On the finite-volume analysis of the stochastic heat equation

Flore Nabet

joint work with Caroline Bauzet, Anne De Bouard, Ludovic Goudenège, Kerstin Schmitz and Aleksandra Zimmermann

CMAP – Ecole polytechnique

Finite Volumes and/or Optimal Transport: past, present and perspectives

19th-21st November 2024

Outline

THE CONTINUOUS PROBLEM

$$\begin{cases} du - \Delta u \, dt = g(u) \, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
    - $W(0) = 0$ almost surely;
    - $W(t)$ is almost surely continuous in t;
    - $\forall 0 \leq s \leq t$, $W(t) - W(s) \sim \mathcal{N}(0, t-s)$;
    - $\forall 0 \leq t^1 \leq t^2 \leq \cdots \leq t^k$, $W(t^1)$, $W(t^2) - W(t^1)$, $\cdots$, $W(t^k) - W(t^{k-1})$ are independent.
- $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is a $\mathcal{F}_0$-measurable random variable.

THE CONTINUOUS PROBLEM

$$\begin{cases} du - \Delta u \, dt = g(u)\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
- $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is a $\mathcal{F}_0$-measurable random variable.

DEFINITION

A variational solution to the heat equation with multiplicative Lipschitz noise is a $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process

$$u \in L^2(\Omega; \mathcal{C}([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

such that, for all $t \in [0,T]$, in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω,

$$u(t) - u_0 - \int_0^t \Delta u(s)\, ds = \int_0^t g(u(s))\, dW_s.$$

$\rightsquigarrow$ From classical results existence and uniqueness of a variational solution is well-known. **Pardoux(1975), Krylov-Rozovskii(1981), Liu-Röckner(2015), ...**

THE CONTINUOUS PROBLEM

$$\begin{cases} du - \Delta u \, dt = g(u) \, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
- $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is a $\mathcal{F}_0$-measurable random variable.

DEFINITION

A variational solution to the heat equation with multiplicative Lipschitz noise is a $(\mathcal{F}_t)_{t\geq 0}$*-adapted stochastic process*

$$u \in L^2(\Omega; \mathcal{C}([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

such that, for all $t \in [0,T]$*, in* $L^2(\Lambda)$*,* $\mathbb{P}$*-a.s. in* Ω*,*

$$u(t) - u_0 - \int_0^t \Delta u(s) \, ds = \int_0^t g(u(s)) \, dW_s.$$

⇝ From classical results existence and uniqueness of a variational solution is well-known. **Pardoux(1975), Krylov-Rozovskii(1981), Liu-Röckner(2015), . . .**

GOAL: Convergence of the TPFA scheme towards the variational solution.

PDE'S FORMULATION:

$$\partial_t \left(u(t,x) - \int_0^t g(u(s,x))\mathrm{d}W_s \right) - \Delta u(t,x) = 0$$

TIME DISCRETIZATION:

$N \in \mathbb{N}^* \Rightarrow$ Time step: $\Delta t = \dfrac{T}{N}$ and $\forall n \in [\![0, N]\!]$, $t^n = n\Delta t$.

UNKNOWNS: We are looking for $u^n \sim u(t^n, \cdot)$

IMPLICIT EULER METHOD:

$$\frac{u^{n+1}(x) - u^n(x)}{\Delta t} - \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} g(u(s,x))\mathrm{d}W_s - \Delta u^{n+1}(x) = 0$$

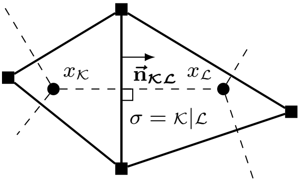
and

$$\int_{t^n}^{t^{n+1}} g(u(s,x))\mathrm{d}W_s \sim g(u^n(x)) \underbrace{W(t^{n+1}) - W(t^n)}_{\sim\mathcal{N}(0,\Delta t)} = \sqrt{\Delta t} g(u^n(x)) \underbrace{\xi^{n+1}}_{\sim\mathcal{N}(0,1)}$$

since

$$\int_0^T X_s dW_s = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} X_{t^n} \left(W(t^{n+1}) - W(t^n) \right)$$

$$\boxed{\frac{u^{n+1}(x) - u^n(x)}{\Delta t} - \Delta u^{n+1}(x) = \frac{1}{\sqrt{\Delta t}} g(u^n(x))\xi^{n+1}}$$



UNKNOWNS:
We are looking for $u_{\mathcal{K}}^n \sim u(t^n, x_{\mathcal{K}})$

Notation: $u_{\mathcal{T}}^n = ((u_{\mathcal{K}}^n)_{\mathcal{K}\in\mathcal{T}})$.

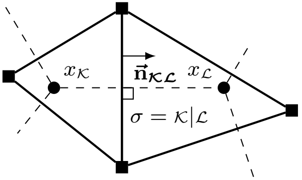
TPFA SCHEME

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be a $\mathcal{F}_{t^n}$-measurable random vector. We look for a $\mathcal{F}_{t^{n+1}}$-measurable random vector $u_{\mathcal{T}}^{n+1} \in \mathbb{R}^{\mathcal{T}}$ such that for almost every $\omega \in \Omega$, for any $\mathcal{K} \in \mathcal{T}$,

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) = m_{\mathcal{K}} g(u_{\mathcal{K}}^n) \left(W(t^{n+1}) - W(t^n)\right),$$

where

$$W(t^{n+1}) - W(t^n) = \sqrt{\Delta t}\xi^{n+1}, \text{ for } n \in \{0, \dots, N-1\}.$$



Unknowns:

We are looking for $u_{\mathcal{K}}^n \sim u(t^n, x_{\mathcal{K}})$

Notation: $u_{\mathcal{T}}^n = ((u_{\mathcal{K}}^n)_{\mathcal{K}\in\mathcal{T}})$.

TPFA scheme

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be a $\mathcal{F}_{t^n}$-measurable random vector. We look for a $\mathcal{F}_{t^{n+1}}$-measurable random vector $u_{\mathcal{T}}^{n+1} \in \mathbb{R}^{\mathcal{T}}$ such that for almost every $\omega \in \Omega$, for any $\mathcal{K} \in \mathcal{T}$,

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) = m_{\mathcal{K}} g(u_{\mathcal{K}}^n)\left(W(t^{n+1}) - W(t^n)\right),$$

where

$$W(t^{n+1}) - W(t^n) = \sqrt{\Delta t}\xi^{n+1}, \text{ for } n \in \{0, \dots, N-1\}.$$

⇝ Existence of a discrete solution.

Need of uniqueness to obtain measurability

Let $(\mathcal{T}_m)_m$ and $(N_m)_m$ be s.t. $\text{size}(\mathcal{T}_m) \xrightarrow[m\to+\infty]{} 0$ and $\Delta t_m = \dfrac{T}{N_m} \xrightarrow[m\to+\infty]{} 0$.

(Bauzet-Nabet-Schmitz-Zimmermann, '22)

Theorem (Convergence)

Let $u_0 \in L^2(\Omega, H^1(\Lambda))$ $\mathcal{F}_0$-measurable, then

$$u_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \;\; \textit{in } L^p(\Omega; L^2(0,T; L^2(\Lambda))) \;\; \textit{with } 1 \leq p < 2,$$

where u is the unique variational solution of the heat equation with multiplicative Lipschitz noise.

