FOSLS for parabolic and instationary Stokes equations

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"Interplay of discretization and algebraic solvers: a posteriori error estimates and adaptivity " at Inria, June 8-11, 2022

University of Amsterdam

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1. Parabolic evolution equations
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   - Minimal residual (least squares) discretization
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   - Applications: RBM and Optimal Control
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   - FOSLS
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Parabolic evolution equations

Model problem: Heat equation[1]. With $I := (0, T)$,

$$\begin{cases} \partial_t u - \Delta_x u = f & \text{on } I \times \Omega \\ u = 0 & \text{on } I \times \partial\Omega \\ u(0, \cdot) = u_0 & \text{on } \Omega \end{cases} \tag{1}$$

Traditional approach is time marching. E.g. method of lines; discretize first in space with e.g. fem, and then in time with say trapezoidal rule: Crank–Nicolson. Vice versa: Rothe's method.

Growing interest in simult. space-time variational methods for parabolic problems (monolithic approach), because they are much better suited for a massively parallel implementation, allow for local refinements simultaneously in space and time, and produce numerical approximations from the employed trial spaces which are quasi-best ('Cea's lemma).
Superior in applications where the full time evolution is needed at the same time, as with problems of optimal control or data assimilation. For parameter-dependent problems, reduced basis methods reduce equally well complexity in time.

[1] $-\Delta_x$ may read as general elliptic 2nd order spatial PDO

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Some references space-time methods (for parabolic)

[Andreev, 2013], [Babuška and Janik, 1989], [Babuška and Janik, 1990], [Beranek, Reinholt, and Urban, 2020], [Boiveau, Ehrlacher, Ern, and Nouy 2019], [Devaud, 2020], [Diening and Storn, 2022] [Dyja, Ganapathysubramanian, and van der Zee, 2018], [Gander and Neumüller, 2016], [Gantner and St., 2021], [Gantner and St., 2022], [Gimperlein and Stocek, 2019], [Führer and Karkulik, 2021], [Griebel and Oeltz, 2007], [Gunzburger and Kunoth, 2011], [Loli, Montardini, Sangalli, and Tani, 2019], [Langer and Zank, 2020], [Hofer, Langer, Neumüller, and Schneckenleitner, 2019], [Kestler, Steih, and Urban, 2016], [Langer, Moore, and Neumüller, 2016], [Larsson and Schwab, 2015], [Messner, Schanz, and Tausch, 2014], [Mollet, 2014], [Rekatsinas, 2018], [Schwab and St., 2009], [Schwab and St., 2017], [Steinbach and Zank, 2020], [Neumüller and Smears, 2019], [Steinbach, 2015], [Steinbach and Yang, 2018], [St. and Westerdiep, 2021b], [St., van Venetië, and Westerdiep, 2021], [St. and Westerdiep, 2021a], [van Venetië and Westerdiep, 2021], [van Venetië and Westerdiep, 2021], [Voulis and Reusken, 2018], ...

Being interested in optimally convergent adaptive methods, we focus on methods that are quasi-best w.r.t. mesh-independent norms.

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Simultaneous space-time variational formulation

For Gelfand triple $V \hookrightarrow H \simeq H' \hookrightarrow V'$ on spatial domain Ω, for a.e. $t \in I$, let $a(t;\cdot,\cdot)$ bilinear form on $V \times V$ s.t. for some $\varrho \in \mathbb{R}$

$$|a(t;\eta,\zeta)| \lesssim \|\eta\|_V \|\zeta\|_V \quad (\eta,\zeta \in V) \quad (\textit{boundedness}), \tag{2}$$

$$a(t;\eta,\eta) + \varrho\langle\eta,\eta\rangle \gtrsim \|\eta\|_V^2 \quad (\eta \in V) \quad (\textit{Gårding inequality}). \tag{3}$$

With $A(t) \in \mathcal{L}(V,V')$ by $(A(t)\eta)(\zeta) := a(t;\eta,\zeta)$, given f and u_0, find $u(t)\colon \Omega \to \mathbb{R}$,

$$\begin{cases} \frac{du}{dt}(t) + A(t)u(t) = f(t) \quad (t \in I), \\ u(0) = u_0. \end{cases} \tag{4}$$

Find $u \in X := L_2(I;V) \cap H^1(I;V')$ with $\gamma_0 u = u_0$, s.t. $\forall v \in Y := L_2(I;V)$,

$$(Bu)(v) := \underbrace{\int_I \langle \tfrac{du}{dt}(t), v(t)\rangle dt}_{(\partial_t u)(v):=} + \underbrace{\int_I (A(t)u(t))(v(t))dt}_{(Au)(v):=} = \underbrace{\int_I \langle f(t), v(t)\rangle\, dt}_{f(v):=}$$

Theorem (e.g. [Dautray and Lions, 1992] or [Wloka, 1982])

$(B,\gamma_0) \in \mathcal{L}\mathrm{is}(X, Y' \times H)$.

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Conditioning of $(B = \partial_t + A, \gamma_0) \in \mathcal{L}\mathrm{is}(X, Y' \times H)$

W.l.o.g. $a(t;\eta,\eta) \gtrsim \|\eta\|_V^2$ (coercivity inst. of Gårding).

Then $A \in \mathcal{L}\mathrm{is}(Y, Y')$. $A_s := \frac{1}{2}(A + A')$, $A_a := \frac{1}{2}(A - A')$.

Equip $Y = L_2(I;V)$, $X = L_2(I;V) \cap H^1(I;V')$ with 'energy-norms'

$$\|\cdot\|_Y := \sqrt{(A_s\cdot)(\cdot)}, \quad \|\cdot\|_X := \sqrt{\|\cdot\|_Y^2 + \|\partial_t \cdot\|_{Y'}^2 + \|\gamma_T \cdot\|^2}.$$

Proposition ([St. and Westerdiep, 2021a])

With $\alpha := \|A_a\|_{\mathcal{L}(Y,Y')} = \rho(A_s^{-1}A_a)$,

$$\frac{\|Bu\|_{Y'}^2 + \|\gamma_0 u\|^2}{\|u\|_X^2} \in \left[\frac{1}{1+\frac{\alpha}{2}(\alpha+\sqrt{\alpha^2+4})}, 1+\frac{\alpha}{2}(\alpha+\sqrt{\alpha^2+4})\right].$$

For $\alpha = 0$: [Jovanović and Süli, 2014, Tantardini and Veeser, 2016, Ern, Smears, and Vohralík 2017].
General α: related result in [Ern and Guermond, 2021] not based on energy-norms.

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Minimal residual discretization I

Recall $(B, \gamma_0) \in \mathcal{L}\mathrm{is}(X, Y' \times H)$, but test $\neq$ trial. 'Stable' Petrov-Galerkin discretizations are difficult to construct.

Clearly

$$u = \underset{w \in X}{\operatorname{argmin}} \|Bw - f\|_{Y'}^2 + \|\gamma_0 w - u_0\|^2.$$

Following [Andreev, 2013], for closed subspaces $X^\delta \subset X$, $Y^\delta \subset Y$ take

$$u^\delta := \underset{w \in X^\delta}{\operatorname{argmin}} \|Bw - f\|_{Y^{\delta\prime}}^2 + \|\gamma_0 w - u_0\|^2.$$

Theorem ([St. and Westerdiep, 2021a])

Let $X^\delta \subseteq Y^\delta$ and $\gamma_\delta := \inf_{w \in X^\delta} \frac{\|\partial_t w\|_{Y^{\delta\prime}}}{\|\partial_t w\|_{Y'}} > 0$. Then

$$\|u - u^\delta\|_X \leq \sqrt{\frac{1+\frac{1}{2}\left(\alpha^2+\alpha\sqrt{\alpha^2+4}\right)}{\frac{1}{2}\left(\gamma_\delta^2+\alpha^2+1-\sqrt{(\gamma_\delta^2+\alpha^2+1)^2-4\gamma_\delta^2}\right)}} \inf_{w \in X^\delta} \|u - w\|_X.^{a}$$

[a] $\sqrt{} = 1$ when $\gamma_\delta = 1$ and $\alpha = 0$

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Minimal residual discretization II

$\inf_{\delta \in \Delta} \gamma_\delta > 0$ has been verified for families $(X^\delta)_{\delta \in \Delta}$, $(Y^\delta)_{\delta \in \Delta}$ where X^δ, $Y^\delta = Y^\delta(X^\delta)$ with $\dim Y^\delta \lesssim \dim X^\delta$ are

1. tensor products of 'temporal' and 'spatial' spaces, or
2. spans of collections of such (adaptively selected) tensor products. E.g. coll. of temporal wavelet $\otimes$ spatial wavelet (with Rekatsinas 2018), or temporal wavelet $\otimes$ spatial finite element space (with van Venetië and Westerdiep, 2021)
3. FEM spaces w.r.t. partitions of type $\cup_i [t_i, t_{i+1}] \times$ 'Ω_{h_i}' ('time-slab setting').

