A-posteriori-steered and adaptive $p$-robust
multigrid solvers

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joint work with
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Setting

A-posteriori-steered multigrid

Adaptivity in a-posteriori-steered solvers

Adaptive finite element setting

Conclusion
Introduction

Geometric multigrid solver with error control for high-order discretization:

- **polynomial degree** $p$-robustness

- **number of levels** $L$-robustness

- **optimal step-sizes**
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Model problem: Find $u \in H^1_0(\Omega)$ such that $\langle u, v \rangle := (K \nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)$

Fix $p \geq 1$, let $\mathbb{P}_p(\mathcal{T}_L) := \{v_L \in L^2(\Omega), v_L|_K \in \mathbb{P}_p(K) \ \forall K \in \mathcal{T}_L\}$

Define
$$V^p_L := \mathbb{P}_p(\mathcal{T}_L) \cap H^1_0(\Omega)$$

Discrete problem: Find $u_L \in V^p_L$ such that
$$\langle u_L, v_L \rangle = (f, v_L) \quad \forall v_L \in V^p_L \quad (FE)$$

By introducing a basis of $V^p_L$: $A_L U_L = F_L$

We work with the basis-independent functional formulation (FE)

Algebraic residual functional: $v_L \mapsto (f, v_L) - \langle u^i_L, v_L \rangle \in \mathbb{R}$, $v_L \in V^p_L$
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Ani Miraçi (TU Wien)
Example: Two different hierarchies with $L = 3$ refinements

Assumptions: The meshes $\{T_\ell\}_{1 \leq \ell \leq L}$ can be generated through uniform or adaptive refinement, satisfying

- $(C_{\text{qu}})$-quasi-uniform $T_0$
- $(\kappa_T)$-shape-regularity
- $(C_{\text{ref}})$-maximum strength of refinement

For given $p$ and $L$, choose increasing polynomial degrees

Define the spaces $V_{\ell}^{p_{\ell}} = P_{p_{\ell}}(T_\ell) \cap H_0^1(\Omega)$

Economical choice: $p_0 = p_1 = \ldots = p_{L-1} = 1, \quad p_L = p$
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Let $\mathcal{V}_\ell$ be the set of vertices of $\mathcal{T}_\ell$

Given a vertex $a \in \mathcal{V}_\ell$, we denote

- $\mathcal{T}_\ell^a$ the patch of elements sharing vertex $a$
- $\omega_\ell^a$ the corresponding patch subdomain
- $\mathcal{V}_\ell^a = \mathbb{P}_{p_\ell}(\mathcal{T}_\ell) \cap H^1_0(\omega_\ell^a)$ the associated local space
A-posteriori-steered multigrid
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  - zero pre- and one single post-smoothing step
  - cheapest $P^1$ coarse solve
  - additive Schwarz / block Jacobi smoothing: fully parallel on each level
  - level-wise step-sizes in multigrid error correction stage: optimally chosen by line search
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\[ u^i_L \in \mathbb{V}_L^p \quad u^{i+1}_L = u_L^i + \sum_{\ell=0}^{L} \lambda^i_\ell \rho^i_\ell \in \mathbb{V}_L^p \quad \eta_{\text{alg}}^i = \left( \sum_{\ell=0}^{L} (\lambda^i_\ell \|\nabla \rho^i_\ell\|)^2 \right)^{\frac{1}{2}} \]

- V-cycle of geometric multigrid: coarse grid solve and level-wise smoothing
- zero pre- and one single post-smoothing step
- cheapest $\mathbb{P}_1$ coarse solve
- additive Schwarz / block Jacobi smoothing: fully parallel on each level
- level-wise step-sizes in multigrid error correction stage: optimally chosen by line search
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\[ u^i_L \in \mathbb{V}^p_L \quad u^{i+1}_L = u^i_L + \sum_{\ell=0}^L \lambda^i_\ell \rho^i_\ell \in \mathbb{V}^p_L \]

\[ \eta_{\text{alg}} = \left( \sum_{\ell=0}^L (\lambda^i_\ell \| \nabla \rho^i_\ell \|^2) \right)^{\frac{1}{2}} \]

**V-cycle of geometric multigrid:** coarse grid solve and level-wise smoothing

- **zero** pre- and one **single** post-smoothing step
- **cheapest** \( P^1 \) coarse solve
- **additive Schwarz / block Jacobi** smoothing: fully parallel on each level
- level-wise step-sizes in multigrid error correction stage: optimally chosen by **line search**
Let \( u^i_L \in \mathbb{V}^p_L \) be arbitrary.

