Block Jacobi, Schwarz, and Discontinuous Galerkin

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Paris, ~~March 30th to April 1st, 2020~~
June 8th-10th, 2022
joint work with Soheil Hajian

GATIPOR:
Interplay of discretization and algebraic solvers ...

Block Jacobi Methods

Poisson equation as model problem:

$$\begin{aligned} -\Delta u &= f, \qquad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0, \qquad \text{on } \partial\Omega. \end{aligned}$$

Discretization leads to a linear system of equations:

$$A\mathrm{u} = \mathrm{f},$$

where u is the vector of degrees of freedom representing approximations of u and possibly ∇u.

Block Jacobi with two non-overlapping subblocks:

$$M\mathrm{u}^{n+1} = N\mathrm{u}^n + \mathrm{f},$$

$$M = \left[\begin{array}{c|c} A_1 & O \\ \hline O & A_2 \end{array}\right], \; N = -\left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & O \end{array}\right].$$

A simple example in 1D

Discretization of the Possion equation with finite differences:

$$Au = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \mathrm{f}$$

Block Jacobi written componentwise:

$$A_1\mathrm{u}_1^{n+1} = \mathrm{f}_1 - A_{12}\mathrm{u}_2^n, \qquad A_2\mathrm{u}_2^{n+1} = \mathrm{f}_2 - A_{21}\mathrm{u}_1^n$$

with the transmission matrices

$$A_{12} = \begin{bmatrix} & & \\ & & \\ \frac{1}{h^2} & & \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} & & \frac{1}{h^2} \\ & & \\ & & \end{bmatrix}.$$

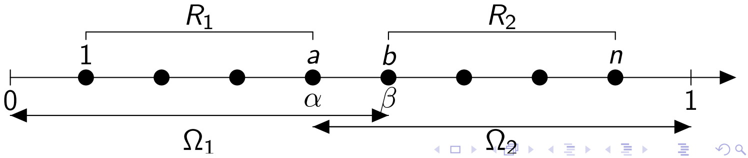
From Block Jacobi to Schwarz

So for the first subblock, $A_1 \mathbf{u}_1^{n+1} = \mathbf{f}_1 - A_{12}\mathbf{u}_2^n$ becomes

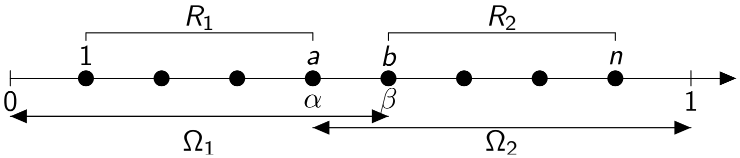
$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_{1,1}^{n+1} \\ u_{1,2}^{n+1} \\ \vdots \\ u_{1,b-1}^{n+1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{b-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{h^2} u_{2,b}^n \end{bmatrix}$$

and for the second subblock, $A_2 \mathbf{u}_2^{n+1} = \mathbf{f}_2 - A_{21}\mathbf{u}_1^n$ becomes

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_{2,a+1}^{n+1} \\ u_{2,a+2}^{n+1} \\ \vdots \\ u_{1,n}^{n+1} \end{bmatrix} = \begin{bmatrix} f_{a+1} \\ f_{a+2} \\ \vdots \\ f_n \end{bmatrix} - \begin{bmatrix} \frac{1}{h^2} u_{1,a}^n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



From Block Jacobi to Schwarz



Result: The non-overlapping block Jacobi method

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{pmatrix} \mathrm{u}_1^{n+1} \\ \mathrm{u}_2^{n+1} \end{pmatrix} = \begin{bmatrix} 0 & -A_{12} \\ -A_{21} & 0 \end{bmatrix} \begin{pmatrix} \mathrm{u}_1^{n} \\ \mathrm{u}_2^{n} \end{pmatrix} + \begin{pmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \end{pmatrix}$$

