y^2}{2}\right)dy = \\
&= \int_{\mathbb{R}} |y|^k \exp\left(-\frac{y^2}{2}\right)dy - \frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\int_{\mathbb{R}}\exp\left(-\frac{y^2}{2}\right)dy = \\
&= 2^{k/2}\sqrt{2}\Gamma\left(\frac{k+1}{2}\right) - 2^{k/2}\sqrt{2}\Gamma\left(\frac{k+1}{2}\right) = 0
\end{aligned}$$

(c')

$$\int_{\mathbb{R}} \mathbb{H}(y)^2\exp\left(-\frac{y^2}{2}\right)dy = \int_{\mathbb{R}}\left(|y|^k - \mathbb{E}_k\right)^2\exp\left(-\frac{y^2}{2}\right)dy =$$

$$\begin{aligned}
&= \int_{\mathbb{R}} |y|^{2k}\exp\left(-\frac{y^2}{2}\right)dy + \frac{2^{k+\frac{1}{2}}\Gamma\left(\frac{k+1}{2}\right)^2}{\sqrt{\pi}} - 2\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}}\int_{\mathbb{R}}|y|^k\exp\left(-\frac{y^2}{2}\right)dy = \\
&= 2^{k+\frac{1}{2}}\Gamma\left(\frac{2k+1}{2}\right) - \frac{2^{k+\frac{1}{2}}\Gamma\left(\frac{k+1}{2}\right)^2}{\sqrt{\pi}} = 2^{k+\frac{1}{2}}\left(\Gamma\left(\frac{2k+1}{2}\right) - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{k+1}{2}\right)^2\right) < \infty
\end{aligned}$$

the function (12) satisfies the conditions (a)-(c) of Fact 3.1.. Hence the series expansion (10) can be calculated; the Hermite polynomial $H_j(x)$ of degree j is defined by the relation (see e.g. Taqqu (1975))

$$\left(\frac{d}{dx}\right)^j e^{-x^2/2} = (-1)^j H_j(x)e^{-x^2/2}, \tag{13}$$

by which one gets $H_0(x) = 1$, $H_1(x) = x$. Then, using the classical recursive relation

$$H_{j+1}(x) = xH_j(x) - jH_{j-1}(x)$$

all the polynomials of higher order can be calculated.

It is well known that the sequence $\left(\frac{H_j(x)}{\sqrt{j!}}\right)_{j\geq 0}$ forms a complete orthonormal system[4] of the Hilbert space with respect to the normal density Φ, $L^2(\mathbb{R},\Phi)$ and therefore the coefficients a_j^k of the series (10) can be calculated applying the following classical formula:

$$a_j^k = \frac{1}{j!}\int_{\mathbb{R}} \mathbb{H}^k(x)H_j(x)\Phi(x)dx, \qquad j\geq 0 \tag{14}$$

Since the functions $(H_{2j+1})_{j\geq 0}$ are odd, for each $j\geq 0$, $a_{2j+1}^k = 0$. Hence, combining (13) and (14), one gets

$$|y|^k - \mathbb{E}_k = \frac{k}{2}\mathbb{E}_k H_2(y) + \frac{k(k-2)}{24}\mathbb{E}_k H_4(y) + \sum_{j=3}^{\infty} a_{2j}^k H_{2j}(y). \tag{15}$$

Notice that $\mathbb{H}^2(y) = |y|^2 - \mathbb{E}_2 = y^2 - \frac{2\Gamma(3/2)}{\Gamma(1/2)} = y^2 - 1 = H_2(y)$. Therefore, substituting in (15) one immediately gets

$$a_2^2 = 1, \text{ and for each } j\neq 2, a_j^2 = 0 \tag{16}$$

so that $m = 2$ in Theorem A. Hence Fact 3.3. justifies the next step.

Step 3. We now combine Theorem A with relation (12).

First observe that $\mathbb{E}^k_{\delta,q,n}(i) = \frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} K^k\left(\frac{q}{n-1}\right)^{kH(i)} = K^k\left(\frac{q}{n-1}\right)^{kH(i)}\mathbb{E}_k$, where $\mathbb{E}_k$ is the same as defined at the beginning of step 2. We can write

