Staggered DG Methods on Polygonal Meshes

Eun-Jae Park

Computational Science & Engineering
School of Mathematics and Computing
Yonsei University, Seoul, Korea

UC-Irvine

ERC Workshop GATIPOR 2020+2
INRIA Paris
June 8-10, 2022
Outline

Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Motivation: Why polygonal meshes?

The interest for general meshes is recently growing:

- Easier/better meshing of domain (and data) features
- Automatic inclusion of "hanging nodes"
- Adaptivity: more efficient mesh refinement/coarsening
- Robustness to mesh distortion
- Topology optimization, Cracks, Fractures
- Interface, Multiphysics
- .......
Some literature

- Mimetic finite difference (MFD) method
- Multipoint flux approximation (MPFA) method
- Polygonal DG (Antonietti, Cangiani, Houston, . . .)
- Hybrid high order (HHO) method (Di Pietro, Ern, . . .)
- Virtual element method (VEM) (Beirão Da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, . . .)
- Weak Galerkin (WG) method (Wang, Ye, . . .)
- Staggered DG method (Zhao and Park, SISC’18)
- .......
Goal: New framework

To develop Staggered DG (SDG) methods of arbitrary polynomial orders on general polygonal meshes that offer the following features:

- Easier/better meshing of domain (and data) features
- Arbitrary shapes of polygon including small edges
- Robust to mesh distortion
- Automatic inclusion of hanging nodes
- No stabilization or penalty terms
- Local and global conservations
- Superconvergence and postprocessing
- Unfitted meshes are allowed
Polygonal SDG

- L. Zhao and E.-J. Park (SISC '18)
  - Inspired from triangular SDG method: Chung and Engquist (SINUM '06,'09)
  - The lowest order polygonal SDG for the Poisson equation
  - Reliable and efficient error estimators

- L. Zhao, E.-J. Park, and D.-w. Shin (CMAME '19)
  - The lowest order polygonal SDG for the Stokes problem
  - Guaranteed error estimators via equilibrated stress recon.

- Dohyun Kim, L. Zhao, and E.-J. Park (SISC '20)
  - Arbitrary high order polygonal SDG for the Stokes problem

- L. Zhao, E.-J. Park (SISC '20)
  - Staggered cell-centered DG for linear elasticity

- L. Zhao, E. Chung, E.-J. Park, and G. Zhou (SINUM '21)
  - Darcy-Forchheimer and Stokes coupling
Polygonal SDG

- L. Zhao, Dohyun Kim, E.-J. Park, and E. Chung (JSC ’22)
  - Darcy flows in **fractured porous media**
- Sanghee Lee, Dohyun Kim, and E.-J. Park - Expanded SDG for anisotropic diffusion: a priori and a posteriori error analysis
- Dohyun Kim, L. Zhao, E. Chung, and E.-J. Park (arXiv’21)
  - **Pressure-robust** SDG for the **Navier-Stokes**
  - Exactly divergence free velocity
  - Arbitrary high order polygonal elements
- L. Zhao, E. Chung, and E.-J. Park (arXiv’20)
  - **Biot’s system** of poroelasticity
  - Arbitrary high order polygonal elasticity elements
Outline

Lowest order SDG method (FVM)
- A priori error estimates
- A posteriori error estimation
- Numerical experiments

Fractured porous media
- A priori error estimates
- Numerical experiments

Concluding remarks and Outlook
Consider the Poisson model problem:

\[-\Delta u = f \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{on } \partial \Omega.\]

By introducing \( p = -\nabla u \), we obtain the first order system

\[p = -\nabla u,
\]

\[\nabla \cdot p = f.\]

- Inspired from standard SDG method on triangular meshes by Chung and Engquist (SINUM 2006,2009)
Staggered grids

Figure: Initial mesh (left) and the resulting mesh (right).