Step 1: Bounds on the discrete solution

$$\sum_{K\in\mathcal{T}} u_K^{n+1} \times \Big(m_K (u_K^{n+1} - u_K^n) + \Delta t \sum_{\sigma\in\mathcal{E}_K^{int}} \frac{m_\sigma}{d_{K,L}} (u_K^{n+1} - u_L^{n+1})$$
$$= m_K g(u_K^n) \left(W(t^{n+1}) - W(t^n) \right) \Big)$$

Step 1: Bounds on the discrete solution

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$= \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} g(u_{\mathcal{K}}^n) u_{\mathcal{K}}^{n+1} \left(W(t^{n+1}) - W(t^n)\right)$$

Step 1: Bounds on the discrete solution

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^{n}\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$

$$= \sum_{K\in\mathcal{T}} m_K g(u_K^n)[u_K^n + (u_K^{n+1} - u_K^n)] \left(W(t^{n+1}) - W(t^n)\right)$$

1. Martingale property:

$$\mathbb{E}\left[g(u_K^n)u_K^n \left(W(t^{n+1}) - W(t^n)\right)\right] = \mathbb{E}\left[g(u_K^n)u_K^n\right]\mathbb{E}\left[\left(W(t^{n+1}) - W(t^n)\right)\right] = 0$$

Step 1: Bounds on the discrete solution

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$= \sum_{K\in\mathcal{T}} m_K g(u_K^n)[u_K^n + (u_K^{n+1} - u_K^n)]\left(W(t^{n+1}) - W(t^n)\right)$$

1. Martingale property:

$$\mathbb{E}\left[g(u_K^n)u_K^n\left(W(t^{n+1}) - W(t^n)\right)\right] = \mathbb{E}\left[g(u_K^n)u_K^n\right]\mathbb{E}\left[\left(W(t^{n+1}) - W(t^n)\right)\right] = 0$$

2. Young inequality:

$$\sum_{K\in\mathcal{T}} m_K g(u_K^n)(u_K^{n+1} - u_K^n)\left(W(t^{n+1}) - W(t^n)\right)$$
$$\leq \frac{1}{4}\left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2 + \sum_{K\in\mathcal{T}} m_K\left|g(u_K^n)\left(W(t^{n+1}) - W(t^n)\right)\right|^2$$

Step 1: Bounds on the discrete solution

$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^{n}\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2$$
$$= \sum_{K\in\mathcal{T}} m_K g(u_K^n)[u_K^n + (u_K^{n+1} - u_K^n)]\left(W(t^{n+1}) - W(t^n)\right)$$

1. Martingale property:

$$\mathbb{E}\left[g(u_K^n)u_K^n\left(W(t^{n+1}) - W(t^n)\right)\right] = \mathbb{E}\left[g(u_K^n)u_K^n\right]\mathbb{E}\left[\left(W(t^{n+1}) - W(t^n)\right)\right] = 0$$

2. Young inequality:

$$\sum_{K\in\mathcal{T}} m_K g(u_K^n)(u_K^{n+1} - u_K^n)\left(W(t^{n+1}) - W(t^n)\right)$$
$$\leq \frac{1}{4}\left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^{n}\right\|_{L^2(\Lambda)}^2 + \sum_{K\in\mathcal{T}} m_K \left|g(u_K^n)\left(W(t^{n+1}) - W(t^n)\right)\right|^2$$

3. Itô isometry:

$$\mathbb{E}\left[\left|g(u_K^n)\left(W(t^{n+1}) - W(t^n)\right)\right|^2\right] = \mathbb{E}\left[\left|\int_{t^n}^{t^{n+1}} g(u_K^n)\mathrm{d}W_s\right|^2\right]$$
$$= \mathbb{E}\left[\int_{t^n}^{t^{n+1}} \left(g(u_K^n)\right)^2 \mathrm{d}s\right] = \Delta t\,\mathbb{E}\left[|g(u_K^n)|^2\right]$$

Proposition (Bounds on the discrete solutions)

For any $n \in \{1, \dots, N\}$,

$$\mathbb{E}\left[\|u^n_{\mathcal{T}}\|^2_{L^2(\Lambda)}\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \|u^{n+1}_{\mathcal{T}} - u^n_{\mathcal{T}}\|^2_{L^2(\Lambda)}\right] + 2\Delta t \sum_{i=0}^{n-1} \mathbb{E}\left[\left|u^{n+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\right] \leq C.$$

Consequences: Weak convergence

There exists $u \in L^2(\Omega; L^2(0,T; H^1(\Lambda))$ such that (up to a subsequence):

$$u_{\mathcal{T},N} \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} u \text{ weakly in } L^2(\Omega; L^2(0,T; L^2(\Lambda)),$$

$$\nabla^{\mathcal{T}} u_{\mathcal{T},N} \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} \nabla u \text{ weakly in } L^2(\Omega; L^2(0,T; L^2(\Lambda)).$$

BUT weak convergence not sufficient for nonlinear term:

$$\exists g_u \in L^2(\Omega; L^2(0,T; L^2(\Lambda)) \text{ s.t. } g(u^l_{\mathcal{T},N}) \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} g_u$$

$$g_u = g(u)??$$

$\Rightarrow$ Need of stochastic compactness.

Step 2: Stochastic compactness argument

Theorem (Skorokhod's representation theorem)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m), \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

STEP 2: STOCHASTIC COMPACTNESS ARGUMENT

THEOREM (SKOROKHOD'S REPRESENTATION THEOREM)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m), \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

- Prokhorov's theorem: If $(\mathbb{P} \circ (u_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

⇒ Up to a subsequence, $(u^l_{\mathcal{T},N})_m$ converges in law to a probability measure $\mu_\infty \in L^2(0,T;L^2(\Lambda))$.

Step 2: Stochastic compactness argument

Theorem (Skorokhod's representation theorem)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m), \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

- Prokhorov's theorem: If $(\mathbb{P} \circ (u_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

$\Rightarrow$ Up to a subsequence, $(u^l_{\mathcal{T},N})_m$ converges in law to a probability measure $\mu_\infty \in L^2(0,T;L^2(\Lambda))$.

- The laws of $(u_{\mathcal{T},N})$ are tight if, for any ε there exists a compact set K_ε s.t.

$$\mathcal{L}(u_{\mathcal{T},N})(K_\varepsilon) = [\mathbb{P} \circ (u_{\mathcal{T},N})^{-1}](K_\varepsilon) \geq 1 - \varepsilon.$$

Step 2: Stochastic compactness argument

Theorem (Skorokhod's representation theorem)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m), \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

- Prokhorov's theorem: If $(\mathbb{P} \circ (u_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

$\Rightarrow$ Up to a subsequence, $(u^l_{\mathcal{T},N})_m$ converges in law to a probability measure $\mu_\infty \in L^2(0,T;L^2(\Lambda))$.

