Finite Elements for Wasserstein $\mathbb{W}_p$ gradient flows

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FV(E)OT Workshop

Gradient flows in Hilbert spaces

- H: Hilbert space
- $X \in C^1(\mathbb{R}_+; H)$
- $E : H \to \mathbb{R}$: energy functional

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\nabla_H E(X) \qquad (+IC) \tag{1}$$

Energy evolution along an arbitrary curve $Y \in C^1(\mathbb{R}_+; H)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E(Y) &= \nabla_H E(Y) \cdot \frac{\mathrm{d}Y}{\mathrm{d}t} \\
&\overset{\text{C.S.}}{\geq} - \|\nabla_H E(Y)\|_H \left\| \frac{\mathrm{d}Y}{\mathrm{d}t} \right\|_H \\
&\overset{\text{Young}}{\geq} - \frac{1}{2} \|\nabla_H E(Y)\|_H^2 - \frac{1}{2} \left\| \frac{\mathrm{d}Y}{\mathrm{d}t} \right\|_H^2
\end{aligned}$$

Equality holds iff (1) holds true

Examples

- $H = L^2(\Omega)$, $E = \int_\Omega \left(\frac{\alpha}{2}|\nabla u|^2 + F(u)\right)$

$$\partial_t u + \mu = 0, \qquad \mu = -\alpha\Delta u + F'(u)$$

- $H = H^{-1}(\Omega)$, $E = \int_\Omega \left(\frac{\alpha}{2}|\nabla u|^2 + F(u)\right)$

$$\partial_t - \Delta\mu = 0, \qquad \mu = -\alpha\Delta u + F'(u)$$

First step beyond Hilbert spaces

Energy evolution along an arbitrary curve

$$\frac{\mathrm{d}}{\mathrm{d}t}E(Y) = DE(Y) \cdot \frac{\mathrm{d}Y}{\mathrm{d}t}$$
$$\geq -\Psi\left(\frac{\mathrm{d}Y}{\mathrm{d}t}\right) - \Psi^*\left(-DE(Y)\right)$$

Equality case

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \in \partial\Psi^*\left(-DE(Y)\right)$$

$\rightsquigarrow$ Nonlinear monotone relation between the forces and the fluxes

Examples

- $\Psi(z) = \frac{1}{p}\|z\|^p_{L^p(\Omega)}$, $\Psi^*(w) = \frac{1}{q}\|w\|^q_{L^q(\Omega)}$, $E = \int_\Omega \left(\frac{\alpha}{2}|\nabla u|^2 + F(u)\right)$

$$\partial_t u + |\mu|^{q-2}\mu = 0, \qquad \mu = -\alpha\Delta u + F'(u)$$

- $\Psi(z) = \frac{1}{p}\|z\|^p_{W^{-1,p}(\Omega)}$, $\Psi^*(w) = \frac{1}{q}\|w\|^q_{W_0^{1,q}(\Omega)}$, $E = \int_\Omega \left(\frac{\alpha}{2}|\nabla u|^2 + F(u)\right)$

$$\partial_t u - \nabla\cdot\left(|\nabla\mu|^{q-2}\nabla\mu\right) = 0, \qquad \mu = -\alpha\Delta u + F'(u)$$

Wasserstein $\mathbb{W}_p$ gradient flows: governing equations

[Agueh (2005), Amborsio, Gigli & Savaré (2005)]

- Conservation in $\Omega \subset \mathbb{R}^d$ (convex, polyhedral and bounded)

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \quad \text{in } \mathbb{R}_{>0} \times \Omega,$$
$$\rho \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \mathbb{R}_{>0} \times \partial\Omega$$

- Expression for the velocity[1]

$$\boldsymbol{v} = |\boldsymbol{u}|^{q-2}\boldsymbol{u} \quad \text{with} \quad \boldsymbol{u} = -\boldsymbol{\nabla}[\eta'(\rho) + \Psi] \quad \text{in } \mathbb{R}_{>0} \times \Omega$$

- Initial profile $\rho^0 \geq 0$ with finite energy

$$\int_\Omega \rho^0 = 1, \qquad \mathcal{E}(\rho^0) < +\infty,$$

with $\mathcal{E}(\rho) = \int_\Omega [\eta(\rho) + \rho\,\Psi]$.