\( \mathcal{T}_u \) denotes the initial (primal) partition of the domain \( \Omega \), 
\( \mathcal{F}_u \) denotes primal edges, \( \mathcal{F}_u^0 \) denotes interior primal edges and \( \mathcal{F}_p \) denotes dual edges. \( \mathcal{T}_h \) denotes the resulting submeshes.
Typical assumptions

1. Every element $S(\nu)$ in $\mathcal{T}_u$ is star-shaped with respect to a ball of radius $\geq \rho h_{S(\nu)}$.

2. For every element $S(\nu) \in \mathcal{T}_u$ and every edge $e \in \partial S(\nu)$, it satisfies $h_e \geq \rho h_{S(\nu)}$, where $h_e$ denotes the length of edge $e$ and $h_{S(\nu)}$ denotes the diameter of $S(\nu)$.

Figure: Shape regularity of a polygon.
Staggered finite element spaces

Figure: Schematic of primal mesh and dual mesh.

Finite element spaces on quadrilateral and polygonal meshes:

\[ S_h := \{ v : v \mid_\tau \in P_0(\tau), \forall \tau \in T_h; [v] |_e = 0, e \in F_u^0, v |_\tau = 0 \]  
\[ \text{if } \partial \tau \cap \partial \Omega = e, e \in F_u \setminus F_u^0 \}, \]

\[ V_h := \{ q : q \mid_\tau \in P_0(\tau)^2, \forall \tau \in T_h; [q \cdot n] |_e = 0, e \in F_p \}, \]

where \( F_u \) denotes primal edges, \( F_u^0 \) denotes interior primal edges and \( F_p \) denotes dual edges.
Degrees of freedom

Figure: Degrees of freedom for $S_h$ (left) and for $V_h$ (right).

$v \in S_h$ is determined by the following degrees of freedom:

$$\phi_e(v) = \int_e v \, ds \quad \forall e \in \mathcal{F}_u.$$

$p \in V_h$ is determined by the following degrees of freedom:

$$\psi_e(p) = \int_e p \cdot n \quad \forall e \in \mathcal{F}_p.$$
Other possible subdivision

Figure: Subdivision into quadrilaterals.

**Disadvantage:** not robust to mesh distortion
SDG formulation

Introduce an auxiliary variable $p = -\nabla u$, we get the first order system:

\[
\begin{align*}
    p &= -\nabla u, \\
    \nabla \cdot p &= f.
\end{align*}
\]

Multiplying by test function $q \in V_h$ and integration by parts over each $S(\nu)$ implies

\[
(p, q)_{S(\nu)} = (u, \nabla \cdot q)_{S(\nu)} - (u, q \cdot n)_{\partial S(\nu)}.
\]

Similarly, we obtain

\[
(p \cdot n, v)_{\partial D(e)} - (p, \nabla v)_{D(e)} = (f, v)_{D(e)}.
\]
Summing the above equations over all $S(\nu)$ and $D(e)$, we can get the discrete formulation: Find $(u_h, p_h) \in S_h \times V_h$ such that

\begin{align*}
(p_h, q) - b^*_h(u_h, q) &= 0 \quad \forall q \in V_h, \\ b_h(p_h, v) &= (f, v) \quad \forall v \in S_h,
\end{align*}

where

\begin{align*}
b^*_h(u_h, q) &= - \sum_{e \in \mathcal{F}_u^0} (u_h, [q \cdot n])_e, \\
b_h(p_h, v) &= \sum_{e \in \mathcal{F}_p} (p_h \cdot n, [v])_e.
\end{align*}
Remark

(Mass conservation) Taking \( v \) in (2) to be identically one in \( D(e) \), we have

\[-(p_h \cdot n_D, 1) \partial D(e) = (f, 1)_{D(e)},\]

where \( n_D \) is the outward unit normal vector of \( D(e) \).
Inf-sup condition

Discrete $H^1$ norm and $H(div, \Omega)$ semi-norm:

$$\|v\|_Z^2 = \sum_{e \in F_p} h_e^{-1} \| [v] \|_{0,e}^2,$$

$$\|q\|_{Z'}^2 = \sum_{e \in F_u^0} h_e^{-1} \| [q \cdot n] \|_{0,e}^2.$$

We have the inf-sup conditions:

$$\inf_{v \in S_h} \sup_{q \in V_h} \frac{b_h(q, v)}{\|v\|_Z \|q\|_0} \geq \beta_1,$$

$$\inf_{q \in V_h} \sup_{v \in S_h} \frac{b^*(v, q)}{\|v\|_0 \|q\|_{Z'}} \geq \beta_2.$$
Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Error estimates

The $L^2$ error estimates with possibly low regularity can be stated in the next theorem.