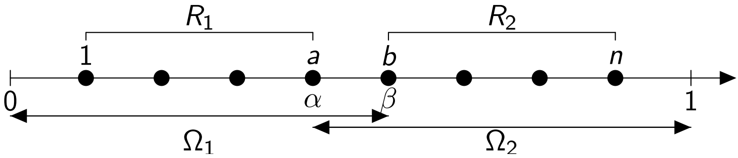
is a classical finite difference discretization of Lions' parallel Schwarz method from 1988 with minimal overlap h,

$$\begin{aligned} \Delta u_1^{n+1} &= f, \text{ in } \Omega_1 \\ u_1^{n+1} &= u_2^n, \text{ at } x = \beta \end{aligned} \qquad \begin{aligned} \Delta u_2^{n+1} &= f, \text{ in } \Omega_2 \\ u_2^{n+1} &= u_1^n, \text{ at } x = \alpha \end{aligned}$$

Holds also for classical FEM discretizations

G.: Schwarz Methods Over the Course of Time, ETNA, 2008

Relation with Additive Schwarz



With the restriction matrices

$$R_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{bmatrix}, \quad R_2 = \begin{bmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

and $A_j = R_j A R_j^T$, Additive Schwarz is defined as

$$\mathrm{u}^{n+1} = \mathrm{u}^n + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(\mathrm{f} - A\mathrm{u}^n)$$

Theorem (G 2008)

If the R_j do not overlap, Additive Schwarz for a Finite Difference or classical Finite Element discretization gives a consistent discretization of the parallel Schwarz method of Lions.

Proof Sketch of this Result

$$u^{n+1} = u^n + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(f - Au^n)$$

contains an interesting cancellation:

$$\begin{aligned} R_2(f - Au^n) &= f_2 - A_{21}u_1^n - A_2 u_2^n \\ A_2^{-1} R_2(f - Au^n) &= A_2^{-1}(f_2 - A_{21}u_1^n) - u_2^n \\ R_2^T A_2^{-1} R_{12}(f - Au^n) &= \begin{pmatrix} 0 \\ A_2^{-1}(f_2 - A_{21}u_1^n) - u_2^n \end{pmatrix} \end{aligned}$$

Similarly

$$R_1^T A_1^{-1} R_1(f - Au^n) = \begin{pmatrix} A_1^{-1}(f_1 - A_{12}u_2^n) - u_1^n \\ 0 \end{pmatrix}$$

Hence

$$u^{n+1} = u^n + \begin{pmatrix} A_1^{-1}(f_1 - A_{12}u_2^n) - u_1^n \\ A_2^{-1}(f_2 - A_{21}u_1^n) - u_2^n \end{pmatrix}$$

Remark: This does not work with more overlap $\Longrightarrow$ **RAS**

Discontinuous Galerkin: SIPG example

$$-\Delta u + \frac{1}{\varepsilon} u = f \quad \text{in } \Omega := (0,1), \qquad u = 0 \quad \text{on } \partial\Omega$$

Variational form $a(u,v) = (f,v)$, and discontinuous approximation space on mesh cells K:

$$V_h := \left\{ v \in L^2(\Omega) \middle| \forall K, v_K \in \mathbb{P}_p(K) \right\}.$$

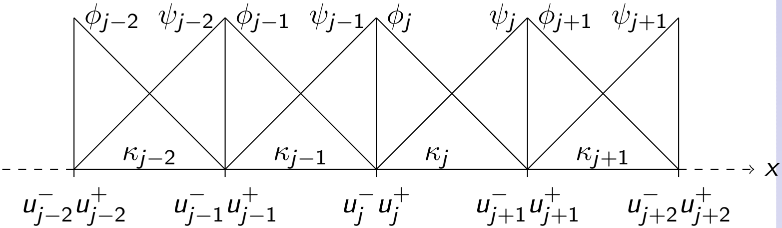
With the jump and average operators

$$[[u]] := u^+ - u^-, \quad \{\{u\}\} := \frac{u^- + u^+}{2},$$

the SIPG bilinear form on $\mathbb{T}$ the union of K is

$$a_h(u,v) := \int_{\mathbb{T}} \nabla u \cdot \nabla v dx + \frac{1}{\varepsilon} \int_{\mathbb{T}} uv dx +$$
$$+ \int_{\mathbb{F}} \left([[u]] \{\{ \tfrac{\partial v}{\partial n} \}\} + \{\{ \tfrac{\partial u}{\partial n} \}\} [[v]] \right) ds + \int_{\mathbb{F}} \mu [[u]] [[v]] ds$$