[1] p and q are conjugate, i.e. $1/p + 1/q = 1$

Pressure, metric slope and velocity in $\mathbb{W}_p$

[Ambrosio, Gigli & Savaré (2005)]

- **Pressure function** $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ increasing

$$\phi(s) = s\eta'(s) - \eta(s) + \eta(0) = \int_0^s r\eta''(r)\mathrm{d}r$$

- **Metric slope:** given $\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega)$ with $\mathcal{E}(\rho) < +\infty$ and $\phi(\rho) \in W^{1,1}(\Omega)$

$$|\partial\mathcal{E}(\rho)|^q = \int_\Omega \rho|\boldsymbol{u}|^q \quad \text{with } \boldsymbol{u} = -\frac{\boldsymbol{\nabla}\phi(\rho)}{\rho} - \boldsymbol{\nabla}\Psi \text{ on } \{\rho > 0\}$$

- **Metric velocity:** if $t \mapsto \rho(t)$ solves

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{v}) = 0 \quad + \text{no-flux BC}$$

then

$$|\rho'|^p \leq \int_\Omega \rho|\boldsymbol{v}|^p$$

Solution concepts

Weak solution

$\rho : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}_{\geq 0}$ is a **weak solution** if

- $\rho(t, \cdot)$ is a probability density at each $t \geq 0$;
- $\phi(\rho) \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, W^{1,1}(\Omega))$;
- the time-dependent vector field $\boldsymbol{u} : \mathbb{R}_{>0} \times \Omega \to \mathbb{R}^d$ defined by

$$\boldsymbol{u} = -\frac{\nabla \phi(\rho)}{\rho} - \nabla \Psi \quad \text{on } \{\rho > 0\}, \qquad \boldsymbol{u} \equiv 0 \quad \text{on } \{\rho = 0\}$$

  satisfies $\rho |\boldsymbol{u}|^q \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \overline{\Omega})$;
- the nonlinear continuity equation holds: $\forall \varphi \in C_c^\infty(\mathbb{R}_{\geq 0} \times \overline{\Omega})$,

$$\int_0^\infty \int_\Omega \left(\rho\, \partial_t \varphi + \rho |\boldsymbol{u}|^{q-2} \boldsymbol{u} \cdot \nabla \varphi\right) \mathrm{d}x \mathrm{d}t + \int_\Omega \rho^0 \varphi(0, x) \mathrm{d}x = 0$$

Solution concepts

p-gradient flow

Assumptions

(A) $\eta \in C^2(\mathbb{R}_{>0}) \cap C(\mathbb{R}_{\geq 0})$ is strictly convex and superlinear at $+\infty$

(B) $\Psi \in W^{1,\infty}(\Omega; \mathbb{R}_{\geq 0})$ is semi-convex if $p \geq 2$ and convex if $p < 2$

(C) η satisfies McCann's and the doubling conditions

Then $|\partial\mathcal{E}(\rho)|$ is a strong upper gradient, cf. [Ambrosio, Gigli & Savaré (2005)]

$\rho : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}_{\geq 0}$ is a ***p*-gradient flow solution** if

- ρ is a weak solution
- $t \mapsto \mathcal{E}(\rho(t))$ is absolutely continuous and

$$\mathcal{E}(\rho(t_*)) + \frac{1}{q}\int_0^{t_*} |\partial\mathcal{E}(\rho)|^q + \frac{1}{p}\int_0^{t_*} |\rho'|^p \leq \mathcal{E}(\rho^0), \qquad \forall t_* \geq 0 \qquad \text{(EDI)}$$

Remark: The Energy Dissipation Inequality (EDI) is in fact an equality (EDE)

Recipe of the numerical approximation

- **Full gradient approximation** required for $p, q \neq 2$

$$\boldsymbol{v} = |\boldsymbol{u}|^{q-2}\boldsymbol{u} \quad \text{with} \quad \boldsymbol{u} = -\boldsymbol{\nabla}[\eta'(\rho) + \Psi] \quad \text{in } \mathbb{R}_{>0} \times \Omega$$

⚠ Strategies based on simple TPFA[2] finite volumes merely approximate $\boldsymbol{u} \cdot \boldsymbol{n}$

$\rightsquigarrow$ Conformal (Lagrange) $\mathbb{P}_1$ **finite elements** space V_h

[2] [Hafiene, Fadili, Chesneau & Moataz (2020)]

[3] see e.g. [Chatzipantelidis, Horváth & Thomée (2015)]