**Coarse solve:** Define \( \rho^0_i \in \mathbb{V}^1_0 \) by:

\[
\langle \rho^0_i, v_0 \rangle = (f, v_0) - \langle u^i_L, v_0 \rangle, \quad \forall v_0 \in \mathbb{V}^1_0 \quad \text{and set } \lambda_0^i := 1
\]

**Level-wise local solves:** For \( \ell = 1:L \), for all \( a \in \mathbb{V}_\ell \), define \( \rho^\ell_{i,a} \in \mathbb{V}^a_\ell \) by

\[
\langle \rho^\ell_{i,a}, v_\ell,a \rangle = (f, v_\ell,a) \omega^a_\ell - \langle u^i_L, v_\ell,a \rangle \omega^a_\ell - \sum_{k=0}^{\ell-1} \lambda_k^i \langle \rho^k_i, v_\ell,a \rangle \omega^a_\ell, \quad \forall v_\ell,a \in \mathbb{V}^a_\ell
\]

**\( \ell \)-level update (correction direction):** Define \( \rho^\ell_i \in \mathbb{V}^p_\ell \) by:

\[
\rho^\ell_i := \sum_{a \in \mathbb{V}_\ell} \rho^\ell_{i,a}
\]

**Level-wise step-sizes by line search:** Set \( \lambda^i_\ell := \frac{(f, \rho^\ell_i) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda_k^i \rho^k_i, \rho^\ell_i \rangle}{\| \rho^\ell_i \|^2} \).
Let $u^i_L \in \mathbb{V}^p_L$ be arbitrary.

**Coarse solve:** Define $\rho^0_i \in \mathbb{V}^1_0$ by:
\[
\langle \rho^0_i , v_0 \rangle = (f, v_0) - \langle u^i_L , v_0 \rangle, \quad \forall v_0 \in \mathbb{V}^1_0 \text{ and set } \lambda^i_0 := 1
\]

**Level-wise local solves:** For $\ell = 1: L$, for all $a \in \mathcal{V}_\ell$, define $\rho^i_{\ell,a} \in \mathbb{V}^a_\ell$ by
\[
\langle \rho^i_{\ell,a} , v_{\ell,a} \rangle = (f, v_{\ell,a})_{\omega_\ell^a} - \langle u^i_L , v_{\ell,a} \rangle_{\omega_\ell^a} - \sum_{k=0}^{\ell-1} \lambda^i_k \langle \rho^i_k , v_{\ell,a} \rangle_{\omega_\ell^a}, \quad \forall v_{\ell,a} \in \mathbb{V}^a_\ell
\]

**$\ell$-level update (correction direction):** Define $\rho^i_{\ell} \in \mathbb{V}^p_\ell$ by:
\[
\rho^i_{\ell} := \sum_{a \in \mathcal{V}_\ell} \rho^i_{\ell,a}
\]

**Level-wise step-sizes by line search:** Set $\lambda^i_{\ell} := \frac{(f, \rho^i_{\ell}) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k , \rho^i_{\ell} \rangle}{\| \rho^i_{\ell} \|^2}$
Let \( u^i_L \in V^p_L \) be arbitrary.

**Coarse solve:** Define \( \rho^0 \in V^1_0 \) by:
\[
\langle \rho^0, v_0 \rangle = (f, v_0) - \langle u^i_L, v_0 \rangle, \quad \forall v_0 \in V^1_0 \quad \text{and set} \quad \lambda^0 := 1
\]

**Level-wise local solves:** For \( \ell = 1 : L \), for all \( a \in V^a_\ell \), define \( \rho^\ell, a \in V^a_\ell \) by
\[
\langle \rho^\ell, a, v^\ell, a \rangle = (f, v^\ell, a)_{\omega^a_\ell} - \langle u^i_L, v^\ell, a \rangle_{\omega^a_\ell} - \sum_{k=0}^{\ell-1} \lambda^i_k \langle \rho^i_k, v^\ell, a \rangle_{\omega^a_\ell}, \quad \forall v^\ell, a \in V^a_\ell
\]