- The laws of $(u_{\mathcal{T},N})$ are tight if, for any ε there exists a compact set K_ε s.t.

$$\mathcal{L}(u_{\mathcal{T},N})(K_\varepsilon) = [\mathbb{P} \circ (u_{\mathcal{T},N})^{-1}](K_\varepsilon) \geq 1 - \varepsilon.$$

- $\mathcal{W} = L^2(0,T;W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T;L^2(\Lambda)) \overset{\text{compact}}{\hookrightarrow} L^2(0,T;L^2(\Lambda))$

STEP 2: STOCHASTIC COMPACTNESS ARGUMENT

THEOREM (SKOROKHOD'S REPRESENTATION THEOREM)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m), \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

- Prokhorov's theorem: If $(\mathbb{P} \circ (u_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

$\Rightarrow$ Up to a subsequence, $(u^l_{\mathcal{T},N})_m$ converges in law to a probability measure $\mu_\infty \in L^2(0,T;L^2(\Lambda))$.

- The laws of $(u_{\mathcal{T},N})$ are tight if, for any ε there exists a compact set K_ε s.t.

$$\mathcal{L}(u_{\mathcal{T},N})(K_\varepsilon) = [\mathbb{P} \circ (u_{\mathcal{T},N})^{-1}](K_\varepsilon) \geq 1 - \varepsilon.$$

- $\mathcal{W} = L^2(0,T;W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T;L^2(\Lambda)) \overset{\text{compact}}{\hookrightarrow} L^2(0,T;L^2(\Lambda))$
- $B_{\mathcal{W}}(0,R) := \{v \in \mathcal{W} : \|v\|_{\mathcal{W}} \leq R\}$ compact in $L^2(0,T;L^2(\Lambda))$

$$[\mathbb{P}\circ(u_{\mathcal{T},N})^{-1}](B_{\mathcal{W}}(0,R)) = 1-[\mathbb{P}\circ(u_{\mathcal{T},N})^{-1}](B_{\mathcal{W}}(0,R)^c) = 1-\int_{\{\|u_{\mathcal{T},N}\|_{\mathcal{W}}>R\}} 1\, d\mathbb{P}$$

Step 2: Stochastic compactness argument

Theorem (Skorokhod's representation theorem)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m),\ \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y),\ \textit{and} \quad Y_m \to Y \quad \mathbb{P}'\textit{-a.s. in } \Omega'.$$

- Prokhorov's theorem: If $(\mathbb{P} \circ (u_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

$\Rightarrow$ Up to a subsequence, $(u^l_{\mathcal{T},N})_m$ converges in law to a probability measure $\mu_\infty \in L^2(0,T;L^2(\Lambda))$.

- The laws of $(u_{\mathcal{T},N})$ are tight if, for any ε there exists a compact set K_ε s.t.

$$\mathcal{L}(u_{\mathcal{T},N})(K_\varepsilon) = [\mathbb{P} \circ (u_{\mathcal{T},N})^{-1}](K_\varepsilon) \geq 1 - \varepsilon.$$

- $\mathcal{W} = L^2(0,T;W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T;L^2(\Lambda)) \overset{\text{compact}}{\hookrightarrow} L^2(0,T;L^2(\Lambda))$
- $B_{\mathcal{W}}(0,R) := \{v \in \mathcal{W} : \|v\|_{\mathcal{W}} \leq R\}$ compact in $L^2(0,T;L^2(\Lambda))$

$$[\mathbb{P}\circ(u_{\mathcal{T},N})^{-1}](B_{\mathcal{W}}(0,R)) = 1 - [\mathbb{P}\circ(u_{\mathcal{T},N})^{-1}](B_{\mathcal{W}}(0,R)^c) = 1 - \int_{\{\|u_{\mathcal{T},N}\|_{\mathcal{W}} > R\}} 1\, d\mathbb{P}$$

- Markov inequality

$$\int_{\{\|u_{\mathcal{T},N}\|_{\mathcal{W}} > R\}} 1\, d\mathbb{P} \leq \frac{1}{R^2} \int_{\{\|u_{\mathcal{T},N}\|_{\mathcal{W}} > R\}} \|u_{\mathcal{T},N}\|^2_{\mathcal{W}}\, d\mathbb{P} \leq \frac{1}{R^2} \mathbb{E}\left[\|u_{\mathcal{T},N}\|^2_{\mathcal{W}}\right]$$

Step 2: Stochastic compactness argument

- Bounds on the Gagliardo seminorm for $\alpha \in (0, \frac{1}{2})$:

$$\mathcal{W} = L^2(0,T; W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T; L^2(\Lambda))$$

  - $L^2(0,T; W^{\alpha,2}(\Lambda))$ bound:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,y)|^2}{|x-y|^{2+2\alpha}} dx\, dy$$

$$= \int_{|\eta|>R} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} dx\, d\eta + \int_{|\eta|<R} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} dx\, d\eta$$

$$\leq 4\|\bar{u}_{h,N}(t)\|^2_{L^2(\mathbb{R})} \int_{|\eta|>R} |\eta|^{-2(1+\alpha)} d\eta + C \left(|u_{\mathcal{T},N}|^2_{1,h} + \|u_{\mathcal{T},N}\|^2_{L^2(\Lambda)} \right) \int_{|\eta|<R} |\eta|^{-2(1+\alpha)+1} d\eta$$

$\Longrightarrow$ Need of space translate estimates.

STEP 2: STOCHASTIC COMPACTNESS ARGUMENT

- Bounds on the Gagliardo seminorm for $\alpha \in (0, \frac{1}{2})$:

$$\mathcal{W} = L^2(0,T; W^{\alpha,2}(\Lambda)) \cap W^{\alpha,2}(0,T; L^2(\Lambda))$$

  - $L^2(0,T; W^{\alpha,2}(\Lambda))$ bound:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,y)|^2}{|x-y|^{2+2\alpha}} dx\, dy$$
$$= \int_{|\eta|>R} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} dx\, d\eta + \int_{|\eta|<R} \int_{\mathbb{R}} \frac{|\bar{u}_{\mathcal{T},N}(t,x) - \bar{u}_{h,N}(t,x+\eta)|^2}{|\eta|^{2(1+\alpha)}} dx\, d\eta$$
$$\leq 4\|\bar{u}_{h,N}(t)\|^2_{L^2(\mathbb{R})} \int_{|\eta|>R} |\eta|^{-2(1+\alpha)} d\eta + C\left(|u_{\mathcal{T},N}|^2_{1,h} + \|u_{\mathcal{T},N}\|^2_{L^2(\Lambda)}\right) \int_{|\eta|<R} |\eta|^{-2(1+\alpha)+1} d\eta$$

$\Longrightarrow$ Need of space translate estimates.

  - $W^{\alpha,2}(0,T; L^2(\Lambda))$-bound:

$$\mathbb{E}\left[\int_0^T \int_0^T \frac{\|\bar{u}_{h,N}(s) - \bar{u}_{h,N}(t)\|^2_{L^2(\Lambda)}}{|t-s|^{1+2\alpha}} ds\, dt\right]$$

$\Longrightarrow$ Need of time translate estimates.