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- **Non-monotone** numerical method:

⚠ Possible undershoots on the approximate density ρ_h mitigated by "**Lagrange multipliers**" for the positivity

$$\mu_h(a) \in \partial\eta(\rho_h(a)) = \begin{cases} \{\eta'(\rho_h(a))\} & \text{if } \rho_h(a) > 0, \\ (-\infty, \eta'(0)] & \text{if } \rho_h(a) = 0 \in \mathrm{Dom}(\partial\eta), \end{cases} \qquad a \in \mathcal{V}_h$$

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- **Mass lumping**[3] provides piecewise constant reconstruction $\overline{\rho}_h, \overline{\mu}_h$ fulfilling

$$\overline{\mu}_h \in \partial\eta(\overline{\rho}_h) \quad \text{for a.e. } (t, x) \in \mathbb{R}_{>0} \times \Omega.$$

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Simplicial mesh and reconstructions

- **Piecewise linear reconstruction**

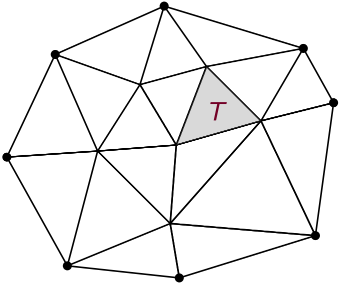
$$V_h = \{f_h \in C(\overline{\Omega}) \mid f_h \in \mathbb{P}_1(T),\ T \in \mathcal{T}_h\}$$

- **Piecewise constant reconstruction**

$$\widetilde{V}_h = \{\widetilde{f}_h \in L^\infty(\Omega) \mid \widetilde{f}_h \in \mathbb{P}_0(T),\ T \in \mathcal{T}_h\}$$

Given $\rho_h \in V_h$, one defines $\widetilde{\rho}_h \in \widetilde{V}_h$ by

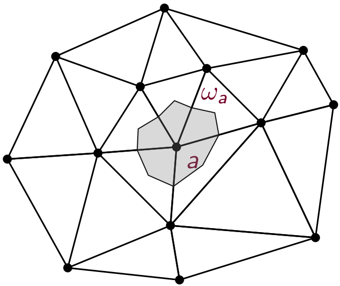
$$\widetilde{\rho}_h(x) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_T} \rho_h(a), \quad x \in T$$



Simplicial mesh cell $T \in \mathcal{T}_h$.

Donald mesh and mass lumping

- **Donald mesh**: To each $a \in \mathcal{V}_h$, we associate $\omega_a \subset \Omega$ by joining the centers of gravity of the simplices and those of the edges (d=2)



Donald mesh cell ω_a for $a \in \mathcal{V}_h$.

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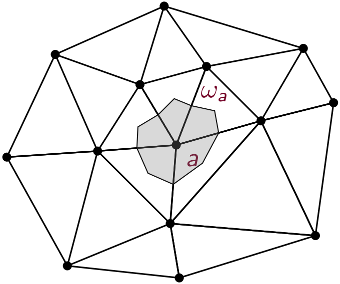
- **Mass lumped reconstruction**

$$X_h = \{\overline{f}_h \in L^\infty \mid \overline{f_h}_{|_{\omega_a}} \text{ is constant}, \ a \in \mathcal{V}_h\}$$

Given $\rho_h \in V_h$, one defines $\overline{\rho}_h$ by

$$\rho_h(a) = \overline{\rho}_h(a), \qquad a \in \mathcal{V}_h$$



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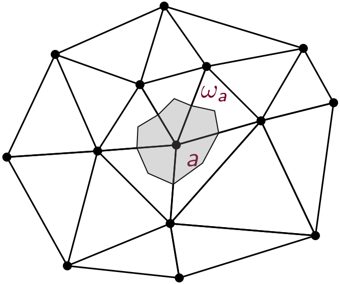

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$$\int_\Omega \rho_h \widetilde{\varphi}_h = \int_\Omega \overline{\rho}_h \widetilde{\varphi}_h = \int_\Omega \widetilde{\rho}_h \widetilde{\varphi}_h$$

For $f : \mathbb{R} \to \mathbb{R}$ continuous and $\rho_h \in V_h$,

$$f(\overline{\rho}_h) = \overline{f(\rho_h)} \in X_h$$

The numerical scheme

- **Data discretization**

$$\rho_h^0(a) = \frac{1}{|\omega_a|}\int_{\omega_a} \rho^0, \qquad \Psi_h(a) = \frac{1}{|\omega_a|}\int_{\omega_a} \Psi, \qquad \forall a \in \mathcal{V}_h$$