$$\begin{aligned}
Z^k_{\delta,q,n}(i) &= \frac{\mathbb{S}^k_{\delta,q,n}(i)}{K^k\left(\frac{q}{n-1}\right)^{kH(i)}\mathbb{E}_k} - 1 = \\
&= \frac{1}{\mathbb{E}_k}\left(\frac{\mathbb{S}^k_{\delta,q,n}(i)}{K^k\left(\frac{q}{n-1}\right)^{kH(i)}} - \mathbb{E}_k\right) = \\
&= \frac{1}{\mathbb{E}_k}\left(\frac{\mathbb{S}^k_{\delta,q,n}(i)}{K^k}\left(\frac{n-1}{q}\right)^{kH(i)} - \mathbb{E}_k\right) = \\
&= \frac{1}{\mathbb{E}_k}\left(\frac{\sum_{j=i-\delta}^{i-q}|X_{j+q,n}-X_{j,n}|^k}{K^k(\delta-q+1)}\left(\frac{n-1}{q}\right)^{kH(i)} - \mathbb{E}_k\right) = \\
&= \frac{1}{(\delta-q+1)\mathbb{E}_k}\sum_{j=i-\delta}^{i-q}\left(|X_{j+q,n}-X_{j,n}|^k\frac{1}{K^k}\left(\frac{n-1}{q}\right)^{kH(i)} - \mathbb{E}_k\right)
\end{aligned} \tag{17}$$

From $X_{j+q,n} - X_{j,n} \sim N\left(0, K^2\left(\frac{q}{n-1}\right)^{2H(i)}\right)$ it follows $Y_{j,q,n} := \frac{1}{K}\left(\frac{n-1}{q}\right)^{H(i)}(X_{j+q,n}-X_{j,n}) \sim N(0,1)$ and therefore the last line of (17) can be written as

$$Z^k_{\delta,q,n}(i) = \frac{1}{(\delta-q+1)\mathbb{E}_k}\sum_{j=i-\delta}^{i-q}\left(|Y_{j,q,n}|^k - \mathbb{E}_k\right) \tag{18}$$

Using (12) we have

$$Z^k_{\delta,q,n}(i)\sqrt{\delta-q+1} = \frac{1}{\mathbb{E}_k\sqrt{\delta-q+1}}\sum_{j=i-\delta}^{i-q}\mathbb{H}^k(Y_{j,q,n})$$

[4] We recall that a collection of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ on H is an orthonormal system on H if:
(*a*) $\langle f_n, f_m\rangle = 0$ if $n\neq m$, and
(*b*) $\langle f_n, f_n\rangle = \|f_n\| = 1$ $(n = 0,1,2,...)$.
As usual, $\langle\cdot,\cdot\rangle$ here denotes the inner product.

namely, setting $T^*_{\delta,q} = (\delta - q + 1)^{-1/2} \sum_{j=i-\delta}^{i-q} \mathbb{H}^k(Y_{j,q,n})$ and combining Theorem A and Step 1

$$(1 + o_P(1))k(H(i) - H^k_{\delta,q,n}(i)) \ln\left(\frac{n-1}{q}\right)\sqrt{\delta - q + 1} \sim \frac{1}{\mathbb{E}_k} T^*_{\delta,q}, \qquad T^*_{\delta,q} \xrightarrow{d} N(0, \sigma^2) \text{ as } \delta - q \to \infty$$

that is

$$(1 + o_P(1))k(H(i) - H^k_{\delta,q,n}(i)) \ln\left(\frac{n-1}{q}\right)\sqrt{\delta - q + 1} \sim \frac{\sqrt{\pi}}{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)} \sigma N(0, 1). \tag{19}$$

Finally, applying Fact 3.4. to (19) one immediately gets

$$k \ln\left(\frac{n-1}{q}\right)\sqrt{\delta - q + 1}(H(i) - H^k_{\delta,q,n}(i)) \sim N\left(0, \frac{\pi}{2^k\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}\sigma^2\right) \tag{20}$$

Remark 3.5. *From (20), setting* $\alpha = \frac{\pi\sigma^2}{k^2 \ln^2\left(\frac{n-1}{q}\right) 2^k \left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}$ *one has*

$$Var\left(H^k_{\delta,q,n}(i)\right) = \alpha(\delta - q + 1)^{-1}$$

and finally

$$\log Var\left(H^k_{\delta,q,n}(i)\right) = \log \alpha - \log(\delta - q + 1). \tag{21}$$

Relation (21) represents a useful by-product stating that, once fixed q, if $H(i) = H$ the estimator's variance is tied to the lenght δ of the window by a log linear relationship whose slope is expected to be -1, independently on the value of H. This condition can be used as a graphical test of the constancy of H with respect to time. The bias discussed in Remark 2.1. does not affect the slope for it only determines shifts of the whole sequence $H^k_{\delta,q,n}(i)$ which do not modify the estimator's variance. In order to check (21), 100 independent samples (with length $n = 4096$) of fBm have been generated for each parameter $H = 0.1$, 0.2, ..., 0.7 (we remind that the distribution has been deduced when $H(i) \in (0, 3/4)$). The simulations have been carried out using the circulant matrix method introduced by Wood and Chan (1994). The algorithm, known to be as one of the most stable, provides in a fast way an excellent approximation of fBm (see Bardet et al. (2002), Coeurjolly (2000), Jennane et al. (2001) for a detailed discussion of several simulating methods).