SIPG Linear System in 1D



$$
Au = \frac{1}{h^2}\begin{pmatrix}
\ddots & \ddots & -\frac{1}{2} & & & \\
\ddots & \ddots & \frac{h^2}{6\varepsilon} & -\frac{1}{2} & & \\
-\frac{1}{2} & \frac{h^2}{6\varepsilon} & \mu_0+\frac{h^2}{3\varepsilon} & 1-\mu_0 & -\frac{1}{2} & \\
 & -\frac{1}{2} & 1-\mu_0 & \mu_0+\frac{h^2}{3\varepsilon} & \frac{h^2}{6\varepsilon} & -\frac{1}{2} \\
 & & -\frac{1}{2} & \frac{h^2}{6\varepsilon} & \ddots & \ddots \\
 & & & -\frac{1}{2} & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix} \vdots \\ u^+_{j-1} \\ u^-_j \\ u^+_j \\ u^-_{j+1} \\ \vdots \end{pmatrix}
=
\begin{pmatrix} \vdots \\ f^+_{j-1} \\ f^-_j \\ f^+_j \\ f^-_{j+1} \\ \vdots \end{pmatrix}
=: \mathrm{f}
$$

where $\mu_0 := \mu h$. **What DD method is Block Jacobi?**

Many More Discontinuous Galerkin Methods

For example based on the mixed form (flux formulation, Arnold, Brezzi SINUM (2002))

$$\boldsymbol{\sigma} = \nabla u, \quad -\nabla \cdot \boldsymbol{\sigma} = f(x), \quad x \in \Omega.$$

Multiplication with test functions $\boldsymbol{\tau}$ and v on each element K and integration by parts leads to

$$\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau})_K &= -(u_h, \nabla \cdot \boldsymbol{\tau})_K + \langle \hat{u}_h, \boldsymbol{\tau} \cdot \mathrm{n}_K \rangle_{\partial K} & \forall \boldsymbol{\tau} \in \mathbb{P}_p(K)^2 \\ (\boldsymbol{\sigma}_h, \nabla v)_K &= (f, v)_K + \langle v, \hat{\boldsymbol{\sigma}}_h \cdot \mathrm{n}_K \rangle_{\partial K} & \forall v \in \mathbb{P}_p(K) \end{aligned}$$

Definition of the numerical fluxes $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}}_h$ leads to many DG methods.

LDG: Local Discontinuous Galerkin method

$$\hat{u}_h = (u_h)_{K_1}, \quad \hat{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_h)_{K_2} - \mu [[u_h]]$$

Hybridizable Variants of DG

Cockburn, Gopalakrishnan, Lazarov, SINUM (2009):

LDG-H: hybridizable variant of LDG

$$\hat{u}_h = \frac{\mu_1}{\mu_1+\mu_2} u_{h,1} + \frac{\mu_2}{\mu_1+\mu_2} u_{h,2} - \frac{1}{\mu_1+\mu_2} [[\boldsymbol{\sigma}_h]]$$

$$\hat{\boldsymbol{\sigma}}_h = \frac{\mu_2}{\mu_1+\mu_2} \boldsymbol{\sigma}_{h,1} + \frac{\mu_1}{\mu_1+\mu_2} \boldsymbol{\sigma}_{h,2} - \frac{\mu_1\mu_2}{\mu_1+\mu_2} [[u_h]]$$

IP-H: a hybridizable variant of Interior Penalty DG

$$\hat{u}_h = \frac{\mu_1}{\mu_1+\mu_2} u_{h,1} + \frac{\mu_2}{\mu_1+\mu_2} u_{h,2} - \frac{1}{\mu_1+\mu_2} [[\nabla u_h]]$$

$$\hat{\boldsymbol{\sigma}}_h = \frac{\mu_2}{\mu_1+\mu_2} \nabla u_{h,1} + \frac{\mu_1}{\mu_1+\mu_2} \nabla u_{h,2} - \frac{\mu_1\mu_2}{\mu_1+\mu_2} [[u_h]]$$

What kind of DD method does one obtain if one applied Block Jacobi to any of these discretizations ???