- **March in time**
  - Discrete conservation law

$$\int_\Omega (\overline{\rho}_h^n - \overline{\rho}_h^{n-1})\,\overline{\varphi}_h - \tau \int_\Omega \rho_h^n\,|\boldsymbol{u}_h^n|^{q-2}\boldsymbol{u}_h^n \cdot \boldsymbol{\nabla}\varphi_h = 0, \qquad \forall \varphi_h \in V_h,\ n \geq 1.$$

  - Force / velocity relation

$$\boldsymbol{u}_h^n = -\boldsymbol{\nabla}(\mu_h^n + \Psi_h) \in (\widetilde{V}_h)^d.$$

  - Chemical potential with positivity constraint

$$\mu_h^n(a) \in \partial\eta(\rho_h^n(a)), \qquad \forall a \in \mathcal{V}_h.$$

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$$\mu_h^n(a) \in \partial\eta(\rho_h^n(a)), \qquad \forall a \in \mathcal{V}_h.$$

Remark: The mobility $\rho_h^n \in V_h$ can be equivalently replaced by $\widetilde{\rho}_h^n \in \widetilde{V}_h$

Elementary a priori estimates

A priori estimates

- Global mass conservation ($\varphi_h \equiv 1$)

$$\int_\Omega \rho_h^n = \int_\Omega \overline{\rho}_h^n = \int_\Omega \overline{\rho}_h^{n-1} = \int_\Omega \rho^0 = 1$$

- Nonnegativity (positivity if $\eta'(0) = -\infty$)

$$\rho_h^n \in \mathrm{Dom}(\partial\eta) \subset \mathbb{R}_{\geq 0}$$

- Energy dissipation ($\varphi_h = \mu_h^n + \Psi_h$)

$$\mathcal{E}_h(\overline{\rho}_h^n) + \tau \int_\Omega \rho_h^n |\boldsymbol{v}_h^n|^p \leq \mathcal{E}_h(\overline{\rho}_h^{n-1}), \quad n \geq 1 \qquad (\star)$$

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Remark: Estimate $(\star)$ is stronger than the usual one provided by JKO

$$\int_\Omega \rho_h^n |\boldsymbol{v}_h^n|^p = \frac{1}{p}\int_\Omega \rho_h^n |\boldsymbol{v}_h^n|^p + \frac{1}{q}\int_\Omega \rho_h^n |\boldsymbol{u}_h^n|^q$$

Uniform positivity for singular energies

(A1) $\eta \in C^2(\mathbb{R}_{>0}) \cap C(\mathbb{R}_{\geq 0})$ is strictly convex and superlinear at $+\infty$ with

$$\lim_{s \searrow 0} \eta'(s) = -\infty$$

Uniform positivity

Under (A1), there exists $\epsilon_h > 0$ not depending on ρ^0 such that

$$\rho_h^n \geq \epsilon_h \quad \text{for all } n \geq 1$$

Sketch of the proof:

- As $\rho_h^n \in \mathcal{P}(\Omega)$, there exists $a_0 \in \mathcal{V}_h$ s.t. $\rho_h^n(a_0) \geq |\Omega|^{-1}$.
- Let $T_0 \in \mathcal{T}_h$ s.t. $a_0 \in \mathcal{V}_{T_0}$, then $\widetilde{\rho}_h^n \geq \frac{1}{(d+1)|\Omega|}$ on T_0
- By the control of the energy dissipation on T_0

$$|\mu_h^n(a_1) - \mu_h^n(a_0)|^q \leq h^q |\nabla \mu_h^n|^q \lesssim q\, h^q (\widetilde{\rho}_h^n |\boldsymbol{u}_h^n|^q + C_\Psi) \lesssim 1$$

- Thanks to (A1), $\mu_h^n(a_1) = \eta'(\rho_h(a_1)) \rightsquigarrow$ bound on $\rho_h(a_1)$
- Induction + finiteness of the graph corresponding to the mesh

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Existence of a discrete solution #1

Under (A1), there exists (at least) one solution ρ_h^n to the scheme corresponding to the previous iterate $\rho_h^{n-1} \in V_h \cap \mathcal{P}(\Omega)$.