**Theorem**

Assume that $(p, u) \in (H^\epsilon(\Omega)^2 \cap H(\text{div}, \Omega)) \times H^{1+\epsilon}(\Omega)$, $0 < \epsilon \leq 1$. Let $(p_h, u_h)$ be the numerical solution, then there exists a positive constant $C$ such that

$$
\|u - u_h\|_0 \leq C(h_{\min}^{\{1,2\epsilon\}} \|u\|_{1+\epsilon} + \left( \sum_{\tau \in \mathcal{T}_h} h_{\tau}^2 \|f\|_{0,\tau}^2 \right)^{1/2}) ,
$$

$$
\|p - p_h\|_0 \leq C h^\epsilon \|u\|_{1+\epsilon} .
$$
Let $S_h^* = \{ v | T \in P_1(\tau) \forall \tau \in T_h ; v |_{\partial \Omega} = 0 \}$, then we can define the postprocessing $u_h^* \in S_h^*$

$$(\nabla u_h^*, \nabla v_h)_\tau = (p_h, \nabla v_h)_\tau \quad \forall v_h \in P_1(\tau)/P_0(\tau),$$

$$\frac{(u_h^*, 1)_e}{|e|} = u_h |_e \quad \forall e \in F_u \cap \partial \tau.$$
Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
A posteriori error estimator

Let the local error estimator be defined as

\[ \eta_\tau^2 = \sum_{e \in \mathcal{F} \cap \partial \tau} h_e \| [p_h \cdot t] \|_{0,e}^2 + \sum_{e \in \mathcal{F}_u^0 \cap \partial \tau} h_e \| [p_h \cdot n] \|_{0,e}^2 + h_\tau^2 \| f \|_{0,\tau}^2. \]

Then, the global error estimator can be defined by

\[ \eta^2 = \sum_{\tau \in \mathcal{T}_h} \eta_\tau^2. \]

Theorem

Let \((p, u)\) be the weak solution and \((p_h, u_h)\) be the numerical solution, then there exists a positive constant \(C_{rel}\) such that

\[ \| p - p_h \|_0 \leq C_{rel} \eta. \]
Local efficiency

**Theorem**

Let $f_h$ be a piecewise constant approximation of $f$. Let $(p, u)$ be the solution of the weak problem and $(p_h, u_h)$ be the numerical solution. Then there exists a positive constant $C$ independent of the meshsize such that

$$\eta_\tau \leq C(\|p - p_h\|_{0,D(e)} + \sum_{\tau \in D(e)} h_\tau^2 \|f - f_h\|_{0,\tau}^{2})^{1/2}.$$

Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Smooth solution on general meshes

\[ \Omega = (0, 1)^2 \] and the exact solution is given by

\[ u = x(1 - x)y(1 - y). \]

Trapezoidal grid:

Figure: Partition of \( \Omega \).
Figure: Convergence history for $\alpha = 0$ (left) and $\alpha = 0.4$ (right).
Figure: Convergence history for $\alpha = 0.8$. 
Perturbed grid

Figure: Grids used for simulations. From left to right: (a): Smooth grid. (b): Random $h^2$-perturbation of the smooth grid. (c): Random $h$-perturbation of the smooth grid.
Figure: Convergence history for $h^2$-perturbation (left) and $h$-perturbation (right).
Polygonal mesh

Figure: Partition of $\Omega$ into polygons (left) and convergence history (right).
Singular solution on L shaped domain

\[ u = r^\frac{2}{3} \sin \left( \frac{2\theta}{3} \right) \]

Figure: Initial mesh (left) and convergence history on uniform refinement (right).
Figure: Convergence history for adaptive refinement (left) and adaptive mesh pattern (right).
Strong internal layer on unit square domain

\[ \Omega = (0, 1)^2 \] and the exact solution is given by

\[
    u = 16x(1-x)y(1-y) \arctan(25x - 100y + 50)
\]

Although \( u \) is smooth, it has a strong internal layer along the line \( y = 1/2 + x/4 \).
Figure: Initial mesh (left) and adaptive mesh (right).
Figure: Convergence history for adaptive refinement (left) and adaptive mesh pattern (right).
Outline

Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Fracture Model: (Joint with L. Zhao, D. Kim, and E. Chung, JSC 2022)

\[ \mathcal{B}_1, \mathcal{B}_2, \mathcal{B} \equiv \mathcal{B}_1 \cup \mathcal{B}_2 \]

Figure: Illustration of bulk and fracture domain.
Fracture Model

In the bulk domain:

\[ u + K \nabla p = 0 \quad \text{in } \Omega_B, \]
\[ \nabla \cdot u = f \quad \text{in } \Omega_B, \]
\[ p = p_0 \quad \text{on } \partial \Omega_B. \]  

(3)

On the fracture \(^1\):

\[ -\nabla_t \cdot (K_\Gamma \nabla_t p_\Gamma) = \ell_\Gamma f_\Gamma + [u \cdot n_\Gamma] \quad \text{in } \Gamma, \]
\[ p_\Gamma = g_\Gamma \quad \text{on } \partial \Gamma. \]  

(4)

The jump condition:

\[ \eta_\Gamma \{u \cdot n_\Gamma\} = [p] \quad \text{on } \Gamma, \]
\[ \alpha_\Gamma [u \cdot n_\Gamma] = \{p\} - p_\Gamma \quad \text{on } \Gamma. \]  

(5)

\(^1\)V. Martin, J. Jaffré, and J. E. Roberts, Modeling fractures and barriers as interfaces for flow in porous media, SISC ’05
Figure: A fitted polygonal mesh to the fractured porous media.
Figure: Schematic of staggered mesh. $S(\nu)$ is a primal element and $D(e)$ is a dual element. Here, —— are primal edges $F_{pr}$ and --- are dual edges $F_{dl}$. 
Figure: Schematic of staggered mesh. $S(\nu)$ is a primal element and $D(e)$ is a dual element. Here, —— are primal edges $\mathcal{F}_{pr}$ and ——- are dual edges $\mathcal{F}_{dl}$. 
Figure: Schematic of staggered mesh. $S(\nu)$ is a primal element and $D(e)$ is a dual element. Here, —— are primal edges $F_{pr}$ and --- are dual edges $F_{dl}$.
Figure: Schematic of staggered mesh. $S(\nu)$ is a primal element and $D(e)$ is a dual element. Here, —— are primal edges $F_{pr}$ and --- are dual edges $F_{dl}$.
Discrete space

We introduce three spaces

\[ u_h \in V_h = [\mathbb{P}_k(\mathcal{T}_h)]^2 \cap H(\text{div}; \mathcal{S}(\mathcal{N})) , \]

\[ p_h \in S_h = \mathbb{P}_k(\mathcal{T}_h) \cap H^1(\mathcal{D}(\mathcal{F}_{\text{dl}})), \]

\[ p_{\Gamma,h} \in W_h = \mathbb{P}_k(\mathcal{F}_h^\Gamma) \cap H^1_0(\Gamma). \]
Discrete space - Velocity

Figure: DOFs of quadratic velocity variable on a primal element.

\[ V_h = \{ \psi \in [P_k(T_h)]^2 : \psi|_{S(\nu)} \in H(\text{div}; S(\nu)) \forall \nu \in \mathcal{N} \} \]

\[ = \{ \psi \in [P_k(T_h)]^2 : [\psi \cdot \mathbf{n}] = 0 \forall e \in \mathcal{F}_{dl} \} \]
Discrete space - Pressure

Figure: DOFs of quadratic pressure variable on a dual element.

\[ S_h = \{ v \in \mathbb{P}_k(\mathcal{T}_h) : v|_{\mathcal{D}(e)} \in C^0(\mathcal{D}(e)) \ \forall e \in \mathcal{F}_{pr} \} \]

\[ = \{ v \in \mathbb{P}_k(\mathcal{T}_h) : [v] = 0 \ \forall e \in \mathcal{F}_{pr} \} \]
Figure: DOFs of quadratic pressure variable on a dual element with $e \subset \Gamma$. 
Discrete Space - Fracture

Figure: DOFs of quadratic pressure variable on a dual element with $e \subset \Gamma$. 