The software needed for the analysis has been developed in S-PLUS®6.1. For each sequence the values $H^k_{\delta,q,n}(i)$ have been calculated and, for each fixed $\delta = 20, 30, ..., 300$, the variances of $\left(H^k_{\delta,q,n}(i)\right)$ have been computed. Table 1 summarizes the results of the regression, which confirm relation (21) since all the slopes are very close to -1 regardless the value of H (notice that in all cases the match is practically perfect).

Table 1. Regression analysis (average values for 100 independent samples)

	Slope	Intercept	r^2
$H = 0.1$	-1.04247	-4.38007	0.99994
$H = 0.2$	-1.04403	-4.43523	0.99972
$H = 0.3$	-1.03889	-4.56483	0.99985
$H = 0.4$	-1.03782	-4.63961	0.99987
$H = 0.5$	-1.03852	-4.64366	0.99982
$H = 0.6$	-1.04278	-4.56705	0.99986
$H = 0.7$	-0.98998	-4.52484	0.99985

Remark 3.6. *Assume $H(i) = H = \frac{1}{2}$ and set for convenience $C = \frac{1}{(\delta-q+1)^2(\mathbb{E}_k)^2}$. From (18) we have*

$$\begin{aligned} Var(Z^k_{\delta,q,n}(i)) &= C \cdot Var\left(\sum_{j=i-\delta}^{i-q} |Y_{j,q,n}|^k - \mathbb{E}_k\right) = C \cdot Var\left(\sum_{j=i-\delta}^{i-q} |Y_{j,q,n}|^k\right) = \\ &= C \cdot \left[\mathbb{E}\left(\left(\sum_{j=i-\delta}^{i-q} |Y_{j,q,n}|^k\right)^2\right) - \left(\mathbb{E}\left(\sum_{j=i-\delta}^{i-q} |Y_{j,q,n}|^k\right)\right)^2\right] \end{aligned}$$

Due to the strong stationarity of the sequences within the window δ we can set for simplicity of notation

$$\begin{aligned} &Var(Z^k_{\delta,q,n}(i)) = Var(Z^k_{\delta,q,n}(\delta+1)) = \\ &\quad = C\left[\mathbb{E}\left(\sum_{k=1}^{\delta-q+1} |Y_{k,q,n}|^{2k}\right) + 2\mathbb{E}\left(\sum_{k=1}^{\delta-q}\sum_{h=1}^{\delta-q+1} |Y_{k,q,n}|^k\, |Y_{h,q,n}|^k\right) - \left((\delta-q+1)\,\mathbb{E}\left(|Y_{1,q,n}|^k\right)\right)^2\right] \end{aligned}$$

When $q = 1$ the sequences $|Y_{k,q,n}|$ and $|Y_{h,q,n}|$ do not overlap, hence – being by hypothesis $H = \frac{1}{2}$ – it follows $\mathbb{E}(|Y_{k,q,n}|^k\, |Y_{h,q,n}|^k) = \mathbb{E}\left(|Y_{k,q,n}|^k\right)^2$

$$\begin{aligned} Var(Z^k_{\delta,1,n}(i)) &= C \cdot \left[\delta\mathbb{E}\left(|Y_{1,q,n}|^{2k}\right) + 2\frac{\delta(\delta-1)}{2}\mathbb{E}\left(|Y_{1,1,n}|^k\right)^2 - \delta^2(\mathbb{E}\left(|Y_{1,1,n}|^k\right)^2\right] = \\ &= \frac{\pi}{\delta 2^k\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} \cdot \left[\mathbb{E}\left(|Y_{1,q,n}|^{2k}\right) - \mathbb{E}\left(|Y_{1,1,n}|^k\right)^2\right] = \\ &= \frac{\pi}{\delta 2^k\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} \cdot \left(\frac{2^k\Gamma\left(\frac{2k+1}{2}\right)}{\sqrt{\pi}} - \frac{2^k\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2}{\pi}\right) \end{aligned}$$

which equals, using (4)

$$Var(Z^k_{\delta,1,n}(i)) = \frac{\sqrt{\pi}}{\delta\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} \cdot \left(\Gamma\left(\frac{2k+1}{2}\right) - \frac{1}{\sqrt{\pi}}\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2\right)$$