Sketch of the proof:

- Well posed convex problem for frozen positive mobility $\max(\alpha, \widetilde{\rho}_h^*)$
- Fixed point argument: existence of a solution with mobility $\max(\alpha, \widetilde{\rho}_h^n)$
- Choose $\alpha \leq \epsilon_h$

Nonsingular energies

(A2) $\eta \in C^2(\mathbb{R}_{>0}) \cap C^1(\mathbb{R}_{\geq 0})$ is strictly convex and superlinear at $+\infty$

$$\eta'(0) > -\infty$$

Approximation by singular energies

- Solution $\rho^n_{h,\epsilon}$ corresponding to the approximate entropy

$$\eta_\epsilon(s) = \eta(s) + \beta_\epsilon(s) \quad \text{with} \quad \beta_\epsilon(s) = s \log\left(\frac{s}{\epsilon}\right) - s + \epsilon$$

- Boundedness + finite dimension

$$\rho^n_{h,\epsilon} \underset{\epsilon \to 0}{\longrightarrow} \rho^n_h \geq 0 \quad \text{pointwise}$$

- ρ^n_h is a solution to the scheme for the entropy η

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- ρ^n_h is a solution to the scheme for the entropy η

Existence of a discrete solution #2

Under (A2), there exists (at least) one solution ρ^n_h to the scheme corresponding to the previous iterate $\rho^{n-1}_h \in V_h \cap \mathcal{P}(\Omega)$.

Control on the discrete pressure

Pressure function

$$\phi(s) = s\eta'(s) - \eta(s) + \eta(0) = \int_0^s r\eta''(r)\mathrm{d}r \geq 0, \qquad s \geq 0.$$

Approximate pressure

$$\phi_h^n(a) = \phi(\rho_h^n(a)), \qquad a \in \mathcal{V}_h.$$

Control on the approximate pressure

There exists C depending only on the dimension of the ambiant space d and on the (Ciarlet's) regularity of the mesh $\mathcal{T}_h$ such that

$$|\boldsymbol{\nabla}\phi_h^n| \leq C\widetilde{\rho}_h^n|\boldsymbol{\nabla}\mu_h^n|.$$

Moreover,

$$\int_\Omega |\phi_h^n| \lesssim \left(1 + \int_\Omega |\boldsymbol{\nabla}\phi_h^n|\right)$$

Space-time approximations

From the sequence $(\rho_h^n)_{n\geq 0}$, we build piecewise constant in time and piecewise linear/constant in space approximations

$$\begin{aligned}
\overline{\rho}_{h\tau}(t,x) &= \overline{\rho}_h^n(x) \in X_h \\
\widetilde{\rho}_{h\tau}(t,x) &= \widetilde{\rho}_h^n(x) \in \widetilde{V}_h \\
\mu_{h\tau}(t,x) &= \mu_h^n(x) \in V_h \\
\phi_{h\tau}(t,x) &= \phi_h^n(x) \in V_h \\
\overline{\delta}_{h\tau}(t,x) &= \tfrac{\overline{\rho}_h^n(x) - \overline{\rho}_h^{n-1}(x)}{\tau} \in X_h \\
\boldsymbol{u}_{h\tau} &= -\boldsymbol{\nabla}(\mu_{h\tau} + \Psi_h) \in \widetilde{V}_h^d \\
\boldsymbol{v}_{h\tau} &= |\boldsymbol{u}_{h\tau}|^{q-2}\boldsymbol{u}_{h\tau} \in \widetilde{V}_h^d
\end{aligned} \qquad t \in (t^{n-1}, t^n]$$

Uniform estimates

- Mass preservation:

$$\int_\Omega \rho_{h\tau}(t,x)\mathrm{d}x = \int_\Omega \overline{\rho}_{h\tau}(t,x)\mathrm{d}x = \int_\Omega \widetilde{\rho}_{h\tau}(t,x)\mathrm{d}x = 1, \qquad t \geq 0$$

- Energy decay

$$\mathcal{E}_h(\overline{\rho}_{h\tau})(t^n) + \int_{t^\ell}^{t^n}\int_\Omega \widetilde{\rho}_{h\tau}|\boldsymbol{u}_{h\tau}|^q \mathrm{d}t\mathrm{d}x \leq \mathcal{E}_h(\overline{\rho}_{h\tau})(t^\ell), \qquad n \geq \ell \geq 0$$