\[ W = \{ q_\Gamma \in H_0^1(\Gamma) : q_\Gamma|_e \in P_k(F_{\Gamma}) \} \]
Figure: DOFs of quadratic pressure variable on a dual element with $e \subset \Gamma$.

$$W_h = \{ q_\Gamma \in H^1_0(\Gamma) : q_\Gamma|_e \in \mathbb{P}_k(F_h^\Gamma) \}.$$
The discrete spaces are equipped with norms

\[\|q\|_{1,h}^2 = \|\nabla q\|_{L^2(\mathcal{T}_h)}^2 + \sum_{\tau \in \mathcal{T}_h} \sum_{e \in \mathcal{F}_{dl} \cap \partial \tau} \frac{h_e}{2|\tau|} \|[[q]]\|_{L^2(e)}^2\]

\[\|v\|_{0,h}^2 = \|v\|_{L^2(\mathcal{T}_h)}^2 + \sum_{\tau \in \mathcal{T}_h} \sum_{e \in \mathcal{F}_{pr} \cap \partial \tau} \frac{2|\tau|}{h_e} \|[[v \cdot n]]\|_{L^2(e)}^2\]

\[\|q_\Gamma\|_{1,\Gamma}^2 = \|\nabla q_\Gamma\|_{L^2(\Gamma)}^2\]
Assumptions on Mesh

We consider the following assumptions on the polygonal mesh:

**Assumption (A)** Every $S(\nu) \in T_u$ is **star-shaped with respect to a ball** of radius $\geq \rho_S h_{S(\nu)}$.

$\Rightarrow$ Guarantees **valid** triangulation $T_h$.

**Assumption (B)** For each $S(\nu) \in T_u$ and $e \in \partial S(\nu)$, it satisfies $h_e \geq \rho_E h_{S(\nu)}$.

$\Rightarrow$ Guarantees **shape-regular** triangulation $T_h$. 
Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
We introduce the SDG interpolations by

\[
\langle I_h q - q, \psi \rangle_e = 0 \quad \forall \psi \in P_k(e), \ e \in \mathcal{F}_{pr} \setminus \mathcal{F}_h^\Gamma, \\
\langle (I_h q - q)|_{\Omega_{B,i}}, \psi \rangle_e = 0 \quad \forall \psi \in P_k(e), \ e \in \mathcal{F}_h^\Gamma, \ i = 1, 2, \\
(I_h q - q, \psi)_{\tau} = 0 \quad \forall \psi \in P_{k-1}(\tau), \ \tau \in \mathcal{T}_h
\]

and

\[
\langle (J_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}, \phi \rangle_e = 0 \quad \forall \phi \in P_k(e), \ e \in \mathcal{F}_{dl}, \\
(J_h \mathbf{v} - \mathbf{v}, \phi)_{\tau} = 0 \quad \forall \phi \in P_{k-1}(\tau)^2, \ \tau \in \mathcal{T}_h.
\]
Discrete Formulation

Find \((u_h, p_h, p_{\Gamma,h})\) satisfying for all \((v, q, q_\Gamma) \in V_h \times S_h \times W_h\)

\[
(K^{-1}u_h, v)_{\Omega_B} + b^*_h(p_h, v) = 0, \\
-b_h(u_h, q) + J_h(p_h, q) + c_h((p_h, p_{\Gamma,h}), (q, 0)) = (f, q)_{\Omega_B}, \\
\langle K_{\Gamma} \nabla_t p_{\Gamma,h}, \nabla_t q_\Gamma \rangle_{\Gamma} + c_h((p_h, p_{\Gamma,h}), (0, q_\Gamma)) = \langle \ell_{\Gamma,f_{\Gamma}}, q_\Gamma \rangle_{\Gamma}. 
\]  \tag{6}

Here,

\[
b_h(u_h, q) = - \sum_{e \in F_{dl}} \langle u_h \cdot n, [q] \rangle_e + \sum_{\tau \in T_h} (u_h, \nabla q)_\tau, \\
b^*_h(p_h, v) = \sum_{e \in F^0_{pr}} \langle p_h, [v \cdot n] \rangle_e - \sum_{\tau \in T_h} (p_h, \nabla \cdot v)_\tau \\
+ \sum_{e \in F^\Gamma_h} \langle [p_h(v \cdot n)], 1 \rangle_e.
\]
Discrete Formulation