With (2) rates as for corr. stationary problem (cf. sparse grids), but implementation quite complex.
Implementation FEM easy, but stability for fully general partitions *not* available.

To get rid of dual norm therefore: FOSLS.

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FOSLS I

Model problem

$$\begin{cases} (\partial_t - \Delta_x)u = f & \text{on } I \times \Omega \\ u = 0 & \text{on } I \times \partial\Omega \\ u(0,\cdot) = u_0 & \text{on } \Omega \end{cases}$$

First order system:

$$G(u, \underset{\sim}{w}) := (\underbrace{\partial_t u + \operatorname{div}_x \underset{\sim}{w}}_{\operatorname{div}(u,\underset{\sim}{w}):=}, -\underset{\sim}{w} - \underset{\sim}{\nabla}_x u, u(0,\cdot)) = (f, \underset{\sim}{0}, u_0).$$

(with $u = 0$ on $I \times \partial\Omega$).

$(B, \gamma_0) \in \mathcal{L}\mathrm{is}(X, Y' \times H)$, $\underset{\sim}{\nabla}_x \in \mathcal{L}(X, \underset{\sim}{L}_2(I \times \Omega))$, $\operatorname{div}_x \in \mathcal{L}(\underset{\sim}{L}_2(I \times \Omega), Y')$

$$\rightsquigarrow \quad G \in \mathcal{L}\mathrm{is}(X \times \underset{\sim}{L}_2(I \times \Omega), Y' \times \underset{\sim}{L}_2(I \times \Omega) \times L_2(\Omega)).$$

[recall: $X := L_2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$, $Y := L_2(I; H_0^1(\Omega))$.]

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FOSLS II

Theorem ([Führer and Karkulik, 2021])

With $U := \{\vec{u} := (u, \underset{\sim}{w}) \in X \times \underset{\sim}{L}_2(I \times \Omega) \colon \operatorname{div} \vec{u} \in L_2(I \times \Omega)\}$
and $L := L_2(I \times \Omega) \times \underset{\sim}{L}_2(I \times \Omega) \times L_2(\Omega)$,

$$\|G\vec{u}\|_L \eqsim \|\vec{u}\|_U.$$

[In [Gantner and St., 2021] gen. ellip. 2nd order spatial PDOs; gen. b.c., $G \in \mathcal{L}\mathrm{is}(U, L)$; replaced $X = L_2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ by $L_2(I; H_0^1(\Omega))$; and by decomposing $f \in Y' = L_2(I; H^{-1}(\Omega))$ as $f = f_1 + \operatorname{div}_x \underset{\sim}{f}_2$, showed

$$(B, \gamma_0)u = (f, u_0) \Longleftrightarrow G\vec{u} = \vec{f} := (f_1, \underset{\sim}{f}_2, u_0). \quad]$$

Advantages:

- For any closed subspace $U^\delta \subset U$, $\vec{u}^\delta := \underset{\vec{v} \in U^\delta}{\operatorname{argmin}} \|G\vec{v} - \vec{f}\|_L$, i.e.,

$$\langle G\vec{u}^\delta, G\vec{v}\rangle_L = \langle \vec{f}, G\vec{v}\rangle_L \quad (\vec{v} \in U^\delta),$$

  is quasi-best approximation from U^δ w.r.t. $\|\cdot\|_U$.
- Bil. form $\langle G\cdot, G\cdot\rangle_L$ is bounded, symmetric and coercive on $U \times U$.
- A post. error estimator $\|\vec{f} - G\vec{u}^\delta\|_L \eqsim \|\vec{u} - \vec{u}^\delta\|_L$.

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Theorem ([Führer and Karkulik, 2021])

With $U := \{\vec{u} := (u, \underset{\sim}{w}) \in X \times \underset{\sim}{L}_2(I \times \Omega) \colon \operatorname{div} \vec{u} \in L_2(I \times \Omega)\}$
and $L := L_2(I \times \Omega) \times \underset{\sim}{L}_2(I \times \Omega) \times L_2(\Omega)$,

$$\|G\vec{u}\|_L \eqsim \|\vec{u}\|_U.$$

[In [Gantner and St., 2021] gen. ellip. 2nd order spatial PDOs; gen. b.c., $G \in \mathcal{L}\mathrm{is}(U, L)$; replaced $X = L_2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ by $L_2(I; H_0^1(\Omega))$; and by decomposing $f \in Y' = L_2(I; H^{-1}(\Omega))$ as $f = f_1 + \operatorname{div}_x \underset{\sim}{f}_2$, showed

$$(B, \gamma_0)u = (f, u_0) \iff G\vec{u} = \vec{f} := (f_1, \underset{\sim}{f}_2, u_0). \quad]$$

Advantages:

- For any closed subspace $U^\delta \subset U$, $\vec{u}^\delta := \underset{\vec{v} \in U^\delta}{\operatorname{argmin}} \|G\vec{v} - \vec{f}\|_L$, i.e.,
$$\langle G\vec{u}^\delta, G\vec{v}\rangle_L = \langle \vec{f}, G\vec{v}\rangle_L \quad (\vec{v} \in U^\delta),$$
is quasi-best approximation from U^δ w.r.t. $\|\cdot\|_U$.
- Bil. form $\langle G\cdot, G\cdot\rangle_L$ is bounded, symmetric and coercive on $U \times U$.
- A post. error estimator $\|\vec{f} - G\vec{u}^\delta\|_L \eqsim \|\vec{u} - \vec{u}^\delta\|_L$.

Appl. FOSLS: Reduced basis method I

Ex. from [Glas Mayerhofer, and Urban, 2017]. $I \times \Omega = (0,1)^2$.
$\partial_t u - \mu_1 \partial_x^2 u + \mu_2 \partial_x u + \mu_3 u = f$,
$f(t,x) := \sin(2\pi x)\big((4\pi^2 + 0.5)\cos(4\pi t) - 4\pi \sin(4\pi t)\big) + \pi \cos(2\pi x)\cos(4\pi t)$,
$u_0(x) := \sin(2\pi x)$ on Ω.

Parameter set $\mathcal{P} := [0.5, 1.5] \times [0,1] \times [0,1] \subset \mathbb{R}^3$. 'Truth' LS solution $\vec{u} = (u, -\partial_x u)$ from U^δ being 2-fold Cartesian product of continuous piecewise bi-cubic functions w.r.t. subdivision of $I \times \Omega$ into squares with mesh-size 2^{-6}.

$\mathcal{P}_{\text{train}}$ is chosen as 17 equidistantly distributed points in $\mathcal{P}$ in each direction.

'Greedy' to construct reduced basis.

Appl. FOSLS: Reduced basis method II

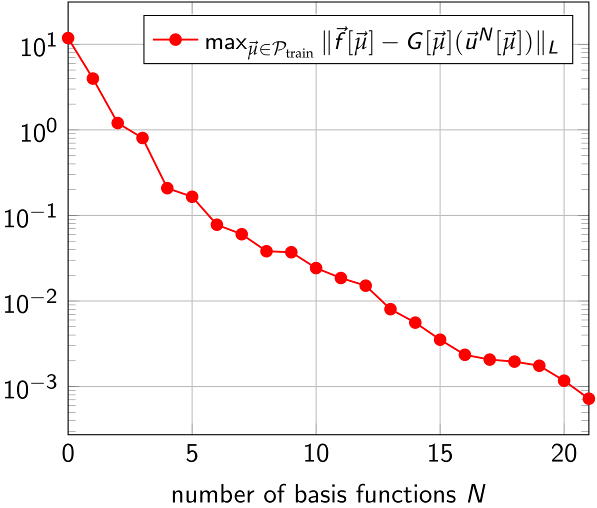

Figure: Offline phase: Exponentially decaying residual norm of greedy algorithm. $\vec{u}^N[\vec{\mu}]$ is Gal. approx. from span$\{\vec{u}^\delta[\vec{\mu}_1], \ldots, \vec{u}^\delta[\vec{\mu}_N]\}$.

Appl. FOSLS: Reduced basis method III

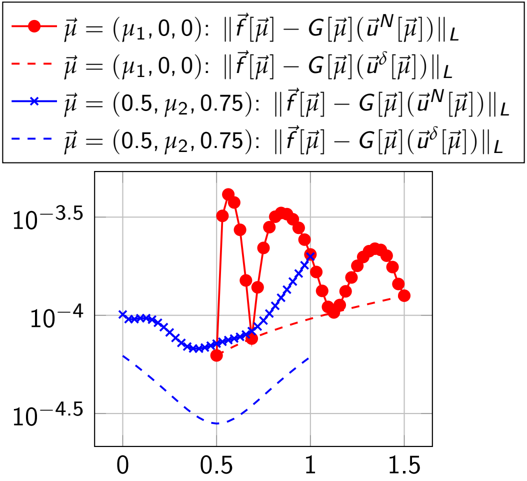

Figure: Online phase: residual norm in 'truth sols' and RB approxs at $\vec{\mu} = (\mu_1, 0, 0)$ with $\mu_1 \in [0.5, 1.5]$ (red), and $\vec{\mu} = (0.5, \mu_2, 0.75)$ with $\mu_2 \in [0, 1]$ (blue) with $N = 21$.