Recall now that $Z^k_{\delta,q,n}(i) = (1 + o_P(1))k(H(i) - H^k_{\delta,q,n}(i))ln\left(\frac{n-1}{q}\right))$, hence

$$Var(H^k_{\delta,1,n}(i)) = \frac{\sqrt{\pi}}{\delta k^2 ln^2(n-1)\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2} \cdot \left(\Gamma\left(\frac{2k+1}{2}\right) - \frac{1}{\sqrt{\pi}}\left(\Gamma\left(\frac{k+1}{2}\right)\right)^2\right) \tag{22}$$

which provides the confidence interval under the null hypothesis $H(i) = \frac{1}{2}$ (and $q = 1$). Relation (22), which simply reduces to $Var(H^1_{\delta,1,n}(i)) = \frac{1}{\delta ln^2(n-1)}\left(\frac{\pi}{2} - 1\right)$ when $k = 1$, has been checked by simulating five hundred independent sequences of fBm with $H = \frac{1}{2}$ (one hundred realizations for each considered length: $n = 1024$, 2048, 4096, 8192, 16384). For each series the values $H^1_{\delta,1,n}(i)$ have been calculated with $\delta = 20, 30, ..., 300$ and then the actual frequencies have been determined for the probability levels $\alpha = 10\%$, 5%, 2.5% and, finally, 1%. Table 2 shows the results of the analysis, which indeed confirms relation (22).

Table 2. Nominal and actual frequencies (average values for 100 independent samples)

	10.0%					5.0%				
	1024	2048	4096	8192	16384	1024	2048	4096	8192	16384
20	10.3	10.0	10.5	10.4	10.4	5.4	5.3	5.4	5.5	5.4
30	10.1	9.9	10.2	10.1	10.2	5.2	5.1	5.3	5.2	5.2
40	9.9	9.7	10.1	10.1	10.2	5.2	4.9	5.2	5.2	5.1
50	9.8	9.6	10.0	10.0	10.1	5.1	4.8	5.1	5.1	5.1
60	9.7	9.6	10.0	10.0	10.0	5.0	4.8	5.0	5.1	5.1
70	9.6	9.7	10.1	10.0	10.0	5.0	4.8	5.1	5.1	5.1
80	9.4	9.7	10.1	10.0	10.0	4.9	4.8	5.1	5.1	5.0
90	9.5	9.6	10.1	10.0	9.9	4.7	4.7	5.1	5.0	5.0
100	9.6	9.6	10.2	9.9	9.9	4.9	4.7	5.2	5.0	5.0
120	9.8	9.5	10.3	9.8	9.8	4.9	4.7	5.3	4.9	5.0
140	10.0	9.4	10.3	9.8	9.9	5.1	4.7	5.4	4.9	4.9
160	10.0	9.5	10.3	9.8	9.9	4.8	4.7	5.4	4.9	5.0
180	9.9	9.5	10.4	9.9	9.9	4.9	4.6	5.5	4.9	4.9
200	9.9	9.5	10.4	9.9	9.8	4.9	4.5	5.5	5.0	4.9
250	9.8	9.1	10.4	9.8	9.6	5.2	4.5	5.6	5.0	4.8
300	10.2	9.0	10.2	9.8	9.5	5.5	4.3	5.3	4.9	4.7

	2.5%					1.0%				
	1024	2048	4096	8192	16384	1024	2048	4096	8192	16384
20	3.0	2.9	2.9	3.0	2.9	1.4	1.3	1.4	1.4	1.4
30	2.9	2.7	2.8	2.8	2.8	1.2	1.1	1.3	1.3	1.2
40	2.7	2.6	2.8	2.7	2.7	1.3	1.0	1.3	1.2	1.1
50	2.8	2.4	2.7	2.6	2.6	1.2	1.0	1.3	1.2	1.1
60	2.7	2.4	2.7	2.6	2.6	1.2	1.0	1.3	1.1	1.1
70	2.6	2.4	2.7	2.6	2.5	1.1	1.0	1.2	1.1	1.1
80	2.6	2.4	2.7	2.6	2.5	1.1	1.0	1.2	1.1	1.1
90	2.5	2.4	2.7	2.6	2.5	1.1	0.9	1.2	1.1	1.0
100	2.6	2.3	2.7	2.6	2.5	1.1	0.9	1.2	1.1	1.0
120	2.7	2.4	2.8	2.5	2.5	1.2	1.0	1.3	1.1	1.0
140	2.5	2.4	2.8	2.5	2.5	1.2	1.0	1.3	1.0	1.0
160	2.4	2.3	2.9	2.5	2.4	1.1	0.9	1.3	1.0	0.9
180	2.4	2.3	2.9	2.5	2.4	1.1	0.8	1.3	1.0	0.9
200	2.5	2.2	2.9	2.5	2.4	0.9	0.8	1.3	1.0	0.9
250	2.9	2.1	3.0	2.6	2.3	1.3	0.9	1.3	1.1	0.9
300	3.0	2.2	2.9	2.6	2.3	1.4	0.8	1.3	1.1	0.9