Equivalence Results

Theorem (G. Hajian 2014)

Block Jacobi applied to **LDG** *is a discretization of the non-overlapping optimized Schwarz method*

$$\begin{aligned} -\Delta u_1^{(n+1)} &= f \quad \text{in } \Omega_1, & -\Delta u_2^{(n+1)} &= f \quad \text{in } \Omega_2 \\ \mathcal{B}_1 u_1^{(n+1)} &= \mathcal{B}_1 u_2^{(n)} \quad \text{on } \Gamma, & \mathcal{B}_2 u_2^{(n+1)} &= \mathcal{B}_2 u_1^{(n)} \quad \text{on } \Gamma \end{aligned}$$

with transmission conditions $\mathcal{B}_1 = \partial_{\mathrm{n}_1} + \mu$ *and* $\mathcal{B}_2 = I$.

Theorem (G. Hajian 2014)

Block Jacobi applied to **LDG-H** *or* **IP-H** *is a discretization of the non-overlapping optimized Schwarz method with transmission conditions* $\mathcal{B}_1 = \partial_{\mathrm{n}_1} + \mu_2$ *and* $\mathcal{B}_2 = \partial_{\mathrm{n}_2} + \mu_1$.

Same results also hold with the reaction term.

Interplay of discretization and algebraic solver

Theorem (G. 2006)

With $\mu = \frac{C}{\sqrt{h}}$, the optimized Schwarz method converges with convergence factor estimate

$$\rho = 1 - O(\sqrt{h}) \quad \textit{(like SOR with } \omega^* \textit{)}$$

With $\mu_1 = \frac{C_1}{h^{1/4}}$ and $\mu_2 = \frac{C_2}{h^{3/4}}$, we get

$$\rho = 1 - O(h^{1/4}) \quad \textit{(much faster than SOR!)}.$$

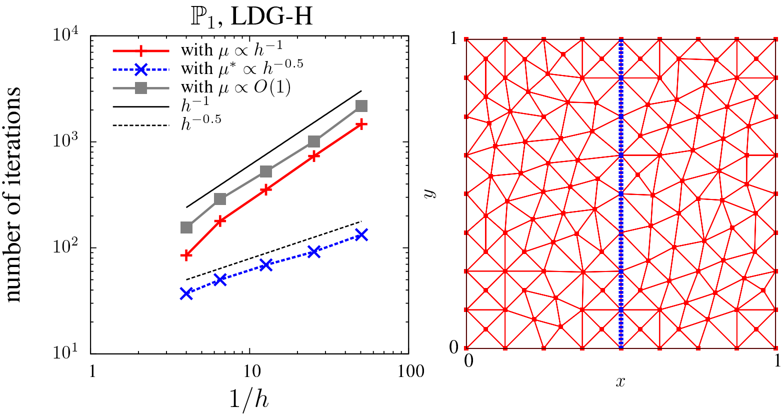
For LDG-H, we can choose $\mu = \frac{C}{\sqrt{h}}$, but for IP-H and LDG we must choose $\mu_j = \frac{C_j}{h}$ for convergence of DG.