Step 2: Stochastic compactness argument

Skorokhod's theorem:

On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$:

- there exist random variables v_0, $(v_m)_m$, u_∞ with $\mathcal{L}(v_0) = \mathcal{L}(u_0)$, $\mathcal{L}(v_m) = \mathcal{L}(u_{\mathcal{T}_m, N_m})$ for all $m \in \mathbb{N}$, $\mathcal{L}(u_\infty) = \mu_\infty$ and

$$v_m \xrightarrow[m\to+\infty]{} u_\infty \text{ in } L^2(0,T;L^2(\Lambda)), \ \mathbb{P}'\text{-a.s. in } \Omega';$$

- there exists a stochastic process W_∞ and a sequence of Brownian motions $(W_m)_m$ such that

$$W_m \xrightarrow[m\to+\infty]{} W_\infty \text{ in } \mathcal{C}([0,T]), \ \mathbb{P}'\text{-a.s. in } \Omega'.$$

STEP 2: STOCHASTIC COMPACTNESS ARGUMENT

Skorokhod's theorem:

On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$:

- there exist random variables v_0, $(v_m)_m$, u_∞ with $\mathcal{L}(v_0) = \mathcal{L}(u_0)$, $\mathcal{L}(v_m) = \mathcal{L}(u_{\mathcal{T}_m, N_m})$ for all $m \in \mathbb{N}$, $\mathcal{L}(u_\infty) = \mu_\infty$ and

$$v_m \xrightarrow[m \to +\infty]{} u_\infty \text{ in } L^2(0,T;L^2(\Lambda)), \ \mathbb{P}'\text{-a.s. in } \Omega';$$

- there exists a stochastic process W_∞ and a sequence of Brownian motions $(W_m)_m$ such that

$$W_m \xrightarrow[m \to +\infty]{} W_\infty \text{ in } \mathcal{C}([0,T]), \ \mathbb{P}'\text{-a.s. in } \Omega'.$$

CONSEQUENCES:

- For $m \in \mathbb{N}^*$, v_m is a step function i.e.

$$\exists v_{\mathcal{T}_m, N_m} \in \mathbb{R}^{\mathcal{T}_m \times N_m} \text{ s.t. } v_m = v_{\mathcal{T}_m, N_m} \, \mathbb{P}'\text{-a.s. in } \Omega'$$

$$\Rightarrow \quad v_m(t,x) = v_K^n, \quad \forall t \in [t^n, t^{n+1}), \forall x \in K.$$

- For $m \in \mathbb{N}^*$, any $n \in \{0, \dots, N_m - 1\}$ and any $K \in \mathcal{T}_m$, the random vector $v_{\mathcal{T}_m}^{n+1}$ is solution to

$$m_K(v_K^{n+1} - v_K^n) + \Delta t \sum_{\sigma \in \mathcal{E}_K^{int}} \frac{m_\sigma}{d_{K,L}}(v_K^{n+1} - v_L^{n+1}) - m_K g(v_K^n)(W_m(t^{n+1}) - W_m(t^n)) = 0.$$

Step 3: Identification of the stochastic integral

$$m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{K}}^{n})+\Delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}}\frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{L}}^{n+1})=m_{\mathcal{K}}g(v_{\mathcal{K}}^{n})\left(W_m(t^{n+1})-W_m(t^n)\right)$$

- For any $m\in\mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq0}$ and $W_m=(W_m(t))_{t\geq0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq0}$.

Step 3: Identification of the stochastic integral

$$m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{K}}^{n})+\Delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}}\frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{L}}^{n+1})=\int_{\mathcal{K}}\int_{t^n}^{t^{n+1}}g(v_{\mathcal{T},N})\,dW_m(t)\,dx$$

- For any $m\in\mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq 0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq 0}$ and $W_m=(W_m(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq 0}$.

Step 3: Identification of the stochastic integral

$$m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1} - v_{\mathcal{K}}^{n}) + \Delta t \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (v_{\mathcal{K}}^{n+1} - v_{\mathcal{L}}^{n+1}) = \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} g(v_{\mathcal{T},N})\, dW_m(t)\, dx$$

- For any $m \in \mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq 0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq 0}$ and $W_m = (W_m(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq 0}$.
- $W_m \xrightarrow[m\to+\infty]{} W_\infty$ in $L^2(\Omega'; C([0,T]))$
- There exists a filtration $(\mathfrak{F}_t^\infty)_{t\geq 0}$ such that u_∞ has a predictable $d\mathbb{P}' \otimes dt$-representative and $W_\infty = (W_\infty(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\geq 0}$.

Step 3: Identification of the stochastic integral

$$m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1} - v_{\mathcal{K}}^{n}) + \Delta t \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v_{\mathcal{K}}^{n+1} - v_{\mathcal{L}}^{n+1}) = \int_{\mathcal{K}} \int_{t^n}^{t^{n+1}} g(v_{\mathcal{T},N})\, dW_m(t)\, dx$$

- For any $m \in \mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq 0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq 0}$ and $W_m = (W_m(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq 0}$.
- $W_m \xrightarrow[m\to+\infty]{} W_\infty$ in $L^2(\Omega'; C([0,T]))$
- There exists a filtration $(\mathfrak{F}_t^\infty)_{t\geq 0}$ such that u_∞ has a predictable $d\mathbb{P}' \otimes dt$-representative and $W_\infty = (W_\infty(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\geq 0}$.
-
  - $W_m \xrightarrow[m\to+\infty]{} W_\infty$ in probability in $C([0,T])$.
  - $v^l_{\mathcal{T}_m,N_m} \xrightarrow[m\to+\infty]{} u_\infty$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'
    $\Rightarrow g(v^l_{\mathcal{T}_m,N_m}) \xrightarrow[m\to+\infty]{} g(u_\infty)$ in probability in $L^2(0,T;L^2(\Lambda))$.

(Debussche,Glatt-Holtz,Temam, '11)

$$\int_0^t g(v_{\mathcal{T},N})\, dW_m(t)\, dx \xrightarrow[m\to+\infty]{} \int_0^t g(u_\infty)\, dW_\infty(t)\, dx$$

in probability in $L^2(0,T;L^2(\Lambda))$.

Step 3: Identification of the stochastic integral

$$\sum_{\mathcal{K}\in\mathcal{T}}\sum_{n=0}^{N-1} m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{K}}^{n}) + \Delta t \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{L}}^{n+1}) = \int_\Lambda\int_0^T g(v_{\mathcal{T},N})\,dW_m(t)\,dx$$

$$\xrightarrow[m\to+\infty]{} \int_\Lambda\int_0^T g(u_\infty)\,dW_\infty(t)\,dx$$

- For any $m\in\mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq 0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq 0}$ and $W_m=(W_m(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq 0}$.
- $W_m \xrightarrow[m\to+\infty]{} W_\infty$ in $L^2(\Omega';C([0,T]))$
- There exists a filtration $(\mathfrak{F}_t^\infty)_{t\geq 0}$ such that u_∞ has a predictable $d\mathbb{P}'\otimes dt$-representative and $W_\infty=(W_\infty(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\geq 0}$.
-
  - $W_m \xrightarrow[m\to+\infty]{} W_\infty$ in probability in $C([0,T])$.
  - $v_{\mathcal{T}_m,N_m}^l \xrightarrow[m\to+\infty]{} u_\infty$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'

  $\Rightarrow g(v_{\mathcal{T}_m,N_m}^l) \xrightarrow[m\to+\infty]{} g(u_\infty)$ in probability in $L^2(0,T;L^2(\Lambda))$.