- Bounded entropy $\rightsquigarrow$ equi-integrability on $\overline{\rho}_{h\tau}$

$$\int_\Omega \eta(\overline{\rho}_{h\tau}(t,x))\mathrm{d}x \;\leq\; \mathcal{E}_h(\overline{\rho}_{h\tau})(t) \;\leq\; \mathcal{E}_h(\overline{\rho}_h^0) \;\leq\; \mathcal{E}(\rho^0) + Ch$$

- $L^q_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, W^{1,1}(\Omega))$ estimate on the pressure:

$$\int_\Omega |\boldsymbol{\nabla}\phi_{h\tau}| \lesssim \int_\Omega \widetilde{\rho}_{h\tau}|\boldsymbol{\nabla}\mu_{h\tau}| \lesssim 1 + \int_\Omega \widetilde{\rho}_{h\tau}|\boldsymbol{u}_{h\tau}| \lesssim 1 + \left(\int_\Omega \widetilde{\rho}_{h\tau}|\boldsymbol{u}_{h\tau}|^q\right)^{1/q}$$

- Time translate estimate

$$\mathbb{W}_1(\overline{\rho}_{h\tau}(t), \overline{\rho}_{h\tau}(s)) \lesssim (|t-s| + \tau)^{1/q}$$

Compactness properties

- Refined Arzelà-Ascoli theorem[4] + equi-integrability

$$\overline{\rho}_{h\tau}(t,\cdot) \underset{h,\tau\to 0}{\longrightarrow} \rho(t,\cdot) \quad \text{weakly in } L^1(\Omega) \quad \text{with } \rho \in C(\mathbb{R}_{\geq 0}, L^1(\Omega)\text{-w})$$

- All reconstructions share the same limit ρ

$$\mathbb{W}_1(\overline{\rho}_{h\tau}(t), \widetilde{\rho}_{h\tau}(t)) + \mathbb{W}_1(\overline{\rho}_{h\tau}(t), \rho_{h\tau}(t)) \lesssim h, \qquad t \geq 0$$

- Nonlinear discrete Aubin-Simon Lemma[5]

$$\overline{\rho}_{h\tau} \underset{h,\tau\to 0}{\longrightarrow} \rho \quad \text{a.e. in } \mathbb{R}_{\geq 0} \times \Omega, \qquad \phi_{h\tau} \underset{h,\tau\to 0}{\longrightarrow} \phi(\rho) \quad \text{weakly in } L^q_{\text{loc}}(W^{1,1})$$

[4] [Ambrosio, Gigli & Savaré (2005)]
[5] [Andreianov, CC & Moussa (2017)]

Identification of the limit

There exists $\boldsymbol{u} : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^d$ such that

$$\widetilde{\rho}_{h\tau} \boldsymbol{u}_{h\tau} \underset{h,\tau \to 0}{\longrightarrow} \rho \boldsymbol{u} = -\boldsymbol{\nabla} \phi(\rho) - \rho \boldsymbol{\nabla} \Psi \quad \text{weakly in } L^q(\mathbb{R}_{\geq 0}; L^1(\Omega))$$

and

$$|\partial \mathcal{E}(\rho)|^q = \int_\Omega \rho |\boldsymbol{u}|^q \in L^1(\mathbb{R}_{\geq 0})$$

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There exists $\boldsymbol{v} : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^d$ such that

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$$

such that

$$\int_0^{t_*} \int_\Omega \rho |\boldsymbol{v}|^p \leq \liminf_{h,\tau \to 0} \int_0^{t_*} \int_\Omega \widetilde{\rho}_{h\tau} |\boldsymbol{v}_{h\tau}|^p,$$

hence

$$|\rho'|^p \leq \int_\Omega \rho |\boldsymbol{v}|^p \in L^1(\mathbb{R}_{\geq 0})$$

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$$\mathcal{E}(\rho)(t) \leq \liminf_{h,\tau \to 0} \mathcal{E}_h(\rho_{h\tau})(t) \qquad \text{and} \qquad \mathcal{E}_h(\rho_h^0) \underset{h \to 0}{\longrightarrow} \mathcal{E}(\rho^0)$$