Corollary (G. Hajian (2014))

With $\mu_j = \frac{C_j}{h}$, the optimized Schwarz method converges like

$$\rho = 1 - O(h) \quad \textit{(like classical Schwarz).}$$

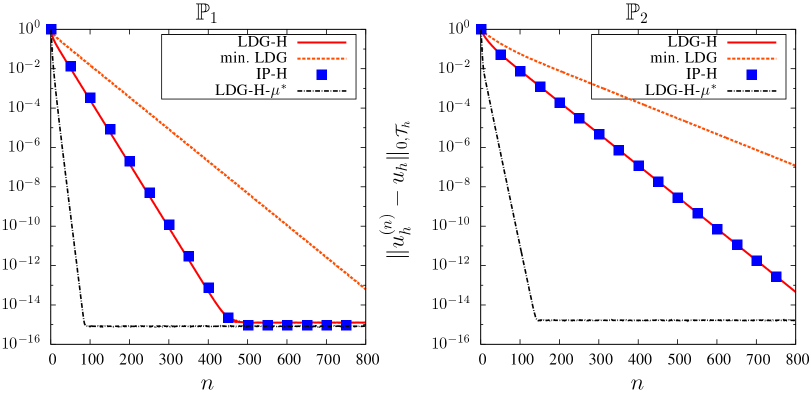
Numerical Experients: LDG-H



Left: asymptotic number of iterations required by the block Jacobi using LDG-H.

Right: unstructured mesh with the interface $\Gamma = \{0.5\} \times (0, 1)$.

LDG-H, LDG and IP-H comparison



Block Jacobi method for LDG-H, LDG , IP-H and LDG-H with μ^* for $\mathbb{P}_1$ and $\mathbb{P}_2$ DG elements.

Is it possible to improve the convergence of block Jacobi applied to LDG and IP-H ?

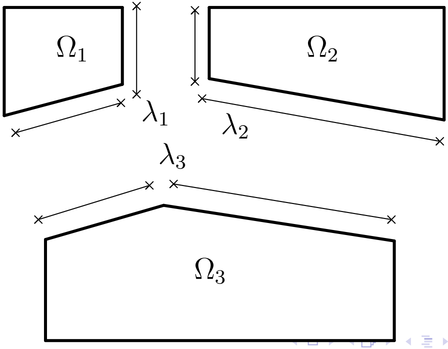
Modified Block Jacobi for IP-H

The continuity condition between K_i and K_j in IP-H states

$$\lambda_h = \frac{1}{2\mu}\left(\mu u_i - \frac{\partial u_i}{\partial \mathsf{n}_i}\right) + \frac{1}{2\mu}\left(\mu u_j - \frac{\partial u_j}{\partial \mathsf{n}_j}\right), \quad \text{on } K_i \cap K_j.$$

On subdomain interfaces Γ_{ij} we introduce the double trace

$$\begin{aligned} \gamma\lambda_i + (1-\gamma)\lambda_j &= \tfrac{1}{2\mu}\left(\mu u_i - \frac{\partial u_i}{\partial \mathsf{n}_i}\right) + \tfrac{1}{2\mu}\left(\mu u_j - \frac{\partial u_j}{\partial \mathsf{n}_j}\right) \\ (1-\gamma)\lambda_i + \gamma\lambda_j &= \tfrac{1}{2\mu}\left(\mu u_i - \frac{\partial u_i}{\partial \mathsf{n}_i}\right) + \tfrac{1}{2\mu}\left(\mu u_j - \frac{\partial u_j}{\mathsf{n}_j}\right) \end{aligned}$$



At the Linear Algebra Level

For a two subdomain example

$$\begin{bmatrix} A_1 & & A_{1\Gamma} \\ & A_2 & A_{2\Gamma} \\ A_{1\Gamma}^\top & A_{2\Gamma}^\top & A_\Gamma \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix},$$

imposing double valued traces $\lambda_1 = \lambda_2 = \lambda$,

$$\begin{array}{rcrcrcrcl} \gamma A_\Gamma \lambda_1 & + & (1-\gamma) A_\Gamma \lambda_2 & + & A_{1\Gamma}^\top u_1 & + & A_{2\Gamma}^\top u_2 & = & 0, \\ (1-\gamma) A_\Gamma \lambda_1 & + & \gamma A_\Gamma \lambda_2 & + & A_{1\Gamma}^\top u_1 & + & A_{2\Gamma}^\top u_2 & = & 0, \end{array}$$

leads to the augmented system

$$\left[\begin{array}{cc|cc} A_1 & A_{1\Gamma} & & \\ A_{1\Gamma}^\top & \gamma A_\Gamma & A_{2\Gamma}^\top & (1-\gamma) A_\Gamma \\ \hline & & A_2 & A_{2\Gamma} \\ A_{1\Gamma}^\top & (1-\gamma) A_\Gamma & A_{2\Gamma}^\top & \gamma A_\Gamma \end{array}\right] \begin{pmatrix} u_1 \\ \lambda_1 \\ u_2 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ f_2 \\ 0 \end{pmatrix}.$$

Can do block Jacobi on this augmented system!