Appl. FOSLS: Reduced basis method IV

Advantages:

- dimension reduction also in time-direction ($\perp$ time marching).
- no POD needed ($\perp$ time marching).
- thanks to coercive bil. form, theory as solid as with Poisson problem.
- faster both in online as in offline phase.

Appl. FOSLS: Optimal control I

Given $\vec{f}^* = (f_1, \underset{\sim}{f}_2, u_0) \in L$, $w^* \in W$ Hilbert, and $F \in \mathcal{L}(U, W)$; Hilbert $Z \hookrightarrow L$ and param $\rho > 0$, minimize

$$J(\vec{u}, \vec{z}) := \tfrac{1}{2}\|F\vec{u} - w^*\|_W^2 + \tfrac{\varrho}{2}\|\vec{z}\|_Z^2 \quad \text{over}$$
$$\big\{(\vec{u}, \vec{z}) \in U \times Z \colon \langle G\vec{u}, G\vec{v}\rangle_L = \langle \vec{f}^* + \vec{z}, G\vec{v}\rangle_L \quad (\vec{v} \in U)\big\}.$$

(latter is FOSLS form. of heat eq. with hom. Dir. bdr. cond., rhs $f_1 + z_1 + \operatorname{div}_x(\underset{\sim}{f}_2 + \underset{\sim}{z}_2)$, and $u(0, \cdot) = u_0 + z_3$)

Rewritten as equiv. saddle-point it yields $(\vec{u}, \vec{z}, \vec{p}) \in U \times Z \times U$.

Discretisation: Replace U, Z by closed subspaces. Thanks to $\langle G\cdot, G\cdot\rangle_L$ coercive, stability uniform in choice subspaces:

$$\begin{aligned}&\|\vec{u} - \vec{u}^\delta\|_U + \|\vec{z} - \vec{z}^\delta\|_Z + \|\vec{p} - \vec{p}^\delta\|_U \\ &\quad\lesssim \frac{1}{\varrho} \inf_{(\vec{v}, \vec{y}, \vec{q}) \in U^\delta \times Z^\delta \times U^\delta} \big(\|\vec{u} - \vec{v}\|_U + \|\vec{z} - \vec{y}\|_Z + \|\vec{p} - \vec{q}\|_U\big).\end{aligned} \tag{5}$$

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Given $\vec{f}^* = (f_1, \underset{\sim}{f}_2, u_0) \in L$, $w^* \in W$ Hilbert, and $F \in \mathcal{L}(U, W)$; Hilbert $Z \hookrightarrow L$ and param $\rho > 0$, minimize

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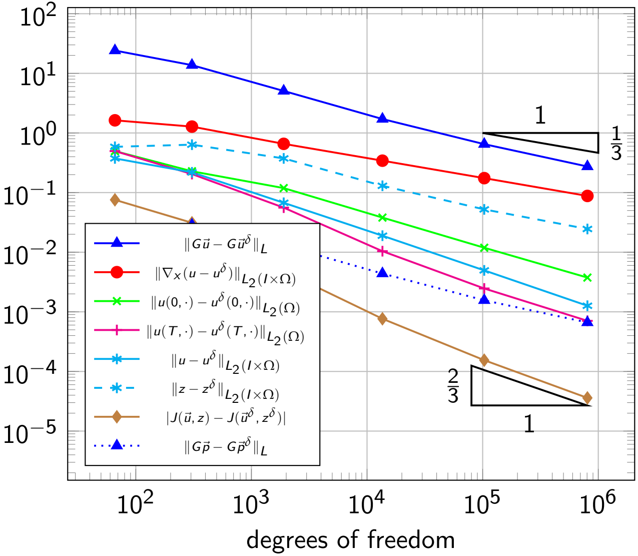
Appl. FOSLS: Optimal control II

Num. ex. FOSLS formulation of opt. control problem that in 2nd order strong form reads as

$$\underset{\{(u,z)\in X\times L_2(I\times\Omega):\, \partial_t u-\Delta_x u=f_1+z\wedge u(0,\cdot)=u_0\}}{\operatorname{argmin}} \tfrac{1}{2}\|u-w^\star\|^2_{L_2(I\times\Omega)}+\tfrac{\varrho}{2}\|z\|^2_{L_2(I\times\Omega)}.$$

Took u, f_1 and $w^\star$ s.t. $\vec{u}$, z, $\vec{p}$ are smooth.
$I\times\Omega=(0,1)^3$. Quasi-uniform subdivision into tetrahedra. U^δ (vectorial) continuous piecewise linears, $Z^\delta\subset Z=L_2(I\times\Omega)$ piecewise constants.

Appl. FOSLS: Optimal control III



FOSLS problem: slow convergence for non-smooth sols

Recall

$$U = \{\vec{u} = (u, \underset{\sim}{w}) \in L_2(I; H_0^1(\Omega)) \times \underset{\sim}{L}_2(I \times \Omega) \colon \operatorname{div} \vec{u} \in L_2(I \times \Omega)\}$$

$$L = L_2(I \times \Omega) \times \underset{\sim}{L}_2(I \times \Omega) \times L_2(\Omega)$$

$$G\vec{u} := (\operatorname{div} \vec{u}, -\underset{\sim}{w} - \nabla_x u, u(0, \cdot)) = (f_1, \underset{\sim}{f}_2, u_0) =: \vec{f}$$

where $\operatorname{div} \vec{u} := \partial_t u + \operatorname{div}_x \underset{\sim}{w}$, and $f_1 + \operatorname{div}_x \underset{\sim}{f}_2 \in Y'$ decomposition rhs parabolic.

Let $\Omega = (0, 1)^d$, U^δ (vectorial) cont. piecewise lin. w.r.t. conf. subdiv. of $I \times \Omega$ into unif. shape reg. $(d + 1)$-simplices. For smooth sols, conv. rate is $\frac{1}{d+1}$.

Take $\underset{\sim}{f}_2 = \underset{\sim}{0}$, $u_0 = 1$. Experiments from [Führer and Karkulik, 2021] show for

- $d = 1$, $f_1 = 2$, rate 0.08 for quasi-unif. part, and 0.17 for adap. refs.
- $d = 2$, $f_1 = 0$, rate 0.07 for quasi-unif. part, and 0.07 for adap. refs.

Trouble maker is $\|\operatorname{div} \vec{\cdot}\|_{L_2(I \times \Omega)}$ in graph norm. With unif. refs., rate $\frac{1}{d+1}$ requires both $\partial_t u$, $\operatorname{div}_x \underset{\sim}{w} = \triangle_x u$ in $H^1(I \times \Omega)$, which *would* reduce to $\operatorname{div} \vec{u} = \partial_t u + \operatorname{div}_x \underset{\sim}{w} = f_1 \in H^1(I \times \Omega)$ when U^δ allows quasi-interp. with comm. diagr.

$H(\text{div}; I \times \Omega)$-elements not applicable.

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$$G\vec{u} := (\operatorname{div} \vec{u}, -\underset{\sim}{w} - \nabla_x u, u(0,\cdot)) = (f_1, \underset{\sim}{f}_2, u_0) =: \vec{f}$$

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$H(\operatorname{div}; I \times \Omega)$-elements not applicable.

Let $\mathcal{P}$ part. of $I \times \Omega$ into prisms $P = J \times K$ for interval J, d-simplex K. Let U^δ space of $(u, \underset{\sim}{w})$ in $\big(H^1(I; L_2(\Omega)) \cap L_2(I; H_0^1(\Omega))\big) \times L_2(I; H(\mathrm{div}; \Omega))$ that restricted to $P \in \mathcal{P}$ are in $P_{\ell+1}(J) \otimes P_{\ell+1}(K) \times P_\ell(J) \otimes RT_{\ell+1}(K)$ for $\ell \in \mathbb{N}_0$.