Identification of the limit

continued

Passing to the $\liminf$ in the discrete energy dissipation inequality

$$\mathcal{E}_h(\rho_{h\tau})(t) + \int_0^t \int_\Omega \widetilde{\rho}_{h\tau} |\boldsymbol{u}_{h\tau}|^q \leq \mathcal{E}_h(\rho_h^0), \qquad t \geq 0$$

provides (EDI):

$$\mathcal{E}(\rho)(t) + \frac{1}{p}\int_0^t |\rho'|^p + \frac{1}{q}\int_0^t |\partial\mathcal{E}(\rho)|^q \leq \mathcal{E}(\rho^0), \qquad t \geq 0.$$

The limit profile ρ is a gradient flow solution

Identification of the limit

continued

Passing to the lim inf in the discrete energy dissipation inequality

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The limit profile ρ is a gradient flow solution

A posteriori enhanced convergence for the fluxes

The discrete flux $\widetilde{\rho}_{h\tau}\boldsymbol{v}_{h\tau}$ converges strongly in $L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \Omega)^d$ towards $\rho\boldsymbol{v}$.

About the practical implementation

- Numerical implementation by Flore Nabet using FreeFEM

- Making the energy singular by replacing $\eta(\rho)$ by

$$\eta(\rho) - \epsilon_k \log(\rho) \quad \text{with} \quad \epsilon_k \to 0$$

- Each step k requires the resolution of a nonlinear system

$$\mathcal{F}^{n,k}(\rho_{h\tau}^{n,k}) = 0_{\mathbb{R}^{\mathcal{V}_h}} \quad \Longleftrightarrow \quad \rho_{h\tau}^{n,k} \in \underset{w_{h\tau}>0}{\operatorname{argmin}} \frac{1}{2} \left\| \mathcal{F}^{n,k}(w_{h\tau}) \right\|^2 \qquad (\star)$$

while $\epsilon_k > \epsilon_\star$ **do**
  $\rho_{h\tau}^{n,k,0} \leftarrow \rho_{h\tau}^{n,k-1}$
  while $\left\| \mathcal{F}^{n,k}(\rho_{h\tau}^{n,k,\ell}) \right\| > \theta$ **do**
    Newton (+ linesearch) solve for $(\star)$
    $\ell \leftarrow \ell + 1$
  end while
  $\epsilon_k \leftarrow \alpha\, \epsilon_k$ and $k \leftarrow k+1$
  $\rho_{h\tau}^{n,k} \leftarrow \rho_{h\tau}^{n,k,\ell}$
end while

The q-Laplace equation and Barenblatt profile

- Classical q-Laplace equation:

$$\partial_t \rho = \nabla \cdot \left(|\nabla \rho|^{q-2} \nabla \rho \right) = \nabla \cdot \left(\rho |\nabla \eta'(\rho)|^{q-2} \nabla \eta'(\rho) \right)$$

with the non-singular energy density

$$\eta(\rho) = \frac{q-1}{q-2} \left(\frac{q-1}{2q-3} \left(\rho^{\frac{2q-3}{q-1}} - 1 \right) - \rho + 1 \right) \qquad (q > 2).$$

- Exact solution[6]

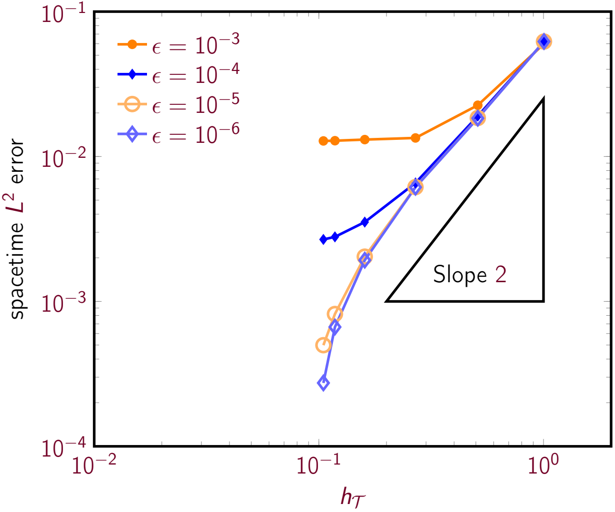
$$\rho(t,x) = (t+t_0)^{-k} \left(\left(M - \alpha \, |\xi|^{p} \right)^{+} \right)^{\frac{1}{2-p}}$$

with $k = \frac{1}{q-2+\frac{q}{d}}$, $\alpha = \frac{q-2}{q} \left(\frac{k}{d} \right)^{\frac{1}{q-1}}$, $\xi = x(t+t_0)^{-\frac{k}{d}}$ and M set to fulfill the mass constraint.