Convergence Estimates

Theorem (G. Hajian 2018)

For the reaction diffusion equation

$$\Delta u - \frac{1}{\varepsilon} u = f,$$

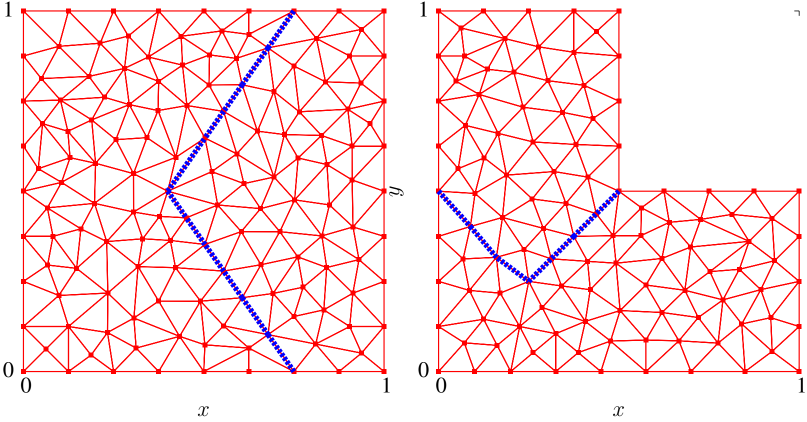
we get the contraction factor estimate

$$\rho_{opt} \leq \begin{cases} 1 - O(\frac{\sqrt{hH}}{p}) & \text{for } \varepsilon = O(1), \quad \text{if } \gamma_{opt} = \frac{1}{2}(1 + \frac{\sqrt{hH}}{p}), \\ 1 - O(\frac{\sqrt{h}}{p}) & \text{for } \varepsilon = O(H), \quad \text{if } \gamma_{opt} = \frac{1}{2}(1 + \frac{\sqrt{h}}{p}), \\ 1 - O(\sqrt{\frac{h}{H}\frac{1}{p}}) & \text{for } \varepsilon = O(H^2), \text{ if } \gamma_{opt} = \frac{1}{2}(1 + \sqrt{\frac{h}{H}\frac{1}{p}}). \end{cases}$$

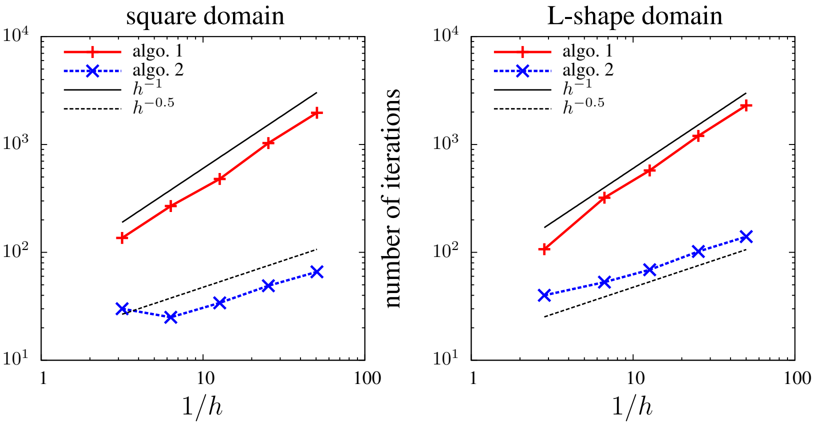
Additive Schwarz applied to the primal formulation of IPH gives