Proposition

If local patch ω_P is conforming, then for $h_K \eqsim h_J$,

$$\begin{aligned}
\|\operatorname{div}(\vec{u} - \mathcal{I}^{\mathcal{P}}\vec{u})\|_{L_2(J\times T)} &\lesssim h_K^{\ell+1}\|\partial_t^{\ell+1}\operatorname{div}\vec{u}\|_{L_2(J\times T)} + \\
&\quad h_K^{\ell+1}\big(\|\operatorname{div}\vec{u}\|_{L_2(J;H^{\ell+1}(T))} + \|\partial_t u\|_{L_2(J;H^{\ell+1}(\omega_T))}\big) \\
\|u - (\mathcal{I}^{\mathcal{P}}\vec{u})_1\|_{L_2(J;H^1(T))} &\lesssim h_K^{\ell+1}\|\partial_t^{\ell+1}u\|_{L_2(J;H^1(\omega_T))} + h_K^{\ell+1}\|u\|_{L_2(J;H^{\ell+2}(\omega_T))} \\
\|\underset{\sim}{w} - (\mathcal{I}^{\mathcal{P}}\vec{u})_2\|_{L_2(J\times T)^d} &\lesssim h_K^{\ell+1}\|\partial_t^{\ell+1}\underset{\sim}{w}\|_{L_2(J\times T)^d} + h_K^{\ell+1}\|\underset{\sim}{w}\|_{L_2(J;H^{\ell+1}(T)^d)}.
\end{aligned}$$

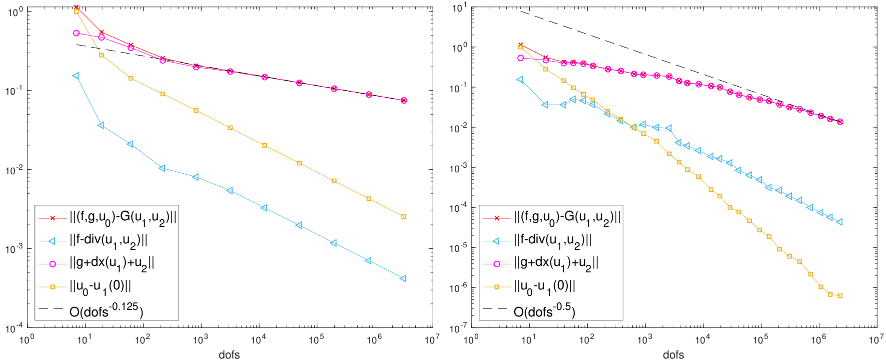
Also when ω_P is not conforming, local quasi-interpolator error $\mathcal{O}(h_K^{\ell+1})$ but under stronger regularity assumptions.

Num. results

Test from [Führer and Karkulik, 2021] for $d = 1$, $u_0 = 1$, $f = 2$.
Lowest order $\ell = 0$, so $P_1(J) \otimes P_1(K) \times P_0(J) \otimes P_2(K)$ $(\star)$.

Rem. $P_1(J) \otimes P_1(K) \times P_1(J) \otimes P_1(K)$ gives rates as in [Führer and Karkulik, 2021], i.e. 0.08 (unif.), 0.17 (adapt).

With $(\star)$, rates 0.125 (unif.), 0.5 (adapt).



Instat. Stokes with slip boundary condition

$\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain. $\underset{\sim}{n} \in \mathbb{R}^n$ normal on $\partial\Omega$. $I := (0, T)$.

$$\begin{cases} \partial_t \underset{\sim}{u} - \nu \Delta_x \underset{\sim}{u} + \nabla_x p = \underset{\sim}{f} & \text{in } I \times \Omega, \\ \operatorname{div}_x \underset{\sim}{u} = 0 & \text{in } I \times \Omega, \\ \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 & \text{on } I \times \partial\Omega, \\ (\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top) \underset{\approx}{T}(\nu \underset{\sim}{u}, p) \underset{\sim}{n} = 0 & \text{on } I \times \partial\Omega, \\ \underset{\sim}{u}(0, \cdot) = \underset{\sim}{u}_0 & \text{on } \Omega. \end{cases}$$

In [Guberovic, Schwab, and St., 2014] well-posed space-time variational 2nd order formulation (but with a co-domain that involves dual spaces).
Deformation and stress tensors $\underset{\approx}{D}(\underset{\sim}{v}) := \underset{\approx}{\nabla}_x \underset{\sim}{v} + (\underset{\approx}{\nabla}_x \underset{\sim}{v})^\top$, $\underset{\approx}{T}(\underset{\sim}{v}, q) := \underset{\approx}{D}(\underset{\sim}{v}) - q\mathrm{Id}$.
2nd bdr. cond. means for $\underset{\sim}{\tau} \perp \underset{\sim}{n}$, $(\underset{\approx}{T}(\nu \underset{\sim}{u}, p)\underset{\sim}{n}) \cdot \underset{\sim}{\tau} = 0 = (\underset{\approx}{D}(\underset{\sim}{u})\underset{\sim}{n}) \cdot \underset{\sim}{\tau}$ on $I \times \partial\Omega$.

From $\underset{\sim}{\operatorname{div}}_x \underset{\approx}{D}(\underset{\sim}{v}) = \underset{\sim}{\Delta}_x \underset{\sim}{v} + \underset{\sim}{\nabla}_x \operatorname{div}_x \underset{\sim}{v} \rightsquigarrow$ first order system

$$G(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p), \underset{\sim}{\partial}_t \underset{\sim}{u} + \underset{\sim}{\operatorname{div}}_x \underset{\approx}{w}, \operatorname{div}_x \underset{\sim}{u}, \underset{\sim}{u}(0, \cdot)) = (0, \underset{\sim}{f}, 0, \underset{\sim}{u}_0),$$

with $\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$ and $(\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{w}\underset{\sim}{n} = 0$ on $I \times \partial\Omega$.

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From $\underset{\sim}{\operatorname{div}}_x \underset{\approx}{D}(\underset{\sim}{v}) = \underset{\sim}{\Delta}_x \underset{\sim}{v} + \underset{\sim}{\nabla}_x \operatorname{div}_x \underset{\sim}{v} \rightsquigarrow$ first order system

$$\mathrm{G}(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p), \partial_t \underset{\sim}{u} + \underset{\sim}{\operatorname{div}}_x \underset{\approx}{w}, \operatorname{div}_x \underset{\sim}{u}, \underset{\sim}{u}(0, \cdot)) = (0, \underset{\sim}{f}, 0, \underset{\sim}{u}_0),$$

with $\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$ and $(\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{w}\underset{\sim}{n} = 0$ on $I \times \partial\Omega$.

Well-posed FOSLS I

Recall $\mathrm{G}(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p), \partial_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}, \mathrm{div}_x\, \underset{\sim}{u}, \underset{\sim}{u}(0, \cdot))$.
Auxiliary spaces:

$$L_{2,0}(\Omega) := \{p \in L_2(\Omega) \colon \int_\Omega p\, dx = 0\}$$
$$\underset{\sim}{\mathbb{H}}^1(\Omega) := \{\underset{\sim}{u} \in \underset{\sim}{H}^1(\Omega) \colon \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Omega\}.$$

Solution space: $\mathscr{X} \times L_2(I; L_{2,0}(\Omega))$, where

$$\mathscr{X} := \{(\underset{\sim}{u}, \underset{\approx}{w}) \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)) \times L_2(I; L_2(\Omega; \underset{\approx}{\mathbb{S}})) \colon \partial_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w} \in \underset{\sim}{L}_2(I \times \Omega),$$
$$\mathrm{div}_x\, \underset{\sim}{u} \in H^1(I; L_{2,0}(\Omega)), (\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{w}|_{I \times \partial\Omega} \underset{\sim}{n} = 0\},$$

equipped with the (squared) graph norm

$$\|(\underset{\sim}{u}, \underset{\approx}{w})\|^2_{\mathscr{X}} := \|\underset{\sim}{u}\|^2_{L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega))} + \|\underset{\approx}{w}\|^2_{\underset{\approx}{L}_2(I \times \Omega)}$$
$$+ \|\partial_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}\|^2_{\underset{\sim}{L}_2(I \times \Omega)} + \|\mathrm{div}_x\, \underset{\sim}{u}\|^2_{H^1(I; L_{2,0}(\Omega))}.$$

Well-posed FOSLS I

Recall $\mathrm{G}(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p), \underset{\sim}{\partial}_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}, \mathrm{div}_x\, \underset{\sim}{u}, \underset{\sim}{u}(0, \cdot))$.
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Solution space: $\mathscr{X} \times L_2(I; L_{2,0}(\Omega))$, where

$$\mathscr{X} := \{(\underset{\sim}{u}, \underset{\approx}{w}) \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)) \times L_2(I; L_2(\Omega; \underset{\approx}{\mathbb{S}})) \colon \underset{\sim}{\partial}_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w} \in \underset{\sim}{L}_2(I \times \Omega),$$
$$\mathrm{div}_x\, \underset{\sim}{u} \in H^1(I; L_{2,0}(\Omega)), (\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{w}|_{I \times \partial\Omega}\underset{\sim}{n} = 0\},$$

equipped with the (squared) graph norm

$$\|(\underset{\sim}{u}, \underset{\approx}{w})\|_{\mathscr{X}}^2 := \|\underset{\sim}{u}\|^2_{L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega))} + \|\underset{\approx}{w}\|^2_{\underset{\approx}{L}_2(I \times \Omega)}$$
$$+ \|\underset{\sim}{\partial}_t \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}\|^2_{\underset{\sim}{L}_2(I \times \Omega)} + \|\mathrm{div}_x\, \underset{\sim}{u}\|^2_{H^1(I; L_{2,0}(\Omega))}.$$

Well-posed FOSLS II

Theorem

$\mathscr{F} := \underset{\approx}{L}_2(I \times \Omega) \times \underset{\sim}{L}_2(I \times \Omega) \times H^1(I; L_{2,0}(\Omega)) \times \underset{\sim}{L}_2(\Omega)$.

