Homogenisation of transport problems on graphs

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Summary

(1) A general class of **dynamical transport problems** in $\mathbb{R}^d$.

(2) The **discrete optimal transport** problem on graphs.

(3) **Gradient flows** via Energy Dissipation Inequality.

(4) **Discrete-to-continuum** limits of transport problems.

(1/4) Dynamical Transport Problems in $\mathbb{R}^d$

Dynamical transport problems in $\mathcal{M}_+(\mathbb{R}^d)$.

For a given measurable, lsc function $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we are interested in

$$C_f(\mu_0, \mu_1) := \inf_{(\mu_t, \xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t, \xi_t)\, \mathrm{d}x\, \mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla \cdot \xi_t = 0}_{\text{continuity equation}},\ \mu_{t=i} = \mu_i \right\}$$

where μ_0, $\mu_1 \in \mathcal{M}_+(\mathbb{R}^d)$ are given initial and final measures, $\xi_t := \mu_t v_t$ is the **flux**.

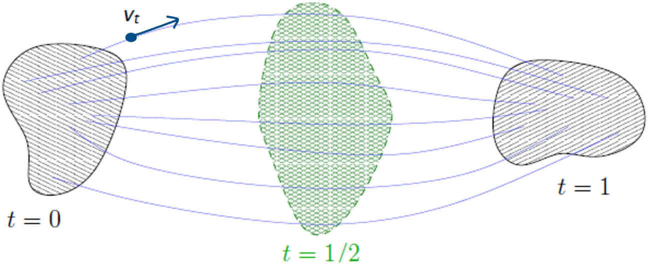

Figure: An evolution $(\mu_t)_t \subset \mathcal{M}_+(\mathbb{R}^d)$ from μ_0 to μ_1 (edited from [Villani, 2009]).

Examples of transport problems (1).

$$C_f(\mu_0,\mu_1) := \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla\cdot\xi_t = 0,\ \mu_{t=i} = \mu_i}_{(\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1)} \right\}$$

- $f(\mu,\xi) = |\xi|^2/\mu$ corresponds to the **(2)-Wasserstein distance** $\mathbb{W}_2$:

$$\mathbb{W}_2(\mu_0,\mu_1)^2 = \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\xi_t|^2}{\mu_t}\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1) \right\}$$

whose dynamical interpretation is due to [Benamou and Brenier, 2000].

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  whose dynamical interpretation is due to [Benamou and Brenier, 2000].

- More general: $f(\mu,\xi) = |\xi|^p/m(\mu)^{p-1}$ for $m : \mathbb{R}^+ \to \mathbb{R}^+$ concave mobility:

  $$\mathbb{W}_{p,m}(\mu_0,\mu_1)^p := \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\xi_t|^p}{m(\mu_t)^{p-1}}\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1) \right\}$$

  are **generalised (p)-Wasserstein distances** [Dolbeault, Nazaret, and Savaré, 2012] .

Examples of transport problems (2).

$$C_f(\mu_0,\mu_1) := \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla\cdot\xi_t = 0,\ \mu_{t=i} = \mu_i}_{(\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1)} \right\}$$

- $f(\mu,\xi) = f(\xi)$ are **flow-based** problems (Beckmann problems). When f is **convex**:

$$\int_0^1 \int_{\mathbb{R}^d} f(\xi_t)\,\mathrm{d}x\,\mathrm{d}t \overset{\text{Jensen}}{\geq} \int_{\mathbb{R}^d} f\left(\underbrace{\int_0^1 \xi_t\,\mathrm{d}t}_{=:\bar\xi} \right) \mathrm{d}x = \int_{\mathbb{R}^d} f(\bar\xi)\,\mathrm{d}x,$$

In this case, one has the equivalent **static** formulation:

$$C_f(\mu_0,\mu_1) = \inf_{\bar\xi} \left\{ \int_{\mathbb{R}^d} f(\bar\xi)\,\mathrm{d}x \ : \ \nabla\cdot\bar\xi = \mu_0 - \mu_1 \right\}.$$

This includes $\mathbb{W}_1$ ($f(\bar\xi) = |\bar\xi|$) and negative Sobolev distance H^{-1} ($f(\bar\xi) = |\bar\xi|^2$).

Motivations.

(1) **Modeling**: optimal transport, traffic flows, congested transport, . . .

(2) Application to PDEs: theory of metric **gradient flows**.

$$\partial_t \mu_t - \nabla \cdot (\mu_t \nabla (\mathsf{D}\mathsf{E}(\mu_t))) = 0, \quad \mathsf{E} : \mathcal{M}_+(\mathbb{R}^d) \to [0, +\infty].$$

[Jordan, Kinderlehrer, and Otto, 1998]: heat flow as gradient flow of the entropy

$$\partial_t \mu_t = \Delta \mu_t, \quad \mathsf{E}(\mu) = \int_{\mathbb{R}^d} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right) \mathrm{d}\mu.$$

(3) Surprising connections with the **Riemannian geometry** (Lott–Villani–Sturm theory).

(4) [Maas, 2011, Mielke, 2011] : generalisation of these ideas to the **discrete setting**.

Discrete-to-continuum problem: the study of the convergence of (rescaled) discrete transport problems (and evolutions) towards a continuous one.

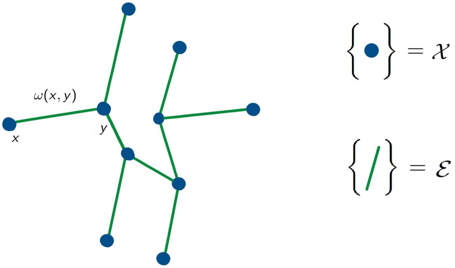
(2/4) **Discrete Optimal Transport**

Optimal transport on discrete spaces.