Find \((u_h, p_h, p_{\Gamma,h})\) satisfying for all \((v, q, q_{\Gamma})\) \(\in V_h \times S_h \times W_h\)

\[
(K^{-1}u_h, v)_{\Omega_B} + b^*(p_h, v) = 0,
\]
\[
-b_h(u_h, q) + J_h(p_h, q) + c_h((p_h, p_{\Gamma,h}), (q, 0)) = (f, q)_{\Omega_B},
\]  
\[
\langle K_\Gamma \nabla_t p_{\Gamma,h}, \nabla_t q_{\Gamma}\rangle_{\Gamma} + c_h((p_h, p_{\Gamma,h}), (0, q_{\Gamma})) = \langle \ell_\Gamma f_{\Gamma}, q_{\Gamma}\rangle_{\Gamma}.
\]  

Here,

\[
J_h(p_h, q) = \sum_{e \in F_h^{\Gamma}} \langle \frac{1}{\eta_{\Gamma}}[p_h], [q]\rangle_e
\]

\[
c_h((p_h, p_{\Gamma,h}), (q, q_{\Gamma})) = \sum_{e \in F_h^{\Gamma}} \langle \frac{1}{\alpha_{\Gamma}}(\{p_h\} - p_{\Gamma,h}), \{q\} - q_{\Gamma}\rangle_e.
\]
Remark on Discrete Operators

- **Discrete adjoint property:**

  \[ b_h(\mathbf{v}, q) = b_h^*(q, \mathbf{v}) \quad \forall \mathbf{v}, q \in V_h \times S_h. \]

- **For given** \( \mathbf{v} \in [H^1(\Omega)]^2 \),

  \[ b_h(\mathbf{v} - J_h \mathbf{v}, q) = 0 \quad \forall q \in S_h \]

  and for given \( q \in H^1(\Omega) \)

  \[ b_h^*(q - I_h q, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in V_h. \]

- **Non-negativity:**

  \[ J_h(q, q) = \sum_{e \in \mathcal{F}_h^\Gamma} \eta_{\Gamma}^{-1} \| [q] \|_{0,e}^2, \]

  \[ c_h(((q, q_\Gamma), (q, q_\Gamma))) = \sum_{e \in \mathcal{F}_h^\Gamma} \alpha_{\Gamma}^{-1} \| \{q\} - q_\Gamma \|_{0,e}^2. \]
Lemma (Discrete inf-sup)

\[
\inf_{q \in S_h} \sup_{v \in V_h} \frac{b_h(v, q)}{\|v\|_{0,h} \|q\|_{1,h}} \geq C.
\]
Stability

Theorem (Stability)

The discrete system (7) admits a unique solution 
\((u_h, p_h, p_{\Gamma,h}) \in V_h \times S_h \times W_h\). Furthermore, there exists a positive constant \(C\) such that

\[
\begin{align*}
&\|K^{-\frac{1}{2}} u_h\|_{0,\Omega_B}^2 + K_{\min} \|p_h\|_{0,\Omega_B}^2 \\
&\quad + \sum_{e \in F^\Gamma_h} \|\eta_{\Gamma}^{-\frac{1}{2}} [p_h]\|_{0,e}^2 + \|K_{\Gamma}^{-\frac{1}{2}} \nabla_t p_{\Gamma,h}\|_{0,\Gamma}^2 \\
&\quad + \sum_{e \in F^\Gamma_h} \|\alpha_{\Gamma}^{-\frac{1}{2}} (\{p_h\} - p_{\Gamma,h})\|_{0,e}^2 \\
&\leq C \left( K_{\min}^{-1} \|f\|_{0,\Omega_B}^2 + K_{\Gamma,\min}^{-1} \|\ell_{\Gamma} f_{\Gamma}\|_{0,\Gamma}^2 \right).
\end{align*}
\]
Theorem (Convergence)