Let Ω be convex or have a C^2 boundary.[a] *Then*

$$\|G(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{F}} \eqsim \|(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{X} \times L_2(I; L_{2,0}(\Omega))}$$

for all $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I; L_{2,0}(\Omega))$.

[a] We assume this from here on.

Some elements of proof

'$\lesssim$' easy. '$\gtrsim$': Let $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I; L_{2,0}(\Omega))$, for convenience $\operatorname{div}_x \underset{\sim}{u} = 0$. Setting

$$\underset{\sim}{f}(\underset{\sim}{v}) := \int_I \int_\Omega \partial_t \underset{\sim}{u} \cdot \underset{\sim}{v} + \tfrac{1}{2} \underset{\approx}{T}(\nu \underset{\sim}{u}, p) : \underset{\approx}{D}(\underset{\sim}{v}) \, dx \, dt,$$

int-by-parts and $\triangle$-ineq. show that

$$\|\underset{\sim}{f}\|_{L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)')} \lesssim \|\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p)\|_{\underset{\approx}{L}_2(I \times \Omega)} + \|\partial_t \underset{\sim}{u} + \underset{\sim}{\operatorname{div}}_x \underset{\approx}{w}\|_{\underset{\sim}{L}_2(I \times \Omega)}. \tag{6}$$

Well-posed FOSLS II

Theorem

$\mathscr{F} := \underset{\approx}{L}_2(I \times \Omega) \times \underset{\sim}{L}_2(I \times \Omega) \times H^1(I; L_{2,0}(\Omega)) \times \underset{\sim}{L}_2(\Omega)$.

Let Ω be convex or have a C^2 boundary.[a] *Then*

$$\|\mathrm{G}(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{F}} \eqsim \|(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{X} \times L_2(I; L_{2,0}(\Omega))}$$

for all $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I; L_{2,0}(\Omega))$.

[a] We assume this from here on.

Some elements of proof

'$\lesssim$' easy. '$\gtrsim$': Let $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I; L_{2,0}(\Omega))$, for convenience $\mathrm{div}_x\, \underset{\sim}{u} = 0$. Setting

$$\underset{\sim}{f}(\underset{\sim}{v}) := \int_I \int_\Omega \underset{\sim}{\partial_t} \underset{\sim}{u} \cdot \underset{\sim}{v} + \tfrac{1}{2} \underset{\approx}{T}(\nu \underset{\sim}{u}, p) : \underset{\approx}{D}(\underset{\sim}{v})\, dx\, dt,$$

int-by-parts and $\triangle$-ineq. show that

$$\|\underset{\sim}{f}\|_{L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)')} \lesssim \|\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p)\|_{\underset{\approx}{L}_2(I \times \Omega)} + \|\underset{\sim}{\partial_t} \underset{\sim}{u} + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}\|_{\underset{\sim}{L}_2(I \times \Omega)}. \tag{6}$$

Well-posed FOSLS II

Some elements of proof (Cont.)

$\mathcal{H}^1(\Omega) := \{u \in \mathbb{H}^1(\Omega) : \operatorname{div} u = 0\}$, $\mathcal{H}^0(\Omega) := \{u \in L_2(\Omega) : \operatorname{div} u = 0,\ u \cdot n = 0 \text{ on } \partial\Omega\}$.
Since for $v \in L_2(I; \mathcal{H}^1(\Omega))$, $f(v) = \int_I \int_\Omega \partial_t u \cdot v + \frac{\nu}{2} D(u) : D(v)\, dx\, dt$, well-posedness of the parabolic PDE for the div-free velocities gives

$$\|u\|_{L_2(I;\mathcal{H}^1(\Omega))} + \|\partial_t u\|_{L_2(I;\mathcal{H}^1(\Omega)')} \lesssim \|f\|_{L_2(I;\mathcal{H}^1(\Omega)')} + \|u(0,\cdot)\|_{\mathcal{H}^0(\Omega)}, \tag{7}$$

For $v \in L_2(I; \mathbb{H}^1(\Omega))$, def. of f and int-by-parts gives

$$\int_I \int_\Omega p \operatorname{div}_x v\, dx\, dt = \int_I \int_\Omega \partial_t u \cdot v + \frac{\nu}{2} D(u) : D(v)\, dx\, dt - f(v).$$

Using that $\inf_{0 \neq p \in L_{2,0}(\Omega)} \sup_{0 \neq v \in \mathbb{H}^1(\Omega)} \frac{|\int_\Omega p \operatorname{div}_x v\, dx|}{\|p\|_{L_{2,0}(\Omega)} \|v\|_{\mathbb{H}^1(\Omega)}} > 0$, one arrives at

$$\|p\|_{L_2(I;L_{2,0}(\Omega))} \lesssim \|\partial_t u\|_{L_2(I;\mathbb{H}^1(\Omega)')} + \|u\|_{L_2(I;\mathbb{H}^1(\Omega))} + \|f\|_{L_2(I;\mathbb{H}^1(\Omega)')}. \tag{8}$$

Since $\|w\|_{L_2(I\times\Omega)} \lesssim \|w + T(\nu u, p)\|_{L_2(I\times\Omega)} + \|u\|_{L_2(I;\mathbb{H}^1(\Omega))} + \|p\|_{L_2(I;L_{2,0}(\Omega))}$, the combination of (8), (7), and, (6) completes proof *if*

$$\|\partial_t u\|_{L_2(I;\mathbb{H}^1(\Omega)')} \lesssim \|\partial_t u\|_{L_2(I;\mathcal{H}^1(\Omega)')}. \tag{9}$$

Well-posed FOSLS II

Some elements of proof (Cont.)

$\underset{\sim}{\mathcal{H}}^1(\Omega) := \{\underset{\sim}{u} \in \underset{\sim}{\mathbb{H}}^1(\Omega) \colon \operatorname{div} \underset{\sim}{u} = 0\}$, $\underset{\sim}{\mathcal{H}}^0(\Omega) := \{\underset{\sim}{u} \in \underset{\sim}{L}_2(\Omega) \colon \operatorname{div} \underset{\sim}{u} = 0,\ \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Omega\}$.

Since for $\underset{\sim}{v} \in L_2(I; \underset{\sim}{\mathcal{H}}^1(\Omega))$, $\underset{\sim}{f}(\underset{\sim}{v}) = \int_I \int_\Omega \partial_t \underset{\sim}{u} \cdot \underset{\sim}{v} + \frac{\nu}{2} \underset{\approx}{D}(\underset{\sim}{u}) : \underset{\approx}{D}(\underset{\sim}{v})\, dx\, dt$, well-posedness of the parabolic PDE for the div-free velocities gives

$$\|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega))} + \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')} \lesssim \|\underset{\sim}{f}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')} + \|\underset{\sim}{u}(0,\cdot)\|_{\underset{\sim}{\mathcal{H}}^0(\Omega)}, \qquad (7)$$

For $\underset{\sim}{v} \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega))$, def. of $\underset{\sim}{f}$ and int-by-parts gives

$$\int_I \int_\Omega p \operatorname{div}_x \underset{\sim}{v}\, dx\, dt = \int_I \int_\Omega \partial_t \underset{\sim}{u} \cdot \underset{\sim}{v} + \frac{\nu}{2} \underset{\approx}{D}(\underset{\sim}{u}) : \underset{\approx}{D}(\underset{\sim}{v})\, dx\, dt - \underset{\sim}{f}(\underset{\sim}{v}).$$

Using that $\inf_{0 \neq p \in L_{2,0}(\Omega)} \sup_{0 \neq \underset{\sim}{v} \in \underset{\sim}{\mathbb{H}}^1(\Omega)} \frac{|\int_\Omega p \operatorname{div}_x \underset{\sim}{v}\, dx|}{\|p\|_{L_{2,0}(\Omega)} \|\underset{\sim}{v}\|_{\underset{\sim}{\mathbb{H}}^1(\Omega)}} > 0$, one arrives at

$$\|p\|_{L_2(I;L_{2,0}(\Omega))} \lesssim \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')} + \|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega))} + \|\underset{\sim}{f}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')}. \qquad (8)$$

Since $\|\underset{\approx}{w}\|_{\underset{\approx}{L}_2(I\times\Omega)} \lesssim \|\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p)\|_{\underset{\approx}{L}_2(I\times\Omega)} + \|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega))} + \|p\|_{L_2(I;L_{2,0}(\Omega))}$, the combination of (8), (7), and, (6) completes proof *if*

$$\|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')} \lesssim \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')}. \qquad (9)$$

Well-posed FOSLS II

Some elements of proof (Cont.)