The dynamical formulation of **(2)-Wasserstein distance** $\mathbb{W}_2$ on $\mathscr{P}_2(\mathbb{R}^d)$:

$$\mathbb{W}_2(\mu_0,\mu_1)^2 = \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\xi_t|^2}{\mu_t} \,\mathrm{d}x\,\mathrm{d}t \;:\; \underbrace{\partial_t\mu_t + \nabla\cdot\xi_t = 0}_{\text{continuity equation}},\ \mu_{t=i} = \mu_i. \right\}$$

Discrete setting: $(\mathcal{X},\mathcal{E},\omega)$ a weighted graph, that is $\mathcal{X}$ finite set of *nodes*, $\mathcal{E}$ set of *edges*, and ω a weight function on $\mathcal{E}$. We fix a reference measure $\pi \in \mathscr{P}(X)$.



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Definition [Maas, 2011] [Mielke, 2011] : for $m_0,m_1\in\mathscr{P}(\mathcal{X})$:

$$\mathcal{W}^\theta(m_0,m_1)^2 := \inf_{(m_t,j_t)}\left\{\int_0^1\frac{1}{2}\sum_{(x,y)\in\mathcal{E}}\frac{1}{\omega(x,y)}\frac{|j_t(x,y)|^2}{\theta\left(\frac{m_t(x)}{\pi(x)},\frac{m_t(y)}{\pi(y)}\right)}\,\mathrm{d}t\right\},$$

where (m_t,j_t) is solution to the **discrete continuity equation** for $x\in\mathcal{X}$:

$$\partial_t m_t(x)+\sum_{y\sim x}j_t(x,y)=0,\quad m_{t=i}=m_i,$$

where $j_t(x,y)=-j_t(y,x)$ (**skew-symmetric**).

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Why the logarithmic average? Maas (2011), Mielke (2011)

$$\mathcal{W}(m_0, m_1)^2 := \inf_{(m_t, j_t)} \left\{ \int_0^1 \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} \frac{1}{\omega(x,y)} \frac{|j_t(x,y)|^2}{\theta_{\log}(r_t(x), r_t(y))} \, \mathrm{d}t \right\}.$$

$$\boxed{\theta_{\log}(r,s) = \frac{r-s}{\log r - \log s}}, \quad r_t(x) := \frac{m_t(x)}{\pi(x)} \text{ (density)}.$$

Consider the discrete entropy functional $\mathcal{E} : (\mathscr{P}(\mathcal{X}), \mathcal{W}) \to \mathbb{R}^+$

$$\mathcal{E}(m) := \sum_{x\in\mathcal{X}} m(x) \log\left(\frac{m(x)}{\pi(x)}\right) = \sum_{x\in\mathcal{X}} r(x) \log r(x)\, \pi(x).$$

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The **gradient flow of** $\mathcal{E}$ in $(\mathscr{P}(\mathcal{X}), \mathcal{W})$ is the graph heat flow

$$\dot{r}_t = \Delta_{\mathcal{X}} r_t, \quad \text{where} \quad \Delta_{\mathcal{X}} r = \sum_{y\sim x} \frac{\omega(x,y)}{\pi(x)} \big(r(y) - r(x)\big) \quad \text{(discrete Laplacian).}$$

(3/4) Gradient flows and Energy Dissipation Inequality (EDI)

Gradient flows: finite dimensional setting

Given a smooth function $E : \mathbb{R}^d \to \mathbb{R}$, its **gradient flow** is described by

$$\begin{cases} \dot{x}_t = -\nabla E(x_t), \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases} \tag{GF}$$

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Energy dissipation: given **any** curve $x = x(t)$, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) = \langle \dot{x}_t, \nabla E(x_t) \rangle \geq -\frac{1}{2} |\dot{x}_t|^2 - \frac{1}{2} |\nabla E(x_t)|^2 .$$

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Curves of maximal slope: x_t solves the ODE in (GF) if and only if for $t > 0$

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Energy Dissipation Inequality (EDI) : solving (GF) is equivalent find x_t such that

$$E(x_T) + \frac{1}{2} \int_0^T |\dot{x}_t|^2 + |\nabla E(x_t)|^2 \,\mathrm{d}t \leq E(x_0) \,.$$

Gradient flows: Wasserstein space

Energy Dissipation Inequality (EDI) formulation of $\partial_t \mu_t - \nabla \cdot (\mu_t \nabla(\mathrm{D}E(\mu_t))) = 0$

$$\boxed{E(\mu_T) + \frac{1}{2}\int_0^T |\dot{\mu}_t|^2_{\mathbb{W}_2} + |\partial_{\mathbb{W}_2} E(\mu_t)|^2 \,\mathrm{d}t \le E(\mu_0)}$$

Let $E : (\mathscr{P}_2(\mathbb{R}^d), \mathbb{W}_2) \to \mathbb{R}^+$ be a given function (for simplicity, convex).

$$|\dot{\mu}_t|_{\mathbb{W}_2} := \lim_{h \to 0} \frac{\mathbb{W}_2(\mu_{t+h}, \mu_t)}{h} \qquad \text{(metric derivative)}$$

$$|\partial_{\mathbb{W}_2} E(\mu)| := \limsup_{\nu \to \mu} \frac{(E(\nu) - E(\mu))_-}{\mathbb{W}_2(\mu, \nu)} \qquad \text{(metric slope)}$$

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Example: $\mathrm{Ent}(\rho \,\mathrm{d}x) = \int \rho \log \rho \,\mathrm{d}x$ corresponds to the heat equation $\partial_t \mu_t = \Delta \mu_t$.

$$|\partial_{\mathbb{W}_2} \mathrm{Ent}(\rho \,\mathrm{d}x)|^2 = \int_{\mathbb{R}^d} |\nabla \log \rho|^2 \,\mathrm{d}\rho \,. \qquad \text{(Fisher info)}$$

Similar in the discrete case, using the discrete entropy $\mathcal{E}$ and distance $\mathcal{W}$.