**\( \ell \)-level update (correction direction):** Define \( \rho^\ell \in V^p_\ell \) by:
\[
\rho^\ell := \sum_{a \in V^a_\ell} \rho^\ell, a
\]

**Level-wise step-sizes by line search:** Set
\[
\lambda^\ell := \frac{\langle f, \rho^\ell \rangle - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^k \rho^k, \rho^\ell \rangle}{\| \rho^\ell \|^2}
\]
Let $u^i_L \in \mathbb{V}_L^p$ be arbitrary.

**Coarse solve:** Define $\rho^i_0 \in \mathbb{V}_0^1$ by:
\[
\langle \langle \rho^i_0, v_0 \rangle \rangle = (f, v_0) - \langle \langle u^i_L, v_0 \rangle \rangle, \quad \forall v_0 \in \mathbb{V}_0^1 \text{ and set } \lambda^i_0 := 1
\]

**Level-wise local solves:** For $\ell = 1 : L$, for all $a \in \mathcal{V}_\ell$, define $\rho^i_{\ell,a} \in \mathbb{V}_\ell^a$ by
\[
\langle \langle \rho^i_{\ell,a}, v_{\ell,a} \rangle \rangle_{\omega^a_\ell} = (f_{\ell,a}, v_{\ell,a})_{\omega^a_\ell} - \langle \langle u^i_L, v_{\ell,a} \rangle \rangle_{\omega^a_\ell} - \sum_{k=0}^{\ell-1} \lambda^i_k \langle \langle \rho^i_k, v_{\ell,a} \rangle \rangle_{\omega^a_\ell}, \quad \forall v_{\ell,a} \in \mathbb{V}_\ell^a
\]

**$\ell$-level update (correction direction):** Define $\rho^i_{\ell} \in \mathbb{V}_\ell^p$ by:
\[
\rho^i_{\ell} := \sum_{a \in \mathcal{V}_\ell} \rho^i_{\ell,a}
\]

**Level-wise step-sizes by line search:** Set
\[
\lambda^i_{\ell} := \frac{(f, \rho^i_{\ell}) - \langle \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k, \rho^i_{\ell} \rangle \rangle_{\omega^a_\ell}}{\| \rho^i_{\ell} \|^2}
\]
Let \( u^i_L \in \mathbb{V}^p_L \) be arbitrary.

**Coarse solve:** Define \( \rho^i_0 \in \mathbb{V}^1_0 \) by:
\[
\langle \rho^i_0 , v_0 \rangle \text{ global lifting } = (f, v_0) - \langle u^i_L , v_0 \rangle \text{ global algebraic residual }, \quad \forall v_0 \in \mathbb{V}^1_0
\]
and set \( \lambda^i_0 := 1 \)

**Level-wise local solves:** For \( \ell = 1 : L \), for all \( a \in \mathbb{V}_\ell \), define \( \rho^i_\ell, a \in \mathbb{V}^a_\ell \) by
\[
\langle \rho^i_\ell, a , v^i_\ell, a \rangle \omega^a_\ell = (f, v^i_\ell, a) \omega^a_\ell - \langle u^i_L , v^i_\ell, a \rangle \omega^a_\ell - \sum_{k=0}^{\ell-1} \lambda^i_k \langle \rho^i_k , v^i_\ell, a \rangle \omega^a_\ell, \quad \forall v^i_\ell, a \in \mathbb{V}^a_\ell
\]

**\( \ell \)-level update (correction direction):** Define \( \rho^i_\ell \in \mathbb{V}^p_\ell \) by:
\[
\rho^i_\ell := \sum_{a \in \mathbb{V}_\ell} \rho^i_\ell, a
\]

**Level-wise step-sizes by line search:** Set
\[
\lambda^i_\ell := \frac{(f, \rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k , \rho^i_\ell \rangle}{\| \rho^i_\ell \|^2}
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in V^p_L \), let \( u^{i+1}_L \in V^p_L \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^{J} (\lambda^i_j \| \rho^i_j \|)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f, \rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k, \rho^i_\ell \rangle}{\| \rho^i_\ell \|^2} \):
**Proposition (Pythagorean error representation of one solver step)**