There exists a positive constant $C$ such that

\[
\| K^{-\frac{1}{2}} (J_h u - u_h) \|_{0, \Omega_B} + \| K^\frac{1}{2} \nabla_t (\Pi_h^p p_\Gamma - p_{\Gamma,h}) \|_{0, \Gamma} \\
+ \left( \sum_{e \in \mathcal{F}_h^\Gamma} \| \eta_{\Gamma}^{-\frac{1}{2}} [I_h p - p_h] \|_{0,e}^2 \right)^{\frac{1}{2}} \\
+ \left( \sum_{e \in \mathcal{F}_h^\Gamma} \| \alpha_{\Gamma}^{-\frac{1}{2}} \{ I_h p - p_h \} - (\Pi_h^p p_\Gamma - p_{\Gamma,h}) \|_{0,e}^2 \right)^{\frac{1}{2}} \\
\leq C \left( \| K^{-\frac{1}{2}} (u - J_h u) \|_{0, \Omega_B} + \| \alpha_{\Gamma}^{-\frac{1}{2}} (p_\Gamma - \Pi_h^p p_\Gamma) \|_{0, \Gamma} \right)
\]

where the Ritz projection $\Pi_h^p : H^1_0(\Gamma) \to W_h$ is defined by

\[
\langle K_{\Gamma} \nabla_t \Pi_h^p p_\Gamma, \nabla_t q_{\Gamma,h} \rangle_\Gamma = \langle K_{\Gamma} \nabla_t p_\Gamma, \nabla_t q_{\Gamma,h} \rangle_\Gamma \quad \forall q_{\Gamma,h} \in W_h.
\]
Corollary

Assume that \((u|_\tau, p|_\tau, p_\Gamma|_e) \in H^{k+1}(\tau)^2 \times H^{k+1}(\tau) \times H^{k+1}(e)\) for \(\tau \in \mathcal{T}_h\) and \(e \in \mathcal{F}_h^\Gamma\). Then there exists a positive constant \(C\) such that

\[
\| K^{-\frac{1}{2}} (u - u_h) \|_{0, \Omega_B} \leq C h^{k+1},
\]

\[
\| p_\Gamma - p_{\Gamma,h} \|_{0, \Gamma} \leq C h^{k+1},
\]

\[
\| p - p_h \|_{0, \Omega_B} \leq C h^{k+1}.
\]
Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Example - 1

Figure: Graphs of solutions $p$ and $p_\Gamma$ for Example 1.

\[ p = \begin{cases} 
\sin(4x) \cos(\pi y) & \text{in } \Omega_{B,1}, \\
\cos(4x) \cos(\pi y) & \text{in } \Omega_{B,2}, 
\end{cases} \]
\[ p_\Gamma = \frac{3}{4} \cos(\pi y)(\cos(2)+\sin(2)), \]
We consider two different configuration for the physical constants.

\[ \kappa_n^\Gamma = \begin{cases} 
0.01 & \text{for impermeable case,} \\
1 & \text{for permeable case.}
\end{cases} \]

Other physical parameters are chosen as \( \xi = 3/4, \ell_\Gamma = 0.01, \)
\( K_\Gamma = 1 \) and
\[ K = \begin{pmatrix} 
\kappa_n^\Gamma / (2\ell_\Gamma) & 0 \\
0 & 1
\end{pmatrix}. \]
Mesh configuration

Figure: Uniform triangular (left), rectangular (center), polygonal (right) meshes with comparable mesh sizes for Example 1. Here, dashed lines represent dual edges and red lines are the fracture $\Gamma$. 
Figure: Convergence history for the impermeable case ($K_\Gamma = 0.01$) of Example 1 with $k = 1, 2, 3$. Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Figure: Convergence history for the impermeable case \((K_\Gamma = 0.01)\) of Example 1 with \(k = 1, 2, 3\). Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Figure: Convergence history for the impermeable case ($K_\Gamma = 0.01$) of Example 1 with $k = 1, 2, 3$. Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Convergence History - Permeable

Figure: Convergence history for the permeable case ($K_\Gamma = 1$) of Example 1 with $k = 1, 2, 3$. Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Convergence History - Permeable

Figure: Convergence history for the permeable case ($K_\Gamma = 1$) of Example 1 with $k = 1, 2, 3$. Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Convergence History - Permeable

Figure: Convergence history for the permeable case ($K_\Gamma = 1$) of Example 1 with $k = 1, 2, 3$. Right triangles indicate theoretical convergence rates. Solid lines, dotted lines, and dashed lines are error with triangular, rectangular, and polygonal meshes, respectively.
Small Edge - Mesh Configuration

Figure: Schematic of perturbation. 2 × 2 squares (left), two rectangles and two pentagons after perturbation with \( d = 0.1 \times h_e \) (center), and a resulting mesh from a uniform rectangular mesh with \( h_e = 2^{-3} \) and \( d = 0.1 \times h_e \). The dashed circle is the ball, described in Assumption (A), of an pentagon.