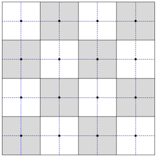
(4/4) Discrete-to-Continuum Limits of Transport Problems and Gradient Flows

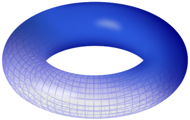
$$\begin{tikzcd}
\begin{array}{c}\textit{Discrete distances}\\ \mathcal{W}_N \textit{ on } \mathscr{P}(\mathcal{X}_N)\end{array} \arrow[r, "\text{conv. distances}", "\mathcal{X}_N \longrightarrow \mathbb{R}^d"'] \arrow[d, dotted, leftrightarrow, "\text{Discrete GF}"] & \begin{array}{c}\textit{continuous distances}\\ \mathbb{W}_2 \textit{ on } \mathscr{P}_2(\mathbb{R}^d)\end{array} \arrow[d, dotted, leftrightarrow, "\text{Continuous GF}"] \\
\begin{array}{c}\textit{Discrete heat flow}\\ \dot{r}_t = \Delta_{\mathcal{X}} r_t\end{array} \arrow[r, "\text{conv. PDEs}"] & \begin{array}{c}\textit{Continuous heat flow}\\ \dot{\rho}_t = \Delta \rho_t\end{array}
\end{tikzcd}$$

- GLADBACH, KOPFER, MAAS, AND P. Homogenisation of one-dimensional discrete optimal transport. *J. Math. Pures Appl. (9)*, 139:204–234, 2020.
- FORKERT, MAAS, P. Evolutionary Γ-convergence of entropic grad. flow structures for Fokker-Planck eq.s in multiple dimensions. *S*IAM Journal on Mathematical Analysis, 2022.
- GLADBACH, KOPFER, MAAS, AND P., *Homogenisation of dynamical optimal transport on periodic graphs*, Calc. Var. PDE, 62(5), Paper No. 143, 75, 2023.
- P. AND F. QUATTROCCHI, *Discrete-to-continuum limits of optimal transport with linear growth on periodic graphs*, to appear in EJAM.
- GLADBACH, MAAS, AND P. , *Stochastic homogenisation of nonlinear minimum-cost flow problems*, in preparation.

Discrete-to-continuum limits of transport problems: some literature.

(1) **First convergence result** [Gigli and Maas, 2013]: transport metrics associated to the cubic mesh on the torus $\mathbb{T}^d$ converge to $\mathbb{W}_2$ in the limit of vanishing mesh size.

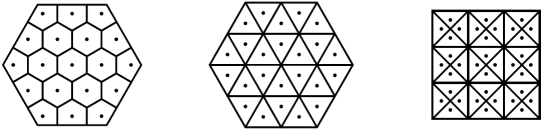




https://en.wikipedia.org/wiki/Torus

(2) **Geometric graphs on point clouds** [García Trillos, 2020]: almost sure convergence of the discrete metrics to $\mathbb{W}_2$, but diverging degree.

(3) **Finite volume partitions** $\mathcal{T}$ in $\mathbb{R}^d$ [Gladbach, Kopfer, and Maas, 2020]: convergence of $\mathcal{W}_{\mathcal{T}}$ to $\mathbb{W}_2$ as size($\mathcal{T}$) $\to 0$ is essentially equivalent to an isotropy condition.



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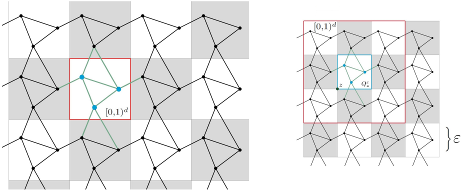
Discrete-to-continuum limits of transport problems: some literature.

(4) **Periodic homogenisation of transport problems** [Gladbach, Kopfer, Maas, and P., 2020 & 2023]: a complete characterisation of the limit costs in a periodic setting.

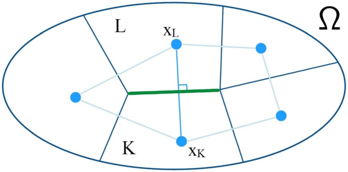


(5) **Convergence of the gradient flows I**: convergence of finite-volume discretisation of diffusions [Disser and Liero, 2015], [Forkert, Maas, and P., 2020] (quadratic) ; [Hraivoronska and Tse, 2023], [Hraivoronska, Schlichting, and Tse, 2023] (cosh); [Cancès, Matthes, Nabet, and Rott, 2022] (nonlinear, p-Wasserstein).

(6) **Convergence of the gradient flows II**: generalised gradient-flow structures associated to jump processes and nonlocal interaction equations [Esposito, Patacchini, Schlichting, and Slepčev, 2021], [Esposito, Patacchini, and Schlichting, 2023b], [Esposito, Heinze, and Schlichting, 2023a].

A typical discretisation: finite-volume partitions of euclidean domains

Standard finite-volume setup (e.g. [Eymard, Gallouët, and Herbin, 2000]) : $\Omega \subset \mathbb{R}^d$ open, bounded and convex, $\mathcal{T} = \{K, x_K\}$ regular partition of Ω.



- Reference measure: $\pi(x_K) = \mathscr{L}^d(K)$.
- Weights: $\omega_{\mathfrak{F}}(x_K, x_L) = \dfrac{\mathscr{H}^{d-1}(\partial K \cap \partial L)}{|x_K - x_L|}$.
- Average: $\theta_{\log}(u, v) = \dfrac{u - v}{\log u - \log v}$.

This uniquely define a discrete distance that we denote by $\mathcal{W}_{\mathcal{T}}$, given by

$$\mathcal{W}_{\mathcal{T}}(m_0, m_1)^2 := \frac{1}{2} \inf \left\{ \int_0^1 \sum_{x \in \mathcal{T}} \sum_{y \sim x} \frac{1}{\omega_{\mathfrak{F}}(x, y)} \frac{|j_t(x, y)|^2}{\theta_{\log}\left(\frac{m_t(x)}{\pi(x)}, \frac{m_t(y)}{\pi(y)}\right)} \mathrm{d}t \ : \ (m_t, j_t)_t \in \mathrm{CE}_{\mathcal{T}}(m_0, m_1) \right\}$$

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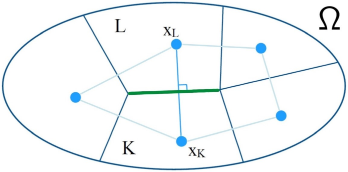

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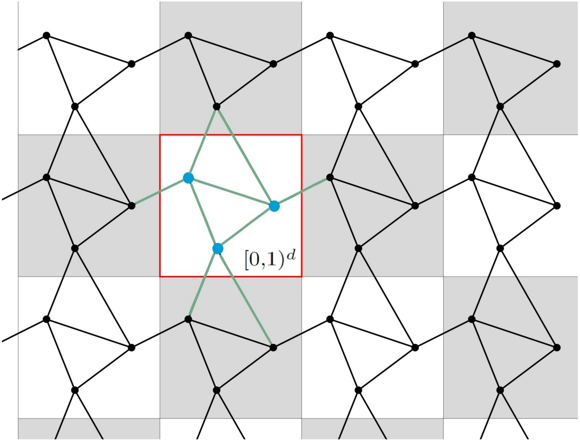
The discrete heat flow converges to the continuous one as size$[\mathcal{T}] \to 0$.