$\underset{\sim}{\mathcal{H}}^1(\Omega) := \{\underset{\sim}{u} \in \underset{\sim}{\mathbb{H}}^1(\Omega) \colon \operatorname{div} \underset{\sim}{u} = 0\}$, $\underset{\sim}{\mathcal{H}}^0(\Omega) := \{\underset{\sim}{u} \in \underset{\sim}{L}_2(\Omega) \colon \operatorname{div} \underset{\sim}{u} = 0,\ \underset{\sim}{u} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Omega\}$.
Since for $\underset{\sim}{v} \in L_2(I; \underset{\sim}{\mathcal{H}}^1(\Omega))$, $\underset{\sim}{f}(\underset{\sim}{v}) = \int_I \int_\Omega \partial_t \underset{\sim}{u} \cdot \underset{\sim}{v} + \frac{\nu}{2} \underset{\approx}{D}(\underset{\sim}{u}) : \underset{\approx}{D}(\underset{\sim}{v})\, dx\, dt$, well-posedness of the parabolic PDE for the div-free velocities gives

$$\|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega))} + \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')} \lesssim \|\underset{\sim}{f}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')} + \|\underset{\sim}{u}(0,\cdot)\|_{\underset{\sim}{\mathcal{H}}^0(\Omega)}, \tag{7}$$

For $\underset{\sim}{v} \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega))$, def. of $\underset{\sim}{f}$ and int-by-parts gives

$$\int_I \int_\Omega p \operatorname{div}_x \underset{\sim}{v}\, dx\, dt = \int_I \int_\Omega \partial_t \underset{\sim}{u} \cdot \underset{\sim}{v} + \frac{\nu}{2} \underset{\approx}{D}(\underset{\sim}{u}) : \underset{\approx}{D}(\underset{\sim}{v})\, dx\, dt - \underset{\sim}{f}(\underset{\sim}{v}).$$

Using that $\inf_{0 \neq p \in L_{2,0}(\Omega)} \sup_{0 \neq \underset{\sim}{v} \in \underset{\sim}{\mathbb{H}}^1(\Omega)} \frac{|\int_\Omega p \operatorname{div}_x \underset{\sim}{v}\, dx|}{\|p\|_{L_{2,0}(\Omega)} \|\underset{\sim}{v}\|_{\underset{\sim}{\mathbb{H}}^1(\Omega)}} > 0$, one arrives at

$$\|p\|_{L_2(I;L_{2,0}(\Omega))} \lesssim \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')} + \|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega))} + \|\underset{\sim}{f}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')}. \tag{8}$$

Since $\|\underset{\approx}{w}\|_{\underset{\approx}{L}_2(I\times\Omega)} \lesssim \|\underset{\approx}{w} + \underset{\approx}{T}(\nu \underset{\sim}{u}, p)\|_{\underset{\approx}{L}_2(I\times\Omega)} + \|\underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega))} + \|p\|_{L_2(I;L_{2,0}(\Omega))}$, the combination of (8), (7), and, (6) completes proof *if*

$$\|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathbb{H}}^1(\Omega)')} \lesssim \|\partial_t \underset{\sim}{u}\|_{L_2(I;\underset{\sim}{\mathcal{H}}^1(\Omega)')}. \tag{9}$$

Well-posed FOSLS III

Some elements of proof (Cont.)

(9) true when $\underset{\sim}{L}_2(\Omega)$-orthogonal projector $\underset{\sim}{\Pi}$ onto $\underset{\sim}{\mathcal{H}}^0(\Omega)$, also known as the *Leray-projector*, satisfies $\underset{\sim}{\Pi} \in \mathcal{L}(\underset{\sim}{\mathbb{H}}^1(\Omega), \underset{\sim}{\mathbb{H}}^1(\Omega))$. Latter is true when Poisson problem with Neumann b.c. is $H^2(\Omega)$-regular. □

Remark

For *no-slip* b.c. $\underset{\sim}{\mathbb{H}}^1(\Omega)$ should read as $\underset{\sim}{H}^1_0(\Omega)$.
Since $\underset{\sim}{L}_2(\Omega)$-orthogonal projector $\underset{\sim}{\Pi}$ onto $\underset{\sim}{\mathcal{H}}^0(\Omega)$ does not preserve no-slip boundary conditions, $\underset{\sim}{\Pi} \notin \mathcal{L}(\underset{\sim}{H}^1_0(\Omega), \underset{\sim}{H}^1_0(\Omega))$.

Well-posed FOSLS IV

W.r.t. a non-Cartesian partition of $I \times \Omega$, a piecewise polynomial subspace of $L_2(I; L_{2,0}(\Omega))$ is hard to construct. Will enforce $\int_\Omega p(t,\cdot)\,dx = 0$ in ls-sense.

Corollary

With $(Mp)(t) := \frac{1}{\sqrt{|\Omega|}} \int_\Omega p(t,x)\,dx$, *and* $\bar{\mathsf{G}}(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\mathsf{G}(\underset{\sim}{u}, \underset{\approx}{w}, p), Mp)$,

$$\|\bar{\mathsf{G}}(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{F} \times L_2(I)} \eqsim \|(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{X} \times L_2(I \times \Omega)}$$

for all $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I \times \Omega)$.

Above corollary shows $\bar{\mathsf{G}}$ is iso with *range*.

Proposition

For $\underset{\sim}{f} \in L_2(I \times \Omega)$, $\underset{\sim}{u}_0 \in \{\underset{\sim}{v} \in L_2(\Omega) \colon \operatorname{div} \underset{\sim}{v} = 0,\ \underset{\sim}{v} \cdot \underset{\sim}{n} = 0$ *on* $\partial\Omega\}$, $(0, \underset{\sim}{f}, 0, \underset{\sim}{u}_0, 0) \in \operatorname{ran} \bar{\mathsf{G}}$.

Well-posed FOSLS IV

W.r.t. a non-Cartesian partition of $I \times \Omega$, a piecewise polynomial subspace of $L_2(I; L_{2,0}(\Omega))$ is hard to construct. Will enforce $\int_\Omega p(t,\cdot)\,dx = 0$ in ls-sense.

Corollary

With $(Mp)(t) := \frac{1}{\sqrt{|\Omega|}} \int_\Omega p(t,x)\,dx$, *and* $\bar{\mathsf{G}}(\underset{\sim}{u}, \underset{\approx}{w}, p) := (\mathsf{G}(\underset{\sim}{u}, \underset{\approx}{w}, p), Mp)$,

$$\|\bar{\mathsf{G}}(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{F} \times L_2(I)} \eqsim \|(\underset{\sim}{u}, \underset{\approx}{w}, p)\|_{\mathscr{X} \times L_2(I \times \Omega)}$$

for all $(\underset{\sim}{u}, \underset{\approx}{w}, p) \in \mathscr{X} \times L_2(I \times \Omega)$.

Above corollary shows $\bar{\mathsf{G}}$ is iso with *range*.

Proposition

For $\underset{\sim}{f} \in \underset{\sim}{L}_2(I \times \Omega)$, $\underset{\sim}{u}_0 \in \{\underset{\sim}{v} \in \underset{\sim}{L}_2(\Omega) \colon \operatorname{div} \underset{\sim}{v} = 0,\ \underset{\sim}{v} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Omega\}$, $(0, \underset{\sim}{f}, 0, \underset{\sim}{u}_0, 0) \in \operatorname{ran} \bar{\mathsf{G}}$.