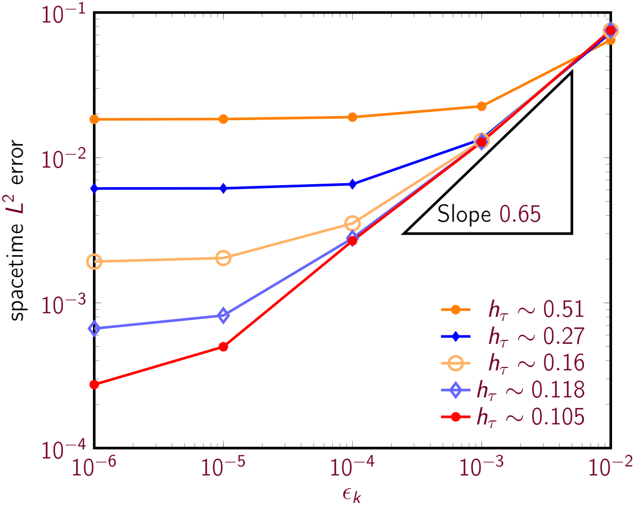
[6] [Kamin & Vazquez (1988)]

Error in function of the mesh size

$p = 5/4$, $q = 5$

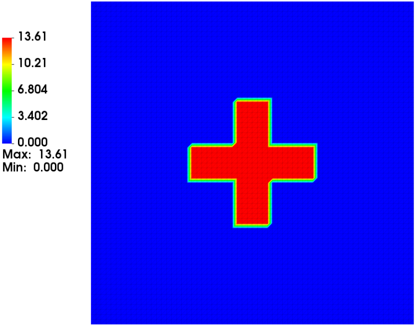


Error in function of the parameter ϵ

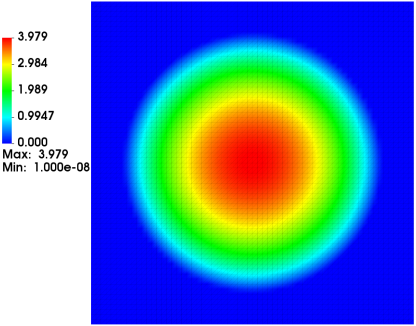


$\mathbb{W}_p$ gradient flow involving a confining potential

- Quadratic entropy density $\eta(\rho) = \frac{1}{2}\rho^2$
- Quadratic confining potential $\Psi(x) = \frac{1}{2}|x - x_\star|^2$

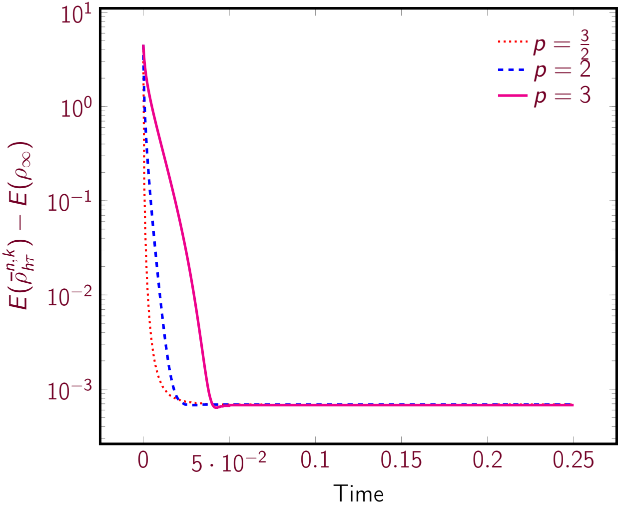


(a) Initial density



(b) Stationary state

Energy dissipation along time



Conclusion and prospects

Conclusion

- Approximation of $\mathbb{W}_p$ gradient flows more involved for $p \neq 2$:
  - more involved continuous theory (displacement convexity)
  - need of reconstructing the whole gradient $\rightsquigarrow$ simple TPFA approach fails
- Backward Euler scheme as a simpler alternative to the JKO scheme
- Difficulty with the non-negativity constraint $\rightsquigarrow$ interior point type approach [Natale-Todeschi (2020)]
- Convergence proof based of [Ambrosio-Gigli-Savaré (2005)]

Prospects

- Extend to (Hybrid?) Finite Volumes
- ...