Remark 3.7. *Assuming $K(t) = K$ in (2) it is possible to use (22) in order to estimate K without reducing the rate of convergence of $H^k_{\delta,q,n}$. In fact, once fixed a sufficiently small radius ϵ, the set of random variables*

$$\mathbb{V}_q = \left\{ (X_{j+q,n} - X_{j,n}) : H^k_{\delta,q,n}(i) \in (H-\epsilon, H+\epsilon)\ ,\ j = i-\delta, ..., i-q; i = \delta+1, ..., n \right\}$$

will be normally distributed with variance $K^2 \left(\frac{q}{n-1}\right)^{2H}$. Therefore K can be estimated by using the following relationship

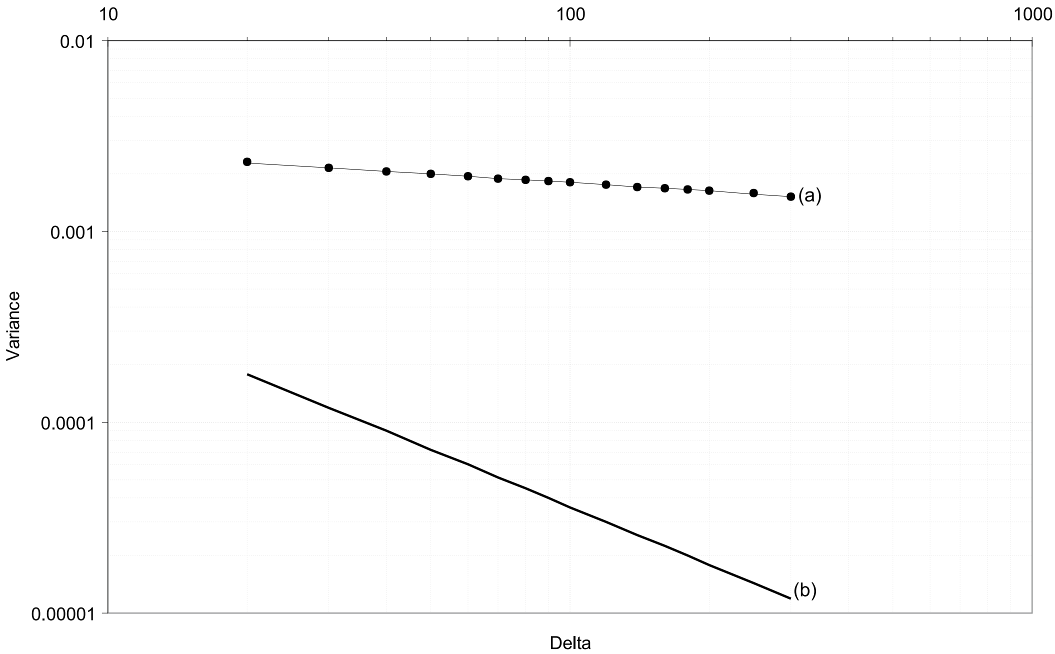
$$\ln \widetilde{\sigma}\left(\mathbb{V}_q\right) = \ln \widetilde{K} + \widetilde{H} \ln \left(\frac{q}{n-1}\right) \tag{23}$$

where $\widetilde{\sigma}$ is the sample standard deviation of the set $\mathbb{V}_q$. The confidence interval defined by (22) can be used to fix the band $\left(\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon\right)$ with respect to the central value $H = \frac{1}{2}$.

4. Empirical application. In this section we discuss an empirical application of the estimation technique above defined. The analysis will concern the closing prices of the U.S. stock index Standard & Poor's (*S&P500*) in the period 20/10/1982 - 27/05/2004 ($n = 5454$ values).

Once set the parameters $q = 1$ and $k = 1$, the constancy over time of H has been tested using relation (21) and considering for δ the values 20, 30, 40, 50, 60, 70, 80, 90, 100, 120, 140, 160, 180, 200, 250 and 300. The slope of the regression line (Figure 1) is about -0.1493 against -1 expected for a H-constant sequence (the value $r^2 = 0.9983$ indicates the goodness of the fit). So, as δ increases the variance of $H^1_{\delta,1,n}(i)$ scales much more fastly than what we would expect if H were constant.