Well-posed FOSLS V

Proposition

Let $\mathscr{Z}^\delta \times P^\delta$ be a closed subspace of $\mathscr{Z} \times L_2(I \times \Omega)$.
Let $F \in \bar{\mathscr{F}} := \mathscr{F} \times L_2(I)$.
Let $(\underset{\sim}{u}, \underset{\approx}{w}, p)$ or $(\underset{\sim}{u}^\delta, \underset{\approx}{w}^\delta, p^\delta)$ minimizers over $\mathscr{Z} \times L_2(I \times \Omega)$ or $\mathscr{Z}^\delta \times P^\delta$ of

$$\tfrac{1}{2}\|F - \bar{\mathsf{G}}(\cdot,\cdot,\cdot)\|^2_{\bar{\mathscr{F}}}.$$

$$\mathfrak{M} := \sup_{0 \neq (\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p}) \in \mathscr{Z} \times L_2(I\times\Omega)} \|\bar{\mathsf{G}}(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\bar{\mathscr{F}}} / \|(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\mathscr{Z}\times L_2(I\times\Omega)}.$$

$$\mathfrak{m} := \inf_{0 \neq (\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p}) \in \mathscr{Z} \times L_2(I\times\Omega)} \|\bar{\mathsf{G}}(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\bar{\mathscr{F}}} / \|(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\mathscr{Z}\times L_2(I\times\Omega)}.$$

$$\begin{aligned}\|(\underset{\sim}{u}, \underset{\approx}{w}, p) - (\underset{\sim}{u}^\delta, \underset{\approx}{w}^\delta, p^\delta)\|_{\mathscr{Z}\times L_2(I\times\Omega)} \\ \leq \frac{\mathfrak{M}}{\mathfrak{m}} \inf_{(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p}) \in \mathscr{Z}^\delta \times P^\delta} \|(\underset{\sim}{u}, \underset{\approx}{w}, p) - (\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\mathscr{Z}\times L_2(I\times\Omega)}.\end{aligned}$$

If $F \in \operatorname{ran} \bar{\mathsf{G}}$, then for $(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p}) \in \mathscr{Z} \times L_2(I \times \Omega)$,

$$\frac{1}{\mathfrak{M}}\|F - \bar{\mathsf{G}}(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\bar{\mathscr{F}}} \leq \|(\underset{\sim}{u}, \underset{\approx}{w}, p) - (\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\mathscr{Z}\times L_2(I\times\Omega)} \leq \frac{1}{\mathfrak{m}}\|F - \bar{\mathsf{G}}(\underset{\sim}{\hat{u}}, \underset{\approx}{\hat{w}}, \hat{p})\|_{\bar{\mathscr{F}}}.$$

FEM I

Let $\underset{\sim}{U} := \{\underset{\sim}{u} \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)) \cap H^1(I; \underset{\sim}{L}_2(\Omega)) \colon \operatorname{div}_x \underset{\sim}{u} \in H^1(I; L_{2,0}(\Omega))\}$,

$H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}}) := \{\underset{\approx}{v} \in L_2(\Omega, \underset{\approx}{\mathbb{S}}) \colon \underset{\sim}{\operatorname{div}} \underset{\approx}{v} \in \underset{\sim}{L}_2(\Omega), (\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{v}|_{\partial\Omega}\underset{\sim}{n} = 0\}$

equipped with graph norms. Then

$$\underset{\sim}{U} \times L_2(I; H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}})) \hookrightarrow \mathscr{X}.$$

Take $\underset{\sim}{U}^\delta \subset \underset{\sim}{U}$, $\underset{\approx}{W}^\delta \subset L_2(I; H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}}))$, $P^\delta \subset L_2(I \times \Omega)$ w.r.t. common partition of $I \times \Omega$. To avoid C^1 elements for $\underset{\sim}{U}^\delta$, partitions into prisms. So far, quasi-uniform, conforming partitions, finite elements of lowest order, and $d = 2$. Let I^δ partition of I into subintervals. Let Ω^δ conforming partition of Ω into unif. shape reg. triangles. Let U_t^δ, U_x^δ cont. piecewise linears w.r.t. I^δ and Ω^δ. Set $\underset{\sim}{U}^\delta := U_t^\delta \otimes \big((U_x^\delta \times U_x^\delta + \text{ span of vectorial edge bubbles}) \cap \underset{\sim}{\mathbb{H}}^1(\Omega)\big)$. Bubbles from [Christiansen and Hu, 2018]. Thanks to a commuting diagram:

Proposition

With $h_\delta := \max(\max_{J \in I^\delta} \operatorname{diam} J, \max_{K \in \Omega^\delta} \operatorname{diam} K)$ *it holds that*

$$\inf_{\underset{\sim}{v} \in \underset{\sim}{U}^\delta} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{\underset{\sim}{U}} \lesssim h_\delta\big(\|\underset{\sim}{u}\|_{\underset{\sim}{H}^2(I\times\Omega)} + \|\underbrace{\operatorname{div}_x \underset{\sim}{u}}_{0}\|_{H^2(I; L_2(\Omega))} + \|\underbrace{\operatorname{div}_x \underset{\sim}{u}}_{0}\|_{H^1(I; H^1(\Omega))}\big).$$

FEM I

Let $\underset{\sim}{U} := \{\underset{\sim}{u} \in L_2(I; \underset{\sim}{\mathbb{H}}^1(\Omega)) \cap H^1(I; \underset{\sim}{L}_2(\Omega)) \colon \operatorname{div}_x \underset{\sim}{u} \in H^1(I; L_{2,0}(\Omega))\}$,

$H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}}) := \{\underset{\approx}{v} \in L_2(\Omega, \underset{\approx}{\mathbb{S}}) \colon \underset{\sim}{\operatorname{div}} \underset{\approx}{v} \in \underset{\sim}{L}_2(\Omega),\ (\mathrm{Id} - \underset{\sim}{n}\underset{\sim}{n}^\top)\underset{\approx}{v}|_{\partial\Omega}\underset{\sim}{n} = 0\}$

equipped with graph norms. Then

$$\underset{\sim}{U} \times L_2(I; H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}})) \hookrightarrow \mathscr{X}.$$

Take $\underset{\sim}{U}^\delta \subset \underset{\sim}{U}$, $\underset{\approx}{W}^\delta \subset L_2(I; H_0(\underset{\sim}{\operatorname{div}}; \Omega, \underset{\approx}{\mathbb{S}}))$, $P^\delta \subset L_2(I \times \Omega)$ w.r.t. common partition of $I \times \Omega$. To avoid C^1 elements for $\underset{\sim}{U}^\delta$, partitions into prisms. So far, quasi-uniform, conforming partitions, finite elements of lowest order, and $d = 2$. Let I^δ partition of I into subintervals. Let Ω^δ conforming partition of Ω into unif. shape reg. triangles. Let U_t^δ, U_x^δ cont. piecewise linears w.r.t. I^δ and Ω^δ. Set $\underset{\sim}{U}^\delta := U_t^\delta \otimes \big((U_x^\delta \times U_x^\delta + \text{ span of vectorial edge bubbles}) \cap \underset{\sim}{\mathbb{H}}^1(\Omega)\big)$. Bubbles from [Christiansen and Hu, 2018]. Thanks to a commuting diagram:

Proposition

With $h_\delta := \max(\max_{J \in I^\delta} \operatorname{diam} J, \max_{K \in \Omega^\delta} \operatorname{diam} K)$ *it holds that*

$$\inf_{\underset{\sim}{v} \in \underset{\sim}{U}^\delta} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{\underset{\sim}{U}} \lesssim h_\delta\big(\|\underset{\sim}{u}\|_{\underset{\sim}{H}^2(I\times\Omega)} + \|\underbrace{\operatorname{div}_x \underset{\sim}{u}}_{0}\|_{H^2(I; L_2(\Omega))} + \|\underbrace{\operatorname{div}_x \underset{\sim}{u}}_{0}\|_{H^1(I; H^1(\Omega))}\big).$$

FEM II

$\underset{\approx}{W}^\delta = W_t^\delta \otimes \underset{\approx}{W}_x^\delta$, where W_t^δ space of piecewise constants w.r.t. I^δ, and $\underset{\approx}{W}_x^\delta$ fem subspace of $H_0(\underset{\sim}{\mathrm{div}};\Omega,\underset{\approx}{\mathbb{S}})$ w.r.t. Ω^δ from [Christiansen and Hu, 2022]. Only 9 DoFs per element. Again thanks to commuting diagram:

Proposition

$$\inf_{\underset{\approx}{v}\in\underset{\approx}{W}^\delta} \|\underset{\approx}{w} - \underset{\approx}{v}\|_{L_2(I;H(\underset{\sim}{\mathrm{div}};\Omega,\underset{\approx}{\mathbb{S}}))} \lesssim h_\delta\Big(\|\underset{\approx}{w}\|_{H^1(I;H(\underset{\sim}{\mathrm{div}};\Omega,\underset{\approx}{\mathbb{S}}))} + \|\underset{\approx}{w}\|_{L_2(I;H^1(\Omega,\underset{\approx}{\mathbb{S}}))} + \|\underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}\|_{L_2(I;\underset{\sim}{H}^1(\Omega))}\Big).$$

With piecewise constants for pressure, using $\underset{\sim}{\mathrm{div}}_x \underset{\approx}{w} = \underset{\sim}{f} - \partial_t \underset{\sim}{u}$ we conclude

Theorem

$$\|(\underset{\sim}{u},\underset{\approx}{w},p)-(\underset{\sim}{u}^\delta,\underset{\approx}{w}^\delta,p^\delta)\|_{\mathscr{X}\times L_2(I\times\Omega)} \lesssim h_\delta\Big(\|\underset{\sim}{u}\|_{\underset{\sim}{H}^2(I\times\Omega)} + \|p\|_{H^1(I\times\Omega)} + \|\underset{\sim}{f}\|_{\underset{\sim}{H}^1(I\times\Omega)}\Big).$$