But: $\mathcal{W}_{\mathcal{T}}$ does NOT always converge to $\mathbb{W}_2$ (**isotropy needed**)!

Discrete-to-continuum: transport on periodic graphs.

Discrete-to-continuum: transport on periodic graphs.

Setting: $\mathbb{Z}^d$-periodic, symmetric, connected, and locally finite graph $(\mathcal{X}, \mathcal{E})$ in $\mathbb{R}^d$.



$$\{\bullet\} =: \mathcal{X} \cap [0,1)^d$$

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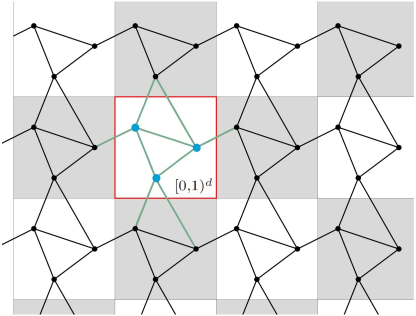
Discrete-to-continuum: transport on periodic graphs.

Given a convex, **local** function $f : \mathcal{M}_+(\mathcal{X}) \times \mathbb{R}^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$, we consider

$$\mathcal{C}_f(m_0, m_1) := \inf \left\{ \int_0^1 f(m_t, j_t)\,\mathrm{d}t \;:\; \partial_t m_t(x) + \sum_{y \sim x} j_t(x,y) = 0,\; j_t \text{ skew-sym.} \right\}$$

among $j_t \in \mathbb{R}^{\mathcal{E}}_{\text{per}}$ and $m_t \in \mathcal{M}_+^{\text{per}}(\mathcal{X})$, satisfying b.c. $m_{t=0} = m_0$, $m_{t=1} = m_1$.



$\{\bullet\} =: \mathcal{X} \cap [0,1)^d$

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Transport on periodic graphs: some examples.

$$\mathcal{C}_f(m_0, m_1) := \inf \left\{ \int_0^1 f(m_t, j_t)\,\mathrm{d}t \ : \ (m_t, j_t)_t \in \mathsf{CE}_{\mathcal{X}}(m_0, m_1) \right\}$$

- The **edge-based** case corresponds to the choice

$$f(m, j) = \frac{1}{2} \sum_{x \in \mathcal{X} \cap [0,1)^d} \sum_{y \sim x} f_{xy}(m(x), m(y), j(x,y)).$$

The $\mathfrak{m}$-Wasserstein-like distances are obtained using quadratic functions

$$f_{xy}(m, n, j) = \frac{1}{\omega(x,y)} \frac{|j|^2}{\mathfrak{m} \circ \theta\left(\frac{m}{\pi(x)}, \frac{n}{\pi(y)} \right)}, \quad m, n \in \mathbb{R}^+, \ j \in \mathbb{R}.$$

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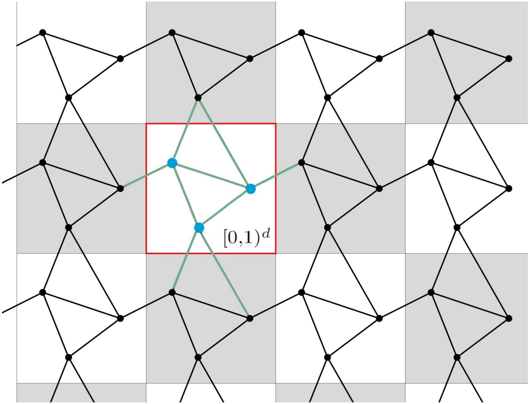
- The **flow-based** case corresponds to the choice $f(m, j) = F(j)$ and

$$\mathcal{C}_f(m_0, m_1) = \inf \left\{ F(j) \ : \ \sum_{y \sim x} j(x, y) = m_0 - m_1 \right\}.$$

Transport on periodic graphs: the convergence result.

$$\mathcal{C}_f(m_0, m_1) := \inf \left\{ \int_0^1 f(m_t, j_t)\,\mathrm{d}t \;:\; (m_t, j_t)_t \in \mathsf{CE}_{\mathcal{X}}(m_0, m_1) \right\}$$



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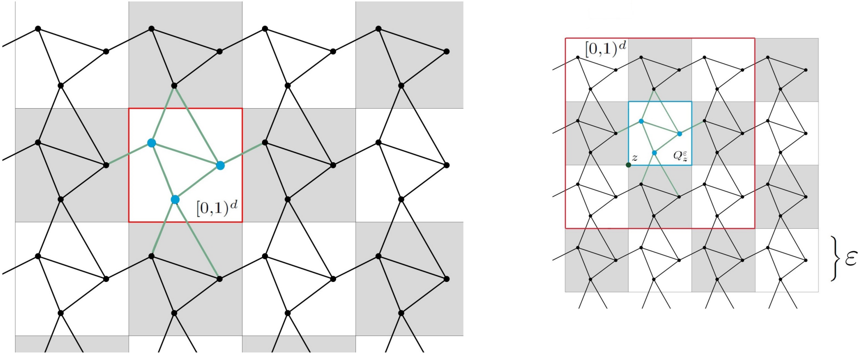
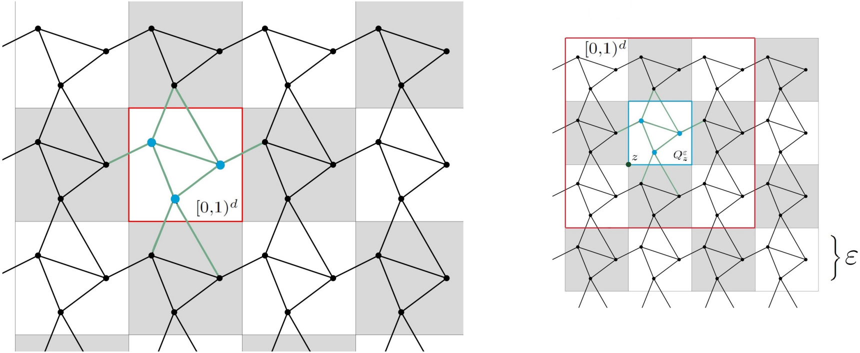