In the following example, we used \( d = 0.001 \times h_e \).
Small Edge vs Rectangle

Figure: Convergence history with uniform rectangular meshes (solid lines) and perturbed meshes with $d = 0.001 \times h_e$ (dashed lines)
Small Edge vs Rectangle

Figure: Convergence history with uniform rectangular meshes (solid lines) and perturbed meshes with $d = 0.001 \times h_e$ (dashed lines)
Small Edge vs Rectangle

Figure: Convergence history with uniform rectangular meshes (solid lines) and perturbed meshes with $d = 0.001 \times h_e$ (dashed lines)
Unfitted Mesh

Figure: Underlying polygonal mesh ($\mathcal{T}_{pr}$, left), modified mesh ($\tilde{\mathcal{T}}_u$) (center) and its magnified view with dual edges (right). The modified mesh contains both sliver elements and small edges.
Figure: Convergence history with fitted (solid lines) and unfitted (dashed lines).
Figure: Convergence history with fitted (solid lines) and unfitted (dashed lines).
Unfitted Mesh - Convergence

Figure: Convergence history with fitted (solid lines) and unfitted (dashed lines).
Numerical Experiments - Curved Fracture

Figure: Fitted mesh using triangles (left) and polygons (right)
Numerical Experiments - Curved Fracture

Figure: Cut mesh from a background mesh
Numerical Experiments - Curved Fracture

Figure: Cut mesh from a background mesh
Figure: Cut mesh and its magnified view
Numerical Experiments - Curved Fracture

Figure: Solution shape (left) and convergence history with respect to degrees of freedom (right)
We set the boundary condition

\[ u \cdot n = 0 \text{ on } \partial \Omega_1 \setminus \Gamma, \quad p = 0 \text{ on } \partial \Omega_2 \setminus \Gamma. \]

We model the injection and production by the source term

\[ f = 10.1 \left( \tanh \left( 200(0.2 - (x^2 + y^2)^{\frac{1}{2}}) \right) - \tanh \left( 200(0.2 - ((x - 1)^2 + (y - 1)^2)^{\frac{1}{2}}) \right) \right). \]
Quarter-Five Spot

We set

\[ K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

and for (1) permeable fracture:

\[ \kappa^\text{n}_\Gamma = 1, \quad \kappa^*_\Gamma = 100 \]

and for (2) impermeable fracture:

\[ \kappa^\text{n}_\Gamma = 0.01, \quad \kappa^*_\Gamma = 1. \]

Background mesh: Uniform rectangular mesh with \( h_e = 2^{-6} \). Cubic polynomials are used.
Figure: Pressure profile for the quarter-five spot problem with permeable (left) and impermeable (right) fracture.
Figure: Pressure profile along $x = y$ for the quarter-five spot problem.
Outline

Lowest order SDG method (FVM)
  A priori error estimates
  A posteriori error estimation
  Numerical experiments

Fractured porous media
  A priori error estimates
  Numerical experiments

Concluding remarks and Outlook
Conclusion and outlook

- Lowest order SDG methods on general meshes (FVM) for Poisson/Stokes/Elasticity problem
- Reliable (and efficient) a posteriori error estimations for Poisson/Stokes equations
- Locking free error estimates for the elasticity problems
- Generalization to high order polynomial approximations (Darcy-Forchheimer and Stokes coupled problem)
- Darcy flows in fractured porous media
- Interface problems and unfitted meshes, small/curved edges
References


Thank You!
ICOSAHOM 2023
August 14-18, 2023
Yonsei University, Seoul, Korea

cf. ICIAM 2023, Tokyo, August 20-25, 2023