FEM II

$\underset{\approx}{W}^\delta = W_t^\delta \otimes \underset{\approx}{W}_x^\delta$, where W_t^δ space of piecewise constants w.r.t. I^δ, and $\underset{\approx}{W}_x^\delta$ fem subspace of $H_0(\underset{\sim}{\mathrm{div}}; \Omega, \underset{\approx}{\mathbb{S}})$ w.r.t. Ω^δ from [Christiansen and Hu, 2022]. Only 9 DoFs per element. Again thanks to commuting diagram:

Proposition

$$\inf_{\underset{\approx}{v} \in \underset{\approx}{W}^\delta} \|\underset{\approx}{w} - \underset{\approx}{v}\|_{L_2(I; H(\underset{\sim}{\mathrm{div}}; \Omega, \underset{\approx}{\mathbb{S}}))} \lesssim h_\delta \Big(\|\underset{\approx}{w}\|_{H^1(I; H(\underset{\sim}{\mathrm{div}}; \Omega, \underset{\approx}{\mathbb{S}}))} + \|\underset{\approx}{w}\|_{L_2(I; H^1(\Omega, \underset{\approx}{\mathbb{S}}))} + \|\underset{\sim}{\mathrm{div}}_x \underset{\approx}{w}\|_{L_2(I; \underset{\sim}{H}^1(\Omega))} \Big).$$

With piecewise constants for pressure, using $\underset{\sim}{\mathrm{div}}_x \underset{\approx}{w} = \underset{\sim}{f} - \partial_t \underset{\sim}{u}$ we conclude

Theorem

$$\|(\underset{\sim}{u}, \underset{\approx}{w}, p) - (\underset{\sim}{u}^\delta, \underset{\approx}{w}^\delta, p^\delta)\|_{\mathscr{X} \times L_2(I \times \Omega)} \lesssim h_\delta \Big(\|\underset{\sim}{u}\|_{\underset{\sim}{H}^2(I \times \Omega)} + \|p\|_{H^1(I \times \Omega)} + \|\underset{\sim}{f}\|_{\underset{\sim}{H}^1(I \times \Omega)} \Big).$$

Numerical results: Stability

$I = (0,1)$, and $\Omega = (0,1)^2$ or $\Omega = (-1,1)^2 \setminus [-1,0]^2$.
$\underset{\sim}{U}^\delta \times \underset{\approx}{W}^\delta \times P^\delta$ w.r.t. unif. part. of $I \times \Omega$ with mesh-size $h_\delta = 2^0, 2^{-1}, \ldots$.
Additionally we investigate replacement of $\|\operatorname{div}_x \underset{\sim}{u}\|^2_{H^1(I;L_{2,0}(\Omega))}$ by $\|\operatorname{div}_x \underset{\sim}{u}\|^2_{L_2(I;L_{2,0}(\Omega))}$, and no-slip by slip b.c.

Table: Ratios $\mathfrak{M}^\delta / \mathfrak{m}^\delta$

h_δ	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
Square (slip, $\partial_t \operatorname{div}_x \underset{\sim}{u} \in L_2$)	3.73	6.75	6.81	6.82	6.82	6.82
L-shape (slip, $\partial_t \operatorname{div}_x \underset{\sim}{u} \in L_2$)	7.65	9.23	10.73	12.22	13.59	14.81
Square (slip, $\operatorname{div}_x \underset{\sim}{u} \in L_2$)	3.73	6.88	7.37	8.21	10.96	18.88
Square (no-slip, $\partial_t \operatorname{div}_x \underset{\sim}{u} \in L_2$)	5.92	7.94	10.62	13.36	15.28	16.72

On both domains, we take $\underset{\sim}{u}(t,x_1,x_2) := \exp(-t)\,\underset{\sim}{\operatorname{curl}}_x \frac{\sin(\pi x_1)\sin(\pi x_2)}{\pi}$, which satisfies no-slip b.c., $p(t,x_1,x_2) := \exp(-t)\sin\big(\pi(x_1 - x_2)\big)$, and $\underset{\approx}{w} := -\underset{\approx}{T}(\nu \underset{\sim}{u}, p)$, and data correspondingly.

Numerical results: Stability

$I = (0,1)$, and $\Omega = (0,1)^2$ or $\Omega = (-1,1)^2 \setminus [-1,0]^2$.
$\underset{\sim}{U}^\delta \times \underset{\approx}{W}^\delta \times P^\delta$ w.r.t. unif. part. of $I \times \Omega$ with mesh-size $h_\delta = 2^0, 2^{-1}, \ldots$.
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On both domains, we take $\underset{\sim}{u}(t, x_1, x_2) := \exp(-t)\, \underset{\sim}{\operatorname{curl}}_x \frac{\sin(\pi x_1)\sin(\pi x_2)}{\pi}$, which satisfies no-slip b.c., $p(t, x_1, x_2) := \exp(-t) \sin\big(\pi(x_1 - x_2)\big)$, and $\underset{\approx}{w} := -\underset{\approx}{T}(\nu \underset{\sim}{u}, p)$, and data correspondingly.

Numerical results: Convergence $I \times \Omega = (0,1)^3$

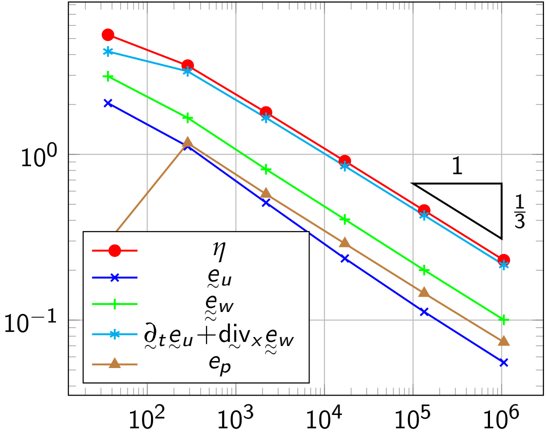

Figure: DoFs vs. estimator $\eta = \|F - \bar{\mathrm{G}}(\underset{\sim}{u}^\delta, \underset{\approx}{w}^\delta, p^\delta)\|_{\bar{\mathscr{F}}}$, and errors $\underset{\sim}{e}_u$, $\underset{\approx}{e}_w$, $\underset{\sim}{\partial}_t \underset{\sim}{e}_u + \underset{\sim}{\mathrm{div}}_x \underset{\approx}{e}_w$, e_p, measured in $\left(\|\cdot\|^2_{L_2(I;\underset{\sim}{H}^1(\Omega))} + \|\mathrm{div}_x \cdot\|^2_{H^1(I;L_2(\Omega))}\right)^{1/2}$, $\|\cdot\|_{\underset{\approx}{L}_2(I\times\Omega)}$, $\|\cdot\|_{\underset{\sim}{L}_2(I\times\Omega)}$ and $\|\cdot\|_{L_2(I\times\Omega)}$, respectively.

Numerical results: $I \times \Omega = (0,1) \times (-1,1)^2 \setminus [-1,0]^2$

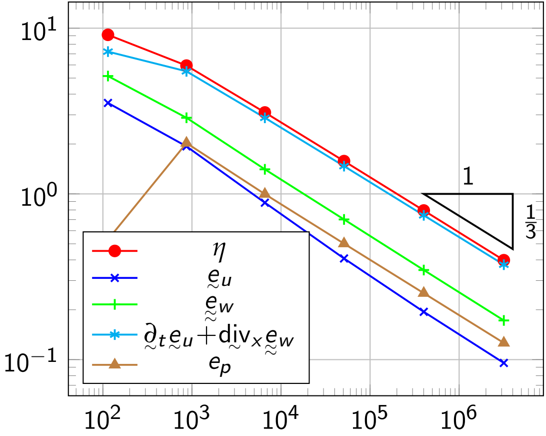

Figure: DoFs vs. estimator $\eta = \|F - \bar{\mathsf{G}}(\underset{\sim}{u}^\delta, \underset{\approx}{w}^\delta, p^\delta)\|_{\bar{\mathscr{F}}}$, and errors $\underset{\sim}{e}_u$, $\underset{\approx}{e}_w$, $\underset{\sim}{\partial}_t \underset{\sim}{e}_u + \underset{\sim}{\text{div}}_x \underset{\approx}{e}_w$, e_p, measured in $\left(\|\cdot\|^2_{L_2(I;\underset{\sim}{H}^1(\Omega))} + \|\operatorname{div}_x \cdot\|^2_{H^1(I;L_2(\Omega))}\right)^{1/2}$, $\|\cdot\|_{\underset{\approx}{L}_2(I\times\Omega)}$, $\|\cdot\|_{\underset{\sim}{L}_2(I\times\Omega)}$ and $\|\cdot\|_{L_2(I\times\Omega)}$, respectively.

Conclusion

Monolithic numerical approximation of parabolic equations and instationary Stokes equations based on a well-posed simultaneous time-space variational formulation is advantageous for

- non-smooth solutions
- parallel computation,
- and for all applications that require the whole time evolution at the same time (RBM, optimal control, data-assimilation).

Thanks for your attention/patience!

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