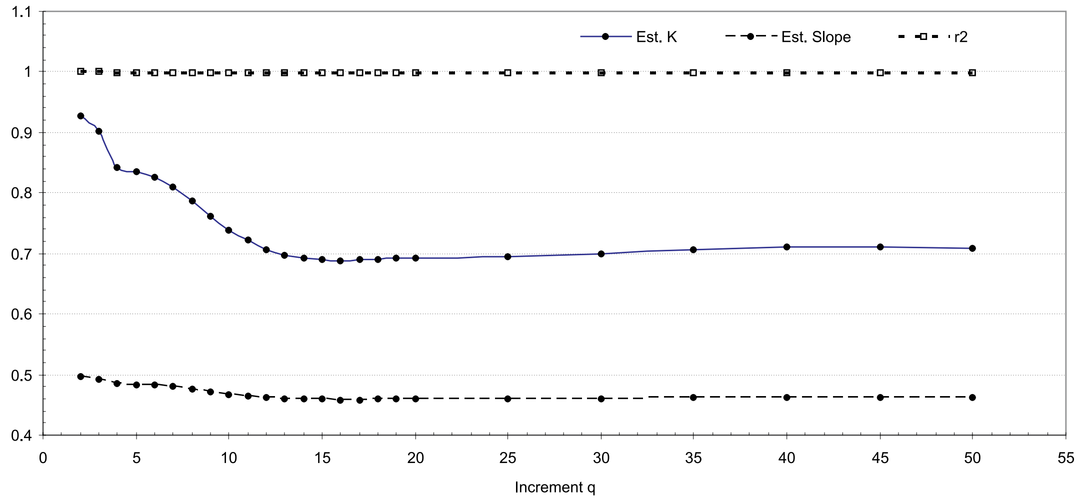
Figure 1. Test of constancy of H. Estimated (a) and theoretical (b) lines for the S&P500.



In order to estimate $H(t)$ we have fixed for the estimator $H^1_{\delta,1,n}$ the parameter $\delta = 30$. Here we will not dwell upon the criterion for choosing the 'optimal' length δ because the trade-off problem which generates between the assumption of local log-normality and the estimator's variance (by increasing δ the latter reduces but the former deteriorates if H changes over time) would deserve an appropriate treatment. Anyway, the choice $\delta = 30$ comes as a rule of thumb obtained by applying sequentially the Shapiro-Wilk normality test to the dataset.

Once fixed δ, we have first estimated $H(t)$ assuming as scale factor $K = 1$. Obtained the sequence $\left\{H^1_{30,1,n}(i)\right\}$ the subset $\mathbb{V}_q$ of log variations corresponding to the values $\frac{1}{2} - 1.96\sqrt{\frac{\pi/2-1}{\delta ln^2(n-1)}} = 0.4686 < H^1_{30,1,n}(i) < 0.5314 = \frac{1}{2} + 1.96\sqrt{\frac{\pi/2-1}{\delta ln^2(n-1)}}$ has been considered. Due to Remark 3.7., if K truly equals 1, $\mathbb{V}_q$ is characterized by the value $H = \frac{1}{2}$. Using relation (23) with $q = 1, ..., 50$, we have found $\widetilde{H} = 0.4628$ and $\widetilde{K} = 0.7096$ (with a value $r^2 = 0.9994$ indicating a practically perfect fit). Figure 2 shows how the parameters of the regression line stabilize as q increases.

Figure 2. Regression analysis for different values of the upper increment q.



The sequence $\left\{H^1_{30,1,n}(i)\right\}$ has been therefore corrected by summing up the value $\frac{\log 0.7096}{\log(5453)} = -0.0399$ deduced by relation (8). Figure 3 and Table 3 summarize the results and the main distributional features of the estimated sequence.

Figure 3. S&P500. Estimated memory function (dotted line $K = 1$, continuous line $K = 0.7096$). Flat lines indicate the confidence interval for a p-level equals to 5%.