$$\rho \leq 1 - O\left(\frac{hH}{p^2}\right).$$

Classical and New Block Jacobi



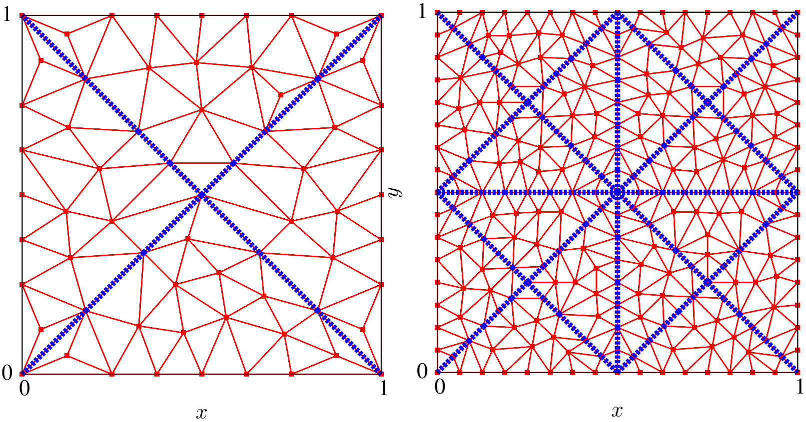
Classical and New Block Jacobi for IP-H



Convergence of classical Block Jacobi (Additive Schwarz, algo. 1) and new Block Jacobi (Optimized Schwarz, algo. 2)

$\mathbb{P}^1$ elements, $\mu_0 = c(p+1)(p+2)$, $c > 0$ a constant independent of h and $p = 1$ (polynomial degree).

The many subdomain case

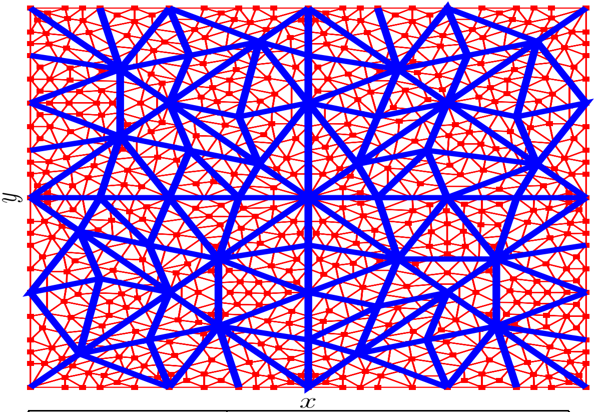


Mesh size (left)	h_0	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
# iterations	25	35	57	82	117

Mesh size (right)	h_0	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
OSM-GMRES	20	52	60	72	87
AS-CG	14	38	55	104	154

Scalability Without Coarse Space

Two Subdomains	h_0	$h_0/2$	$h_0/4$	$h_0/8$
case $\varepsilon = O(1)$	103	214	405	820
case $\varepsilon = O(h)$	41	60	83	115
case $\varepsilon = O(h^2)$	16	16	15	14



H/h constant	h_0	$h_0/2$	$h_0/4$	$h_0/8$
$\varepsilon = O(H^2)$	105	95	99	104

Conclusions

- Block Jacobi for Finite Difference and FEM discretizations are parallel Schwarz methods with minimal overlap of one mesh size.
- Block Jacobi for DG discretizations are optimized Schwarz methods with the penalization parameter as Robin parameter.
- If penalization must be $O(1/h)$, a minor modificatin of the Block Jacobi matrices still permits fast convergence of the corresponding iteration.

Analysis of Schwarz methods for a hybridizable discontinuous Galerkin discretization: the many subdomain case, M.J. Gander, S. Hajian, Math. of Comp., 2018.

Analysis of Schwarz methods for a hybridizable discontinuous Galerkin discretization, M.J. Gander, S. Hajian, SINUM, 2015.

Block Jacobi for discontinuous Galerkin discretizations: no ordinary Schwarz methods, M.J. Gander and S. Hajian, DD21, 2014.