Figure: One the right, the rescaled graph $\mathcal{X}_\varepsilon = \varepsilon\mathcal{X}$, $\mathcal{E}_\varepsilon = \varepsilon\mathcal{E}$, for $\frac{1}{\varepsilon} \in \mathbb{N}$.

Transport on periodic graphs: the convergence result.

$$\mathcal{C}_f^\varepsilon(m_0, m_1) := \inf\left\{ \int_0^1 \sum_{z\in\mathbb{T}_\varepsilon^d} \varepsilon^d f\left(\frac{m_t(\cdot - z)}{\varepsilon^d}, \frac{j_t(\cdot - z)}{\varepsilon^{d-1}}\right) \mathrm{d}t \ : \ (m_t, j_t)_t \in \mathsf{CE}_{\mathcal{X}_\varepsilon}(m_0, m_1) \right\}$$



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Theorem (Gladbach, Kopfer, Maas, and P., 2020; 2023)

Assume f is convex, lower semicontinuous, with superlinear growth$^{(*)}$ in j. Then $\mathcal{C}_f^\varepsilon$ Γ-converges in the weak*-topology as $\varepsilon \to 0$ to a continuous problem

$$C_{\text{hom}}(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{T}^d} f_{\text{hom}}\left(\frac{\mathrm{d}\mu_t}{\mathrm{d}x}, \frac{\mathrm{d}\xi_t}{\mathrm{d}x} \right) \mathrm{d}x \, \mathrm{d}t \ : \ \partial_t \mu_t + \nabla \cdot \xi_t = 0, \ \mu_{t=i} = \mu_i \right\},$$

where f_{hom} is given by a cell problem depending on f and the initial graph $(\mathcal{X}, \mathcal{E})$.

- The $d = 1$, quadratic case: [Gladbach, Kopfer, Maas, and P., JMPA (2020)], with very different techniques (interpolation).

Application: periodic finite-volume partitions.

$$\mathcal{W}_\theta(m_0,m_1)^2 := \frac{1}{2}\inf\left\{ \int_0^1 \sum_{x\in\mathcal{X}}\sum_{y\sim x} \frac{1}{\omega_{\mathfrak{F}}(x,y)} \frac{|j_t(x,y)|^2}{\theta\left(\frac{m_t(x)}{\pi(x)},\frac{m_t(y)}{\pi(y)}\right)}\,\mathrm{d}t \ : \ (m_t,j_t)_t \in \mathsf{CE}_{\mathcal{X}}(m_0,m_1)\right\}$$

where we choose: $\omega_{\mathfrak{F}}(x,y) := \dfrac{\mathscr{H}^{d-1}(\partial K_x \cap \partial K_y)}{|y-x|}\,, \quad \pi(x) := \mathscr{L}^d(K_x)\,.$

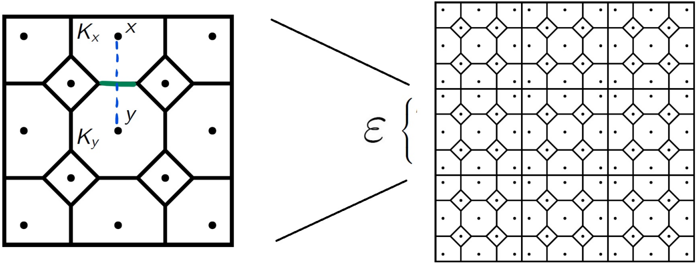

Figure: Periodic finite-volume partition of $\mathbb{T}^d$.

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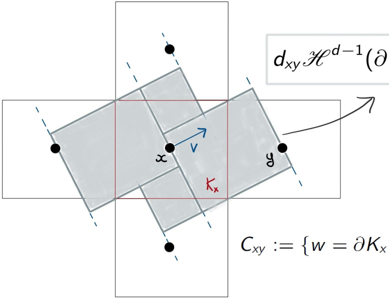
In this setting, the **isotropy condition** reads as, $n_{xy} := (y-x)/|y-x|$,

$$\frac{1}{2}\sum_{y\sim x} d_{xy}\mathscr{H}^{d-1}(\partial K_x\cap\partial K_y)n_{xy}\otimes n_{xy} = |K_x|\mathrm{id},\quad \forall x\in\mathcal{X}.$$



$$d_{xy}\mathscr{H}^{d-1}(\partial K_x\cap\partial K_y)(n_{xy}\cdot v)^2$$

$$C_{xy} := \{w = \partial K_x\cap\partial K_y + \mathbb{R}v \;:\; w\cdot v\in \mathrm{conv}(x\cdot v, y\cdot v)\}$$

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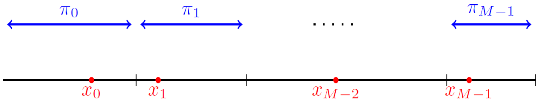
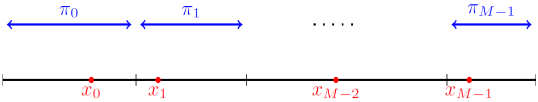


One-dimensional: $\mathcal{W}_\theta$ converges as $\varepsilon \to 0$ to $\mathbb{W}_{\mathrm{hom}} = f_{\mathrm{hom}}(1,1)\mathbb{W}_2$, where

$$f_{\mathrm{hom}}(\mu, \xi) = \frac{|\xi|^2}{\mu} f_{\mathrm{hom}}(1,1), \quad f_{\mathrm{hom}}(1,1) = \inf \left\{ \sum_{k=0}^{M-1} \frac{|x_{k+1} - x_k|}{\theta\left(\frac{m_k}{\pi_k}, \frac{m_{k+1}}{\pi_{k+1}}\right)} \ : \ \|m\| = 1 \right\} \leq 1.$$