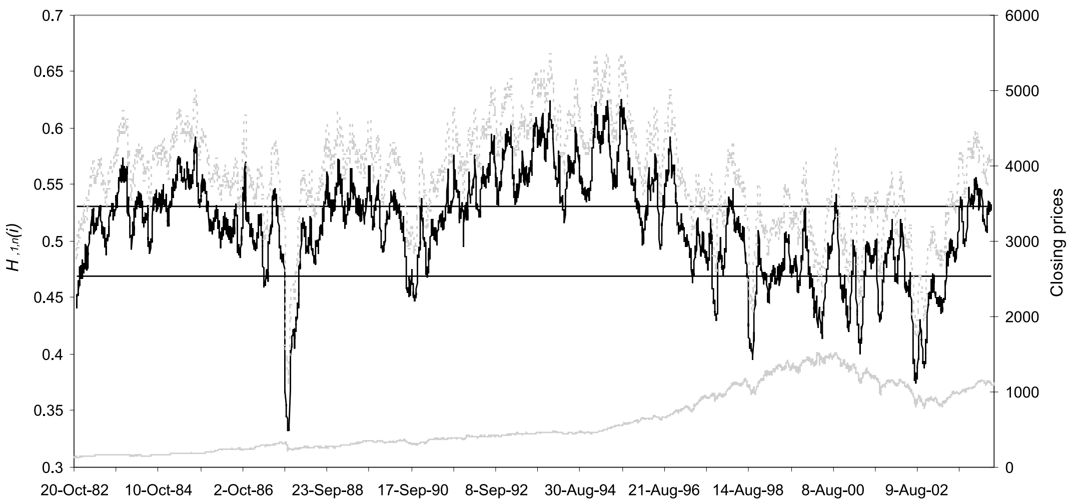


Table 3. Main distributional features of $H^1_{30,1,5454}$

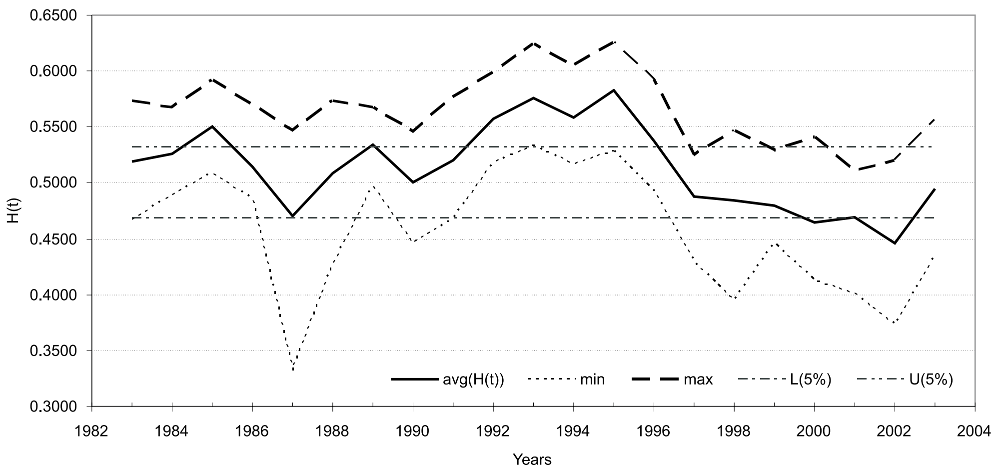
	Mean	St.Dev.	Min	Max	Skewness	Kurtosis
Values	0.5135	0.0464	0.3320	0.6257	-0.4921	0.6297

As shown by the values of skewness and kurtosis in Table 3, the distribution of $H^1_{30,1,5454}$ is far from normality: (the value of two-sided Kolmogorov goodness of fit statistic is 0.0406 significant for any p-value).

4.1. Discussion of the results. Even if the mean value of the sequence $\left\{H^1_{30,1,n}(i)\right\}$ is very close to 0.5, the estimator ranges from significant antipersistence (min $\left(H^1_{30,1,n}\right) = 0.33$) to significant persistence (max $\left(H^1_{30,1,n}\right) = 0.62$).

An evident feature is that quite flat and relatively long periods alternate to sudden and large downward movements of H, followed in their turn by more or less gradual upward movements which often take H back to the previous level. The weight of the downward variations, which tend to occur when H is above 0.5, reflects in the left skewness of the estimator's distribution and lends itself to a suggestive financial interpretation. Measuring the local dependence, H is an indicator of the confidence that markets nourish in the past (this is why it is unlikey to think of H as a constant). Since H is influenced by the new information, when this brings uncertainty in the market the confidence in the past vanishes and H itself declines toward 0.5 or toward even lower values. In these cases the mean-reversion effect can be explained by the short buy-and-sell activity typical of the periods in which uncertainty predominates. The more shocking the new information is, the faster and larger these movements are. Clear examples of this mechanism are the shocks of October 19th, 1987 (-20.5%, -57.9 points down), October 27th, 1997 (-6.9%, -64.7 points down), August 31st, 1998 ($-6,8\%$, $-69,9$ points down). When the market has digested the impact of the new information H begins raising again to restore the previous level.