Up to a subsequence **(Debussche,Glatt-Holtz,Temam, '11)**

$$\int_0^t g(v_{\mathcal{T},N})\,dW_m(t)\,dx \xrightarrow[m\to+\infty]{} \int_0^t g(u_\infty)\,dW_\infty(t)\,dx \text{ in } L^2(0,T;L^2(\Lambda)),\ \mathbb{P}'\text{-a.s. in } \Omega'.$$

Proposition (Weak martingale solution)

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W^\infty, u_\infty, v_0)$ *is a martingale solution for the heat equation with multiplicative Lipschitz noise.*

Proposition (Weak martingale solution)

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^{\infty})_{t\geq 0}, W^{\infty}, u_{\infty}, v_0)$ *is a martingale solution for the heat equation with multiplicative Lipschitz noise.*

Step 4: Strong convergence of finite-volume approximations

1. Pathwise uniqueness: Let u_1, u_2 be two solutions w.r.t. the $\mathcal{F}_0$-initial values $u_0^1, u_0^2 \in L^2(\Omega; L^2(\Lambda))$ on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\geq 0})$, then

$$\mathbb{E}\left[\|u_1(t) - u_2(t)\|^2_{L^2(\Lambda)}\right] \leq C\mathbb{E}\left[\|u_0^1 - u_0^2\|^2_{L^2(\Lambda)}\right], \quad \forall t \in [0, T].$$

2. **(Gyöngy-Krylov, '96)**

$$u_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in probability in } L^2(0, T; L^2(\Lambda)).$$

3. Up to a subsequence,

$$u_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \quad \mathbb{P}\text{-a.s. in } L^2(0, T; L^2(\Lambda)).$$

4. Uniform bounds in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ $\oplus$ Vitali's theorem:

$$u_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in } L^p(\Omega; L^2(0, T; L^2(\Lambda))) \text{ for } 1 \leq p < 2.$$

u is the unique variational solution of the heat equation with multiplicative Lipschitz noise.

$$\begin{cases} du - \Delta u \, dt = Lu \, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda. \end{cases}$$

INITIAL DATA

$$u_0(x,y) = 2\cos(\pi x)\cos(\pi y) \Rightarrow \Delta u_0 = -2\pi^2 u_0 = -\lambda_2 u_0$$

BLACK SCHOLES MODEL

$u(t,(x,y)) = \mathcal{S}(t) u_0(x,y)$ solution iff $\mathcal{S}(t)$ satisfies

$$\begin{cases} d\mathcal{S}(t) = -\lambda_2 \mathcal{S}(t) \, dt + L\mathcal{S}(t) \, dW(t); \\ \mathcal{S}(0) = 1. \end{cases}$$

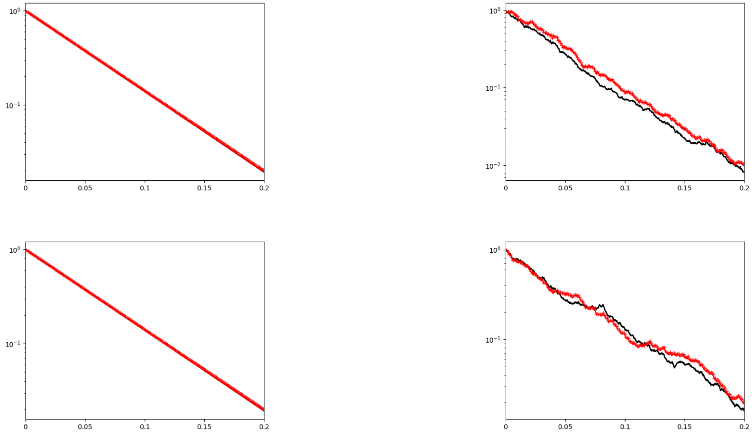
$$\mathcal{S}(t) = \mathcal{S}(0) e^{(-\lambda_2 - \frac{L^2}{2})t} e^{LW(t)} \text{ and } \|u(t,(x,y))\|_{L^2(\Lambda)} = \mathcal{S}(t)$$

COMPARISON BETWEEN $\mathcal{S}(t^n)$ AND $\|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$

- $W(t^n) = \underbrace{W(0)}_{=0} + \sum_{i=1}^{n} \underbrace{(W(t^i) - W(t^{i-1}))}_{=\sqrt{\Delta t}\xi^i} = \sqrt{\Delta t} \sum_{i=1}^{n} \xi^i.$
- Discretization:

$$T = 0.2, \quad \text{Mesh size } \sim 1.66.10^{-2}$$

On a given trajectory

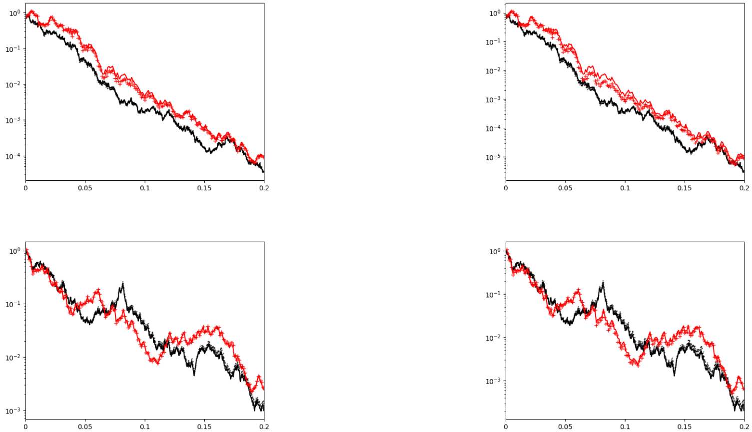


$L = 0$ $\qquad$ $L = 1$

$\mathcal{S}(t^n)$, $+++$ $\|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\Delta t = 10^{-3}$

$\mathcal{S}(t^n)$, $+++$ $\|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\Delta t = 2.10^{-4}$

On a given trajectory

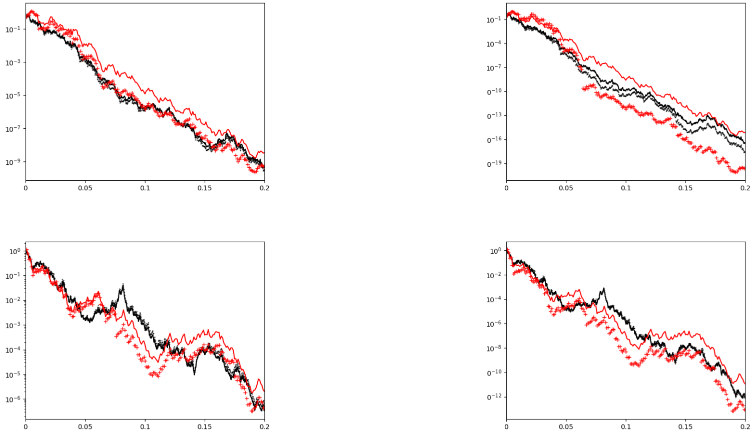


$L = 5$ $\qquad$ $L = 2\pi$

$\mathcal{S}(t^n)$, $\quad +++ \quad \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\qquad \Delta t = 10^{-3}$ (red)