Application: periodic finite-volume partitions.

$$\mathcal{W}_\theta(m_0,m_1)^2 := \frac{1}{2}\inf\left\{\int_0^1 \sum_{x\in\mathcal{X}}\sum_{y\sim x}\frac{1}{\omega_{\mathfrak{F}}(x,y)}\,\frac{|j_t(x,y)|^2}{\theta\left(\frac{m_t(x)}{\pi(x)},\frac{m_t(y)}{\pi(y)}\right)}\,\mathrm{d}t \;:\; (m_t,j_t)_t\in \mathsf{CE}_{\mathcal{X}}(m_0,m_1)\right\}$$



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Multidimensional: $\mathcal{W}_\theta$ converges as $\varepsilon \to 0$ to $\mathbb{W}_{\mathrm{hom}}$, where

$$\mathbb{W}^2_{\mathrm{hom}}(\mu_0,\mu_1) = \left\{\int_0^1\int_{\mathbb{T}^d} f_{\mathrm{hom}}(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \;:\; (\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1)\right\}$$

and $f_{\mathrm{hom}}(\mu,\xi) = \frac{\|\xi\|^2_{\mathrm{hom}}}{\mu} \leq \frac{|\xi|^2}{\mu}$ with $\mathbb{W}_{\mathrm{hom}} = \mathbb{W}_2$ if and only if the mesh is isotropic.

Discrete flow problems in a random setting.

We study flow-based problems with **random energy density** on a **random graph**:

(1) a **stationary random graph** : $\omega \in (\Omega, \mathcal{F}, \mathbb{P}) \mapsto (\mathcal{X}_\omega, \mathcal{E}_\omega)$ (vertices, edges) so that

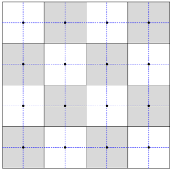
$$\forall z \in \mathbb{Z}_d \,, \quad \mathrm{Law}(\mathcal{X}_\omega + z, \mathcal{E} + z) = \mathrm{Law}(\mathcal{X}_\omega, \mathcal{E}_\omega) \qquad (\textit{periodic in law})\,.$$

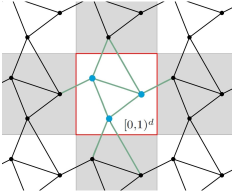
(2) a **stationary energy**: $\omega \in (\Omega, \mathcal{F}, \mathbb{P}) \mapsto F_\omega = F_\omega(J, A)$, $A \subset \mathbb{R}^d$, and study

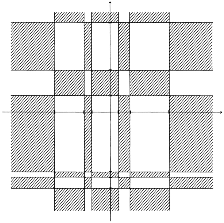
$$C_{\omega,A}(m_0, m_1) = \inf\left\{F_\omega(J, A) \ : \ \mathrm{Div} J = m_0 - m_1\right\}\,, \quad m_0, m_1 \in \mathscr{P}(\mathcal{X}_\omega)\,.$$

Typical example are $\mathbb{W}_1$ is random environment, i.e.

$$\mathrm{F}_\omega(J, A) := \sum_{(x,y)\in\mathcal{E}_\omega} \omega_{xy} \|J(x,y)\| \mathscr{H}^1([x,y] \cap A)\,, \qquad \omega_{xy} \text{ iid conductances}\,.$$







Stochastic homogenisation of linear growth problems.

Rescaling: for $\varepsilon > 0$, set $\mathcal{X}_{\omega,\varepsilon} := \varepsilon\mathcal{X}_\omega$, $\mathcal{E}_{\omega,\varepsilon} := \varepsilon\mathcal{E}_\omega$ and define

$$F_{\omega,\varepsilon}(J, A) := \varepsilon^d \mathsf{F}_\omega \left(\frac{J(\varepsilon\cdot, \varepsilon\cdot)}{\varepsilon^{d-1}}, \frac{1}{\varepsilon} A \right) .$$

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Theorem (Gladbach, Maas, P. (2024+))

Let $m_{i,\varepsilon} \in \mathscr{P}(\mathcal{X}_\varepsilon)$ so that $m_{i,\varepsilon} \to \mu_i \in \mathscr{P}(\mathbb{R}^d)$. Assume that

$$\forall A \subset \mathbb{R}^d, \quad F_\omega(\cdot, A) \text{ is Lipschitz and with linear growth.}$$

Then, $\mathbb{P}$-almost surely, $C_{\omega,\varepsilon,A}$ **Γ-converge** *as $\varepsilon \to 0$ (weak topology) to $\mathbb{C}_{\omega,A,\mathsf{hom}}$, where*

$$\mathbb{F}_{\omega,\mathsf{hom}}(\xi, A) = \int_A f_{\omega,\mathsf{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) \mathrm{d}\mathscr{L}^d + \int_A f^\infty_{\omega,\mathsf{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}\right) \mathrm{d}|\xi|^s .$$

where $f_{\omega,\mathsf{hom}} : \mathbb{R}^d \to \mathbb{R}$ is some homogenised energy density (cell formula).

Main tool: the blow-up method à la Fonseca–Müller.

Multi-cell formula in the stochastic setting: computing $f_{\omega,\mathrm{hom}}$.