Figure 4. S&P500. Average $H(t)$ per year, minimum, maximum and confidence intervals.



The dynamical analysis of the function $H(t)$ offers a very parsimonious explanation for the variety of apparently contradictory results achieved in literature about the estimation of the LRD. To realize this it should be considered that the traditional estimators of the parameter H are mostly characterized by very low rates of convergence which compel to use large samples to get only one estimate. To give an idea of the consequence of this, we have aggregated the data by calculating the yearly mean value of the estimated memory function. The results are summarized in Figure 4 and Table 4 (which lists the averaged values with other statistical features): finding or not LRD depends on the sample one looks at. If the period $1992 - 1995$ is taken into account, significant LRD can be found ($H = 0.569$). The conclusion is just the opposite if the periods $1988 - 1991$ ($H = 0.516$) or $1996 - 2001$ ($H = 0.487$) are considered. Even more ambiguous results can be obtained by analysing non homogeneous periods ($H = 0.538$ in $1993 - 1998$). The rough resolution of the traditional estimators of the LRD parameter could therefore provide a good explanation able to conciliate the controversy about the intensity of the long-term memory in financial markets.

Table 4. S&P500: Main statistical features of $H^1_{30,1,n}$ (yearly average)

Year	Avg(H)	St.Dev.	Min(H)	Max(H)	Skewness	Kurtosis
1983	0.5192	0.0240	0.4673	0.5740	0.1650	−0.3825
1984	0.5264	0.0155	0.4885	0.5672	−0.3864	−0.1869
1985	0.5500	0.0166	0.5089	0.5925	−0.2763	0.0625
1986	0.5143	0.0139	0.4852	0.5706	1.0356	2.2443
1987	0.4700	0.0598	0.3320	0.5468	−0.9953	−0.0716
1988	0.5084	0.0291	0.4258	0.5734	−0.5198	0.4759
1989	0.5346	0.0138	0.4975	0.5672	0.1179	−0.3167
1990	0.5009	0.0298	0.4466	0.5455	−0.4041	−1.1785
1991	0.5196	0.0213	0.4678	0.5771	−0.1135	0.0805
1992	0.5574	0.0203	0.5181	0.5995	0.0890	−0.8063
1993	0.5762	0.0222	0.5325	0.6249	−0.0698	−1.0802
Year	Avg(H)	St.Dev.	Min(H)	Max(H)	Skewness	Kurtosis
1994	0.5589	0.0195	0.5162	0.6044	0.4199	−0.4706
1995	0.5829	0.0251	0.5285	0.6257	−0.1880	−1.1969
1996	0.5379	0.0233	0.4928	0.5925	0.0453	−0.8055
1997	0.4882	0.0212	0.4291	0.5245	−1.1120	0.8726
1998	0.4843	0.0393	0.3946	0.5472	−0.7210	−0.3317
1999	0.4791	0.0169	0.4455	0.5294	0.6415	−0.0193
2000	0.4648	0.0311	0.4130	0.5416	0.6341	−0.4639
2001	0.4686	0.0287	0.4004	0.5106	−0.6718	−0.8300
2002	0.4465	0.0397	0.3735	0.5203	−0.1893	−1.0287
2003	0.4947	0.0346	0.4361	0.5567	−0.0941	−1.1882

5. Conclusion. In this work we have suggested a dynamical approach for the estimation of the LRD parameter of a (multi)fractional process. The estimator's distribution for $H \in (0, 3/4)$ has been deduced as well as a scaling law which can be used as a test of constancy of the parameter H along the process' trajectories. The empirical analysis performed on the stock index S&P500 shows that, under the assumption of lognormality of the δ-dimensional variations, H *changes* over time, swinging from usually short antipersistent spells to generally longer persistent periods. This indicates that the hypothesis of multifractality for the price process deserves further investigations, both empirical and theoretical. Concerning the former, as far as we know, there are still few contributions and empirical evidence of the multifractal nature of the markets hasn't yet been provided; for the latter, the empirical analyses suggesting that H is time-dependent mostly use adaptations of asymptotic methods with rates of convergence not fully satisfactory. On the contrary, the main advantage of the approach used in this work is its efficiency: the good rate of convergence makes the estimation of the sequence $\{H(t)\}$ reliable also for short spans of time. This allows to have a timely information about the local dynamics of H. Toward this end, significant improvements could come by defining in a proper way the neighbourhood δ or, more likely, allowing δ itself to change over time in order to fit the data.

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