$\mathcal{S}(t^n)$, $\quad +++ \quad \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\qquad \Delta t = 2.10^{-4}$ (black)

On a given trajectory



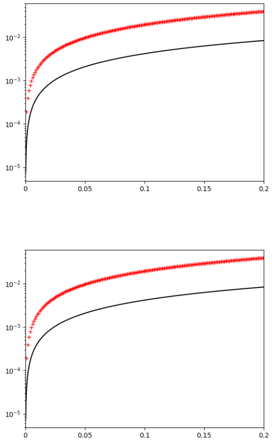
$L = 10$ $\qquad\qquad$ $L = 15$

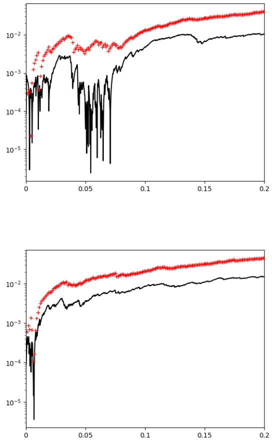
——— $\mathcal{S}(t^n)$, $\quad +++ \quad \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\qquad \Delta t = 10^{-3}$

——— $\mathcal{S}(t^n)$, $\quad +++ \quad \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)}$, $\qquad \Delta t = 2.10^{-4}$

RELATIVE ERROR $\text{size}(\mathcal{T}) \sim 0.016$

$$\frac{\left|\mathcal{S}(t^n) - \|u_{\mathcal{T}}^n\|\right|}{\mathcal{S}(t^n)}$$



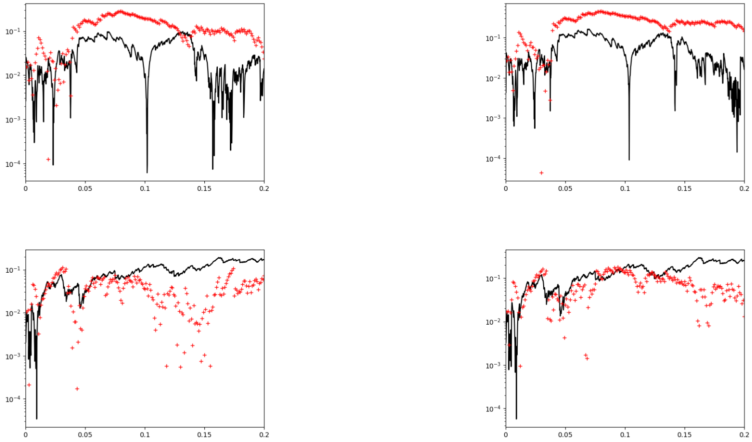


$L = 0$ $L = 1$

$\Delta t = 10^{-3} \Rightarrow \sqrt{\Delta t} \sim 0.03$ $\Delta t = 2.10^{-4} \Rightarrow \sqrt{\Delta t} \sim 0.014$

RELATIVE ERROR $\quad \text{size}(\mathcal{T}) \sim 0.016$

$$\frac{\left|\mathcal{S}(t^n) - \|u_{\mathcal{T}}^n\|\right|}{\mathcal{S}(t^n)}$$

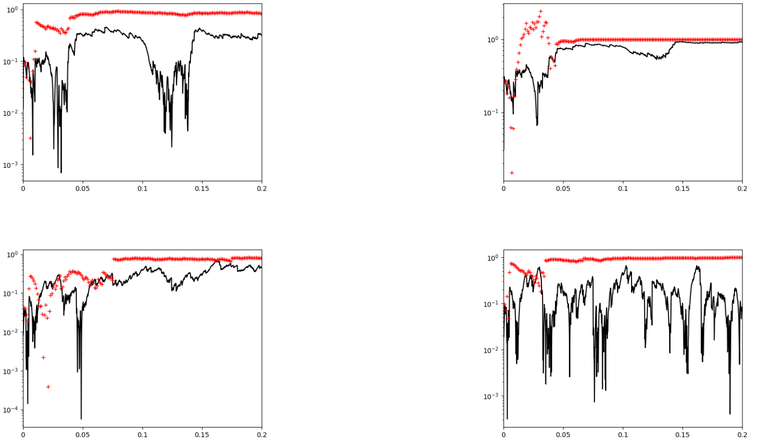


$L = 5$ $\qquad$ $L = 2\pi$

$\Delta t = 10^{-3} \Rightarrow \sqrt{\Delta t} \sim 0.03 \qquad \Delta t = 2.10^{-4} \Rightarrow \sqrt{\Delta t} \sim 0.014$

Relative error size$(\mathcal{T}) \sim 0.016$

$$\frac{\left|\mathcal{S}(t^n) - \|u^n_{\mathcal{T}}\|\right|}{\mathcal{S}(t^n)}$$



$L = 10$ $L = 15$

$\Delta t = 10^{-3} \Rightarrow \sqrt{\Delta t} \sim 0.03$ $\Delta t = 2.10^{-4} \Rightarrow \sqrt{\Delta t} \sim 0.014$

$$\begin{cases} du - \Delta u\,dt = Lu\,dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda. \end{cases}$$

Black Scholes model

$u(t,(x,y)) = \mathcal{S}(t)u_0(x,y)$ with $\|u_0\|_{L^2(\Lambda)} = 1$

$$\mathcal{S}(t) = e^{(-\lambda_2 - \frac{L^2}{2})t} e^{LW(t)} \text{ and } \|u(t,(x,y))\|_{L^2(\Lambda)} = \mathcal{S}(t)$$

- $e^{-\frac{L^2}{2}t + LW(t)}$ is a martingale:

$$\frac{1}{J}\sum_J \|u^n_{\mathcal{T}}\|_{L^2(\Lambda)} \sim \mathbb{E}\left[\|u(t^n,(x,y))\|_{L^2(\Lambda)}\right] = e^{-\lambda_2 t^n}$$

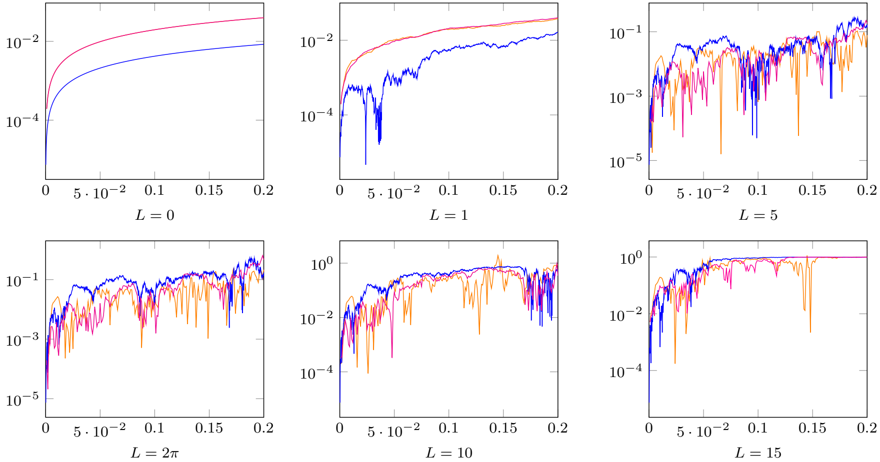
- $\|u(t,(x,y))\|^2_{L^2(\Lambda)} = e^{(-2\lambda_2 - L^2)t} e^{2LW(t)}$ and $e^{-2L^2 t + 2LW(t)}$ is a martingale:

$$\frac{1}{J}\sum_J \|u^n_{\mathcal{T}}\|^2_{L^2(\Lambda)} \sim \mathbb{E}\left[\|u(t^n,(x,y))\|^2_{L^2(\Lambda)}\right] = e^{(L^2 - 2\lambda_2)t^n}$$