$f_{\omega,\mathrm{hom}}$: limit of **cell problems** on **on large cubes**. For $\xi \in \mathbb{R}^n \otimes \mathbb{R}^d$ and $A \subset \mathbb{R}^d$,

$$f_\omega(\xi, A) = \inf \{ \mathsf{F}_\omega(J, A) \ : \ J \in \mathrm{Rep}(\xi, A) \} \ ,$$

where the set of representatives of ξ on A is given by

$$\mathrm{Rep}(\xi, A) := \left\{ J \in \mathbb{R}_a^{\mathcal{E}_\omega} \ : \ \mathrm{Div} J = 0 \quad \text{and} \quad "J = \xi" \text{ on } \partial A \right\} .$$

The **homogenised energy density** is computed as

$$f_{\omega,\mathrm{hom}}(\xi) := \lim_{N \to \infty} \frac{f_\omega(\xi, NQ)}{|NQ|} \ . \tag{1}$$

Existence by **subadditive ergodic theorem** [Akcoglu-Krengel '81; Dal-Maso Modica '86]:

$$f_\omega(\xi, A) \leq \sum_{i \in \mathbb{N}} f_\omega(\xi, A_i) \, , \quad A = \bigcup_{i \in \mathbb{N}} , \quad \{A_i\}_{i \in \mathbb{N}} \text{ disjoint} \, , \quad \xi \in \mathbb{R}^n \otimes \mathbb{R}^d \, .$$

Possible future directions

- Discrete-to-continuum limits of (generalised) gradient flows.
- Stochastic homogenisation for time dependent transport problems.
- Beyond the periodic case and optimal transport on manifolds.

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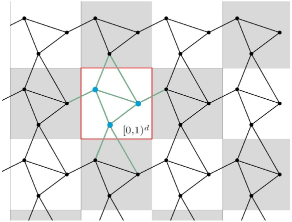

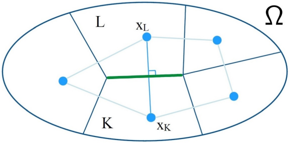

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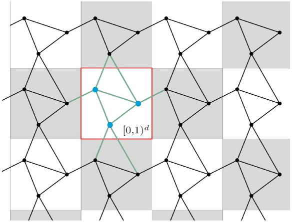

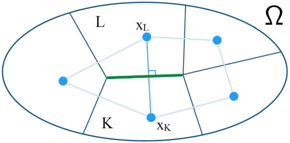

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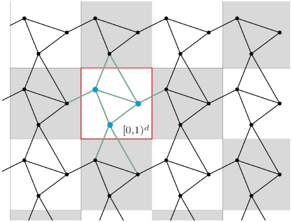
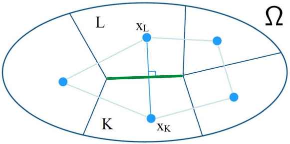




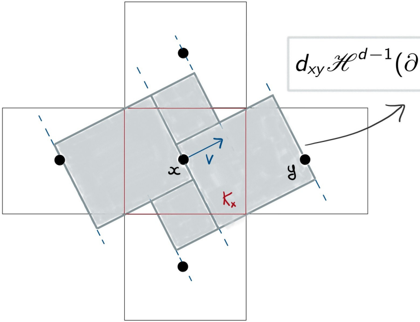
The role of isotropy in the periodic setting

Theorem (multidimensional): $\mathcal{W}_\theta$ converges as $\varepsilon \to 0$ to $\mathbb{W}_{\mathrm{hom}}$, where

$$\mathbb{W}^2_{\mathrm{hom}}(\mu_0,\mu_1) = \left\{ \int_0^1 \int_{\mathbb{T}^d} f_{\mathrm{hom}}(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t,\xi_t)_t \in \mathrm{CE}(\mu_0,\mu_1) \right\}, \quad \text{where}$$

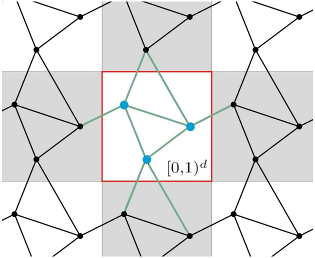
- $\mathbb{W}_{\mathrm{hom}} = \mathbb{W}_2$ if and only if the mesh is isotropic: in the periodic setting, it reads

$$\frac{1}{2}\sum_{y\sim x} d_{xy}\mathscr{H}^{d-1}(\partial K_x \cap \partial K_y) n_{xy} \otimes n_{xy} = |K_x|\mathrm{id}, \quad \forall x \in \mathcal{X}.$$



$d_{xy}\mathscr{H}^{d-1}(\partial K_x \cap \partial K_y)(n_{xy}\cdot v)^2$

The cell problem: a formula for the limit f_{hom}.



For $m \in \mathcal{M}_+^{\mathrm{per}}(\mathcal{X})$ and $\mathbb{Z}^d$-periodic $j \in \mathbb{R}_a^{\mathcal{E}}$, define:

$$\|m\| := \sum_{x \in \mathcal{X} \cap [0,1)^d} m(x) \in \mathbb{R}^+,$$

$$\mathrm{Eff}(j) := \frac{1}{2} \sum_{x \in \mathcal{X} \cap [0,1)^d} \sum_{y \sim x} j(x,y)(y-x) \in \mathbb{R}^d,$$

$$\operatorname{div} j(x) := \sum_{y \sim x} j(x,y)\,.$$

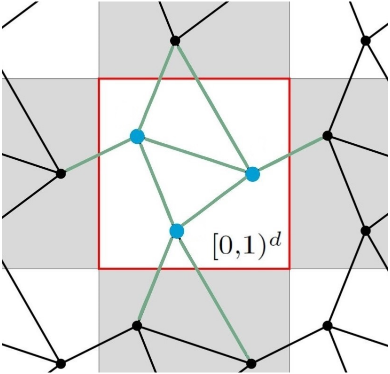
Cell problem: for any $\rho \in \mathbb{R}^+$, $\xi \in \mathbb{R}^d$, the limit cost is given by

$$\boxed{f_{\mathrm{hom}}(\rho,\xi) := \inf_{m,j} \left\{ f(m,j) \;:\; \|m\| = \rho,\ \mathrm{Eff}(j) = \xi,\ \operatorname{div} j = 0 \right\}}$$

where the inf is taken over $m \in \mathcal{M}_+^{\mathrm{per}}(\mathcal{X})$ and $\mathbb{Z}^d$-periodic, skew-sym. $j \in \mathbb{R}^{\mathcal{E}}$.