J: total number of trajectories

COMPARISON BETWEEN THE MEAN OF THE NORM

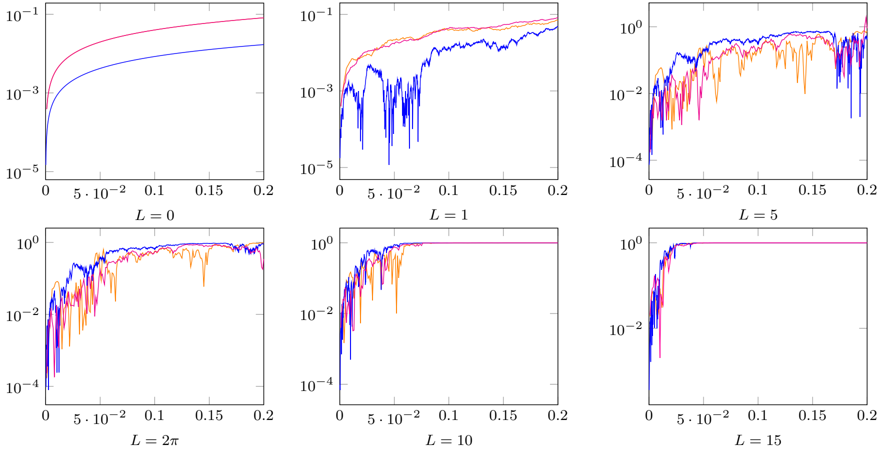
$$\frac{\left|\frac{1}{J}\sum_J \|u_T^n\|_{L^2(\Lambda)} - e^{-\lambda_2 t^n}\right|}{e^{-\lambda_2 t^n}}$$



$\Delta t = 2.10^{-4}, J = 2048$ $\Delta t = 10^{-3}, J = 2048$ $\Delta t = 10^{-3}, J = 5000$

Comparison between the mean of the square of the norm

$$\frac{\left|\frac{1}{J}\sum_J \|u^n_{\mathcal{T}}\|^2_{L^2(\Lambda)} - e^{(L^2-\lambda_2)t^n}\right|}{e^{(L^2-\lambda_2)t^n}}$$



$\Delta t = 2.10^{-4}, J = 2048$ $\Delta t = 10^{-3}, J = 2048$ $\Delta t = 10^{-3}, J = 5000$

The continuous problem

$$\begin{cases} du - \Delta u \, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \quad u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \quad \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

where:

- W is independent standard Brownian motion;
- $\mathrm{div}(\mathbf{b}) = 0$ in $[0,T] \times \Lambda$ and $\mathbf{b} \cdot \mathbf{n} = 0$ on $[0,T] \times \partial\Lambda$.

De Bouard - Goudenège - Nabet

The continuous problem

$$\begin{cases} du - \Delta u\, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

Difference with Itô integral

- integrand evaluated at the midpoint instead of the left-end point:

$$\int_0^T X_s \circ dW_s = \lim_{\Delta t \to 0} \sum_{n=0}^{N-1} \frac{X_{t^n} + X_{t^{n+1}}}{2} \left(W(t^{n+1}) - W(t^n)\right)$$

De Bouard - Goudenège - Nabet

The continuous problem

$$\begin{cases} du - \Delta u \, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

Difference with Itô integral

- evaluated at the midpoint;
- constructed in a way so that the standard-calculus chain rule holds:

$$f(W_t) - f(W_0) = \int_0^t f'(W_s) \circ dW_s$$

instead of

$$f(W_t) - f(W_0) = \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds.$$

$\rightsquigarrow$ natural choice for a variety of models in physics and computational biology.

The continuous problem

$$\begin{cases} du - \Delta u\, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

Difference with Itô integral

- evaluated at the midpoint;
- the standard-calculus chain rule holds;
- the martingale property does not hold :

$$\mathbb{E}\left[\int_0^t X_s \circ dW_s\right] \neq 0$$

$\rightsquigarrow$ the Itô integral widely used in stochastic financial analysis.

The continuous problem

$$\begin{cases} du - \Delta u\, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

Difference with Itô integral

- evaluated at the midpoint;
- the standard-calculus chain rule holds;
- the martingale property does not hold;
- the Itô isometry does not hold :

$$\mathbb{E}\left[\left|\int_0^t X_s \circ dW_s\right|^2\right] \neq \mathbb{E}\left[\int_0^t |X_s|^2 ds\right]$$

De Bouard - Goudenège - Nabet

The continuous problem

$$\begin{cases} du - \Delta u \, dt = \mathbf{b} \cdot \nabla u \circ dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

Difference with Itô integral

- evaluated at the midpoint;
- the standard-calculus chain rule holds;
- the martingale property does not hold;
- the Itô isometry does not hold.

Interest of Stratonovich transport noise

- Regularization: additive noises do not provide enough regularization by noise (for uniqueness of solution)
  $\rightsquigarrow$ introduction to multiplicative noises of transport type (for ex. for 3D incompressible Navier-Stokes equation).
- Fluid dynamical problems: effect of small scales on large scale
  $\rightsquigarrow$ small scale transport noise produces in the limit an extra dissipative term: the "eddy dissipation".
  $\rightsquigarrow$ Turbulent flow: allow to describe the motion of large scale structures, where the noise replaces part of the influence of small scale structures on large scale ones

Time discretization:

$$u^{n+1}(x) - u^n(x) - \Delta t \Delta u^{n+\frac{1}{2}}(x) = \sqrt{\Delta t}\xi^{n+1} b(x) \cdot \nabla u^{n+\frac{1}{2}}(x)$$

A TPFA scheme:

$$\int_{\mathcal{K}} \mathbf{b} \cdot \nabla u = \int_{\mathcal{K}} \operatorname{div}(\mathbf{b}u) = \int_{\partial\mathcal{K}} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} = \sum_{\sigma\in\partial\mathcal{K}} \int_{\sigma} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} \sim \sum_{\sigma\in\partial\mathcal{K}} u_\sigma \underbrace{\int_{\sigma} \mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}}}_{:=b_{\sigma\mathcal{K}}}$$

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^{n}) + \Delta t \sum_{\sigma\in\partial\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(u_{\mathcal{K}}^{n+\frac{1}{2}} - u_{\mathcal{L}}^{n+\frac{1}{2}} \right) = \sqrt{\Delta t} \sum_{\sigma\in\partial\mathcal{K}} b_{\sigma\mathcal{K}} u_\sigma^{n+\frac{1}{2}} \xi^{n+1}$$

TIME DISCRETIZATION:

$$u^{n+1}(x) - u^n(x) - \Delta t \Delta u^{n+\frac{1}{2}}(x) = \sqrt{\Delta t}\xi^{n+1} b(x) \cdot \nabla u^{n+\frac{1}{2}}(x)$$

A TPFA SCHEME:

$$\int_{\mathcal{K}} \mathbf{b} \cdot \nabla u = \int_{\mathcal{K}} \operatorname{div}(\mathbf{b}u) = \int_{\partial\mathcal{K}} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} = \sum_{\sigma\in\partial\mathcal{K}} \int_{\sigma} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} \sim \sum_{\sigma\in\partial\mathcal{K}} u_\sigma \underbrace{\int_{\sigma} \mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}}}_{:=b_{\sigma\mathcal{K}}}$$

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^{n}) + \Delta t \sum_{\sigma\in\partial\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(u_{\mathcal{K}}^{n+\frac{1}{2}} - u_{\mathcal{L}}^{n+\frac{1}{2}}\right) = \sqrt{\Delta t} \sum_{\sigma\in\partial\mathcal{K}} b_{\sigma\mathcal{K}} u_\sigma^{n+\frac{1}{2}} \xi^{n+1}$$

CHOICE FOR u_σ?