An example of a competitor for the cell problem

Example: $\rho = 5$, and $\xi = (2,3) \in \mathbb{R}^2$. We can obtain a representative of ρ, ξ as follows:



Mass, effective flux, and discrete divergence:

$$\|m\| := \sum_{x \in \mathcal{X} \cap [0,1)^d} m(x) \in \mathbb{R}^+,$$

$$\mathrm{Eff}(j) := \frac{1}{2} \sum_{x \in \mathcal{X} \cap [0,1)^d} \sum_{y \sim x} j(x,y)(y-x) \in \mathbb{R}^d,$$

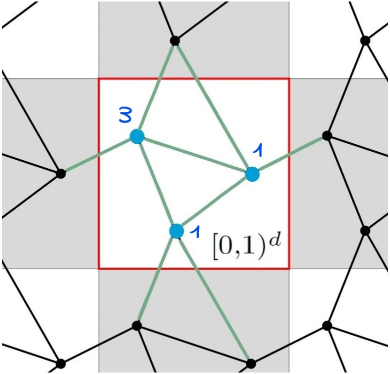
$$\operatorname{div} j(x) := \sum_{y \sim x} j(x,y)\,.$$

Cell problem:

$$\boxed{f_{\mathrm{hom}}(\rho,\xi) := \inf_{m,j} \left\{ f(m,j) \ : \ \|m\| = 5,\ \mathrm{Eff}(j) = (2,3),\ \operatorname{div} j = 0 \right\}}$$

An example of a competitor for the cell problem

Example: $\rho = 5$, and $\xi = (2,3) \in \mathbb{R}^2$. We can obtain a representative of ρ, ξ as follows:



Mass, effective flux, and discrete divergence:

$$\|m\| := \sum_{x \in \mathcal{X} \cap [0,1)^d} m(x) \in \mathbb{R}^+,$$

$$\mathrm{Eff}(j) := \frac{1}{2} \sum_{x \in \mathcal{X} \cap [0,1)^d} \sum_{y \sim x} j(x,y)(y-x) \in \mathbb{R}^d,$$

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Cell problem:

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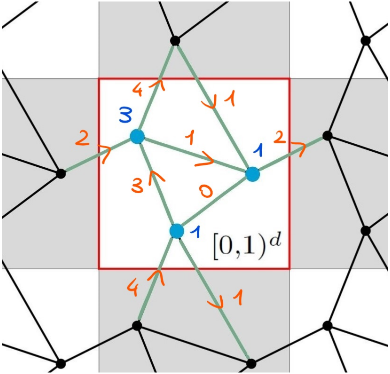

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About the proof: the blow-up method.

Liminf: on a bounded domain $U \subset \mathbb{R}^d$, for $J_\varepsilon \to \xi$, $\mathrm{Div} J_\varepsilon = m_\varepsilon \to \mu$, we must show

$$\liminf_{\varepsilon \to 0} F_{\omega,\varepsilon}(J_\varepsilon, \overline{U}) \geq \mathbb{F}_{\omega,\mathrm{hom}}(\xi, \overline{U}) = \int_{\overline{U}} f_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) \mathrm{d}\mathscr{L}^d + \int_{\overline{U}} f^\infty_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}\right) \mathrm{d}|\xi|^s \,.$$

Blow-up technique á la Fonseca–Müller:

$$\nu_\varepsilon := F_{\omega,\varepsilon}(J_\varepsilon, \cdot) \to \nu \in \mathcal{M}_+(\overline{U}) \quad \Longrightarrow \quad \nu(\overline{U}) = \lim_{\varepsilon \to 0} \nu_\varepsilon(\overline{U}) = \liminf_{\varepsilon \to 0} F_{\omega,\varepsilon}(J_\varepsilon, \overline{U}) \,.$$

We write the Radon–Nikodym decomposition of ν and ξ

$$\xi = \frac{\mathrm{d}\xi}{\mathrm{d}x}\mathscr{L}^d + \xi^s \qquad \text{and} \qquad \nu = \frac{\mathrm{d}\nu}{\mathrm{d}x}\mathscr{L}^d + \frac{\mathrm{d}\nu}{\mathrm{d}|\xi|}|\xi|^s + \nu^{ss} \,.$$

The liminf inequality $\nu(\overline{U}) \geq F_{\omega,\mathrm{hom}}(\xi, \overline{U})$ would follow if

$$f_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) \leq \frac{\mathrm{d}\nu}{\mathrm{d}x} \qquad \mathscr{L}^d - \text{a.e.}\,, \tag{\textbf{AC}}$$

$$f^\infty_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}\right) \leq \frac{\mathrm{d}\nu}{\mathrm{d}|\xi|} \qquad |\xi|^s - \text{a.e.}\,. \tag{\textbf{S}}$$

The role of isotropy in the periodic setting

Theorem (multidimensional): $\mathcal{W}_\theta$ converges as $\varepsilon \to 0$ to $\mathbb{W}_{\mathsf{hom}}$, where

$$\mathbb{W}^2_{\mathsf{hom}}(\mu_0, \mu_1) = \left\{ \int_0^1 \int_{\mathbb{T}^d} f_{\mathsf{hom}}(\mu_t, \xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t, \xi_t)_t \in \mathsf{CE}(\mu_0, \mu_1) \right\}, \quad \text{where}$$

- $f_{\mathsf{hom}}(\mu, \xi) = \frac{\|\xi\|^2_{\mathsf{hom}}}{\mu} \leq \frac{|\xi|^2}{\mu}$, where $\|\cdot\|_{\mathsf{hom}}$ is a norm (possibly not Riemannian!)

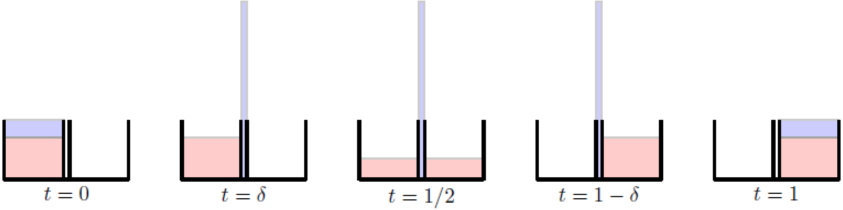

Figure: Strongly oscillating measures on the graph scale can be cheaper.