- Upwind:

$$u_\sigma = \begin{cases} u_{\mathcal{K}} \text{ if } b_{\sigma\mathcal{K}}\xi^{n+1} \geq 0 \\ u_{\mathcal{L}} \text{ otherwise.} \end{cases}$$

Time discretization:

$$u^{n+1}(x) - u^n(x) - \Delta t \Delta u^{n+\frac{1}{2}}(x) = \sqrt{\Delta t}\xi^{n+1} b(x) \cdot \nabla u^{n+\frac{1}{2}}(x)$$

A TPFA scheme:

$$\int_{\mathcal{K}} \mathbf{b} \cdot \nabla u = \int_{\mathcal{K}} \operatorname{div}(\mathbf{b}u) = \int_{\partial\mathcal{K}} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} = \sum_{\sigma \in \partial\mathcal{K}} \int_{\sigma} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} \sim \sum_{\sigma \in \partial\mathcal{K}} u_\sigma \underbrace{\int_{\sigma} \mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}}}_{:=b_{\sigma\mathcal{K}}}$$

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\sigma \in \partial\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(u_{\mathcal{K}}^{n+\frac{1}{2}} - u_{\mathcal{L}}^{n+\frac{1}{2}} \right) = \sqrt{\Delta t} \sum_{\sigma \in \partial\mathcal{K}} b_{\sigma\mathcal{K}} u_\sigma^{n+\frac{1}{2}} \xi^{n+1}$$

Choice for u_σ?

- Upwind:

$$u_\sigma = \begin{cases} u_{\mathcal{K}} & \text{if } b_{\sigma\mathcal{K}}\xi^{n+1} \geq 0 \\ u_{\mathcal{L}} & \text{otherwise.} \end{cases}$$

- Centered:

$$u_\sigma = \frac{u_{\mathcal{K}} + u_{\mathcal{L}}}{2}$$

Time discretization:

$$u^{n+1}(x) - u^n(x) - \Delta t \Delta u^{n+\frac{1}{2}}(x) = \sqrt{\Delta t}\xi^{n+1} b(x) \cdot \nabla u^{n+\frac{1}{2}}(x)$$

A TPFA scheme:

$$\int_{\mathcal{K}} \mathbf{b} \cdot \nabla u = \int_{\mathcal{K}} \operatorname{div}(\mathbf{b}u) = \int_{\partial\mathcal{K}} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} = \sum_{\sigma\in\partial\mathcal{K}} \int_{\sigma} u\mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} \sim \sum_{\sigma\in\partial\mathcal{K}} u_\sigma \underbrace{\int_{\sigma} \mathbf{b} \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}}}_{:=b_{\sigma\mathcal{K}}}$$

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\sigma\in\partial\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(u_{\mathcal{K}}^{n+\frac{1}{2}} - u_{\mathcal{L}}^{n+\frac{1}{2}}\right) = \sqrt{\Delta t} \sum_{\sigma\in\partial\mathcal{K}} b_{\sigma\mathcal{K}} u_\sigma^{n+\frac{1}{2}} \xi^{n+1}$$

Choice for u_σ?

- Upwind:

$$u_\sigma = \begin{cases} u_{\mathcal{K}} & \text{if } b_{\sigma\mathcal{K}}\xi^{n+1} \geq 0 \\ u_{\mathcal{L}} & \text{otherwise.} \end{cases}$$

- Centered:

$$u_\sigma = \frac{u_{\mathcal{K}} + u_{\mathcal{L}}}{2}$$

- Scharfetter-Gummel??

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1}-u_{\mathcal{K}}^{n})+\Delta t\sum_{\sigma\in\partial\mathcal{K}}\frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}\left(u_{\mathcal{K}}^{n+\frac{1}{2}}-u_{\mathcal{L}}^{n+\frac{1}{2}}\right)=\sqrt{\Delta t}\sum_{\sigma\in\partial\mathcal{K}}b_{\sigma\mathcal{K}}u_\sigma^{n+\frac{1}{2}}\xi^{n+1}$$

On the convergence analysis

- Energy estimates $\Longrightarrow$ $L^\infty(0,T;L^2(\Lambda))$ and $L^2(0,T;H^1(\Lambda))$ (on $u_{\mathcal{T}}^{n+\frac{1}{2}}$) bounds ✓
- Existence ✓
- Link with Itô formulation: $\mathbf{b}\cdot\nabla u\circ dW=\mathbf{b}\cdot\nabla u\,dW+\frac{1}{2}\mathbf{b}\cdot\nabla(\mathbf{b}\cdot\nabla u)\,dt.$
- Semi-discrete scheme: $u^{n+1}=u^n+\Delta t\Delta u^{n+\frac{1}{2}}+\sqrt{\Delta t}b\cdot\nabla u^{n+\frac{1}{2}}\xi^{n+1}$

$$\begin{aligned}\sum_{n=0}^{l}\sqrt{\Delta t}\xi^{n+1}\left(b\cdot\nabla u^{n+\frac{1}{2}},\varphi\right)_{L^2(\Lambda)}&=-\sum_{n=0}^{l}\sqrt{\Delta t}\xi^{n+1}\left(u^{n+\frac{1}{2}},\operatorname{div}(b\varphi)\right)_{L^2(\Lambda)}\\&=-\sum_{n=0}^{l}\sqrt{\Delta t}\xi^{n+1}\left(u^{n},\operatorname{div}(b\varphi)\right)_{L^2(\Lambda)}\\&\quad-\frac{1}{2}\sum_{n=0}^{l}\sqrt{\Delta t}\xi^{n+1}\left(u^{n+1}-u^{n},\operatorname{div}(b\varphi)\right)_{L^2(\Lambda)}\end{aligned}$$

Stochastic PDEs with Itô noise

- Convergence in the general case:

$$\begin{cases} du - \Delta u \, dt + \operatorname{div}\big(\mathbf{v} f(u)\big) \, dt = g(u) \, dW(t) + \beta(u) \, dt, & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda. \end{cases}$$

- More numerical results:
  - with smaller time step;
  - with non linear convertion term;
  - ...

Stochastic PDEs with Stratonovich noise

- Numerical results
- Convergence

Thank you for your attention