For \( u^i_L \in \mathbb{V}^p_L \), let \( u^{i+1}_L \in \mathbb{V}^p_L \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\Rightarrow \quad \| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^{J} \left( \lambda^i_{\ell} \| \rho^i_{\ell} \| \right)^2.
\]

\( \lambda^i_{\ell} \) = computable error decrease

**Proof:** From finest to coarsest level and by the optimal step-sizes \( \lambda^i_{\ell} : \frac{(f, \rho^i_{\ell}) - \langle u^i_L + \sum_{k=0}^{L-1} \lambda^i_k \rho^i_k, \rho^i_{\ell} \rangle}{\| \rho^i_{\ell} \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \| u_L - (u^i_L + \sum_{\ell=0}^{L} \lambda^i_{\ell} \rho^i_{\ell}) \|^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_{\ell} \rho^i_{\ell} \|^2 - 2 \lambda^i_L
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_{\ell} \rho^i_{\ell} \|^2 - (\lambda^i_L \| \rho^i_L \|)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^{L} (\lambda^i_{\ell} \| \rho^i_{\ell} \|)^2
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in \mathbb{V}_L^p \), let \( u^{i+1}_L \in \mathbb{V}_L^p \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^{J} \left( \lambda^i_{\ell} \| \rho^i_{\ell} \| \right)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda^i_{\ell} := \frac{(f, \rho^i_{\ell}) - \langle u^i_L + \sum_{k=0}^{L-1} \lambda^i_k \rho^i_k, \rho^i_{\ell} \rangle}{\| \rho^i_{\ell} \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \| u_L - (u^i_L + \sum_{\ell=0}^{L} \lambda^i_{\ell} \rho^i_{\ell}) \|^2
\]
\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_{\ell} \rho^i_{\ell} \|^2 - 2 \lambda^i_L
\]
\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_{\ell} \rho^i_{\ell} \|^2 - (\lambda^i_L \| \rho^i_L \|)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^{L} (\lambda^i_{\ell} \| \rho^i_{\ell} \|)^2.
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in \mathbb{V}^p_L \), let \( u^{i+1}_L \in \mathbb{V}^p_L \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\left\| u_L - u^{i+1}_L \right\|^2 = \left\| u_J - u^i_J \right\|^2 - \sum_{j=0}^{J} \left( \lambda^i_j \| \rho^i_j \| \right)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f,\rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k : \rho^i_\ell \rangle}{\| \rho^i_\ell \|^2} \):

\[
\left\| u_L - u^{i+1}_L \right\|^2 = \left\| u_L - \sum_{\ell=0}^{L} \lambda^i_\ell \rho^i_\ell \right\|^2
\]

\[
= \left\| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \right\|^2 - 2 \lambda^i_L
\]

\[
= \left\| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \right\|^2 - (\lambda^i_L \| \rho^i_L \|)^2 = \ldots = \left\| u_L - u^i_L \right\|^2 - \sum_{\ell=0}^{L} \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2
\]
Proposition (Pythagorean error representation of one solver step)

For $u_L^i \in \mathbb{V}_L^p$, let $u_L^{i+1} \in \mathbb{V}_L^p$ be the next iterate constructed from $u_L^i$ by our solver.

\[
\begin{align*}
\|u_L - u_L^{i+1}\|^2 &= \|u_J - u_J^i\|^2 - \sum_{j=0}^{J} (\lambda_i^j \rho_j^i)^2, \\
&= (\eta_{\text{alg}}^i)^2 \text{ computable error decrease}
\end{align*}
\]

Proof: From finest to coarsest level and by the optimal step-sizes $\lambda_i^\ell := \frac{(f,\rho_i^\ell) - \langle u_L^i + \sum_{k=0}^{\ell-1} \lambda_k^i \rho_k^i, \rho_i^\ell \rangle}{\|ho_i^\ell\|^2}$:

\[
\begin{align*}
\|u_L - u_L^{i+1}\|^2 &= \left\| u_L - \left( u_L^i + \sum_{\ell=0}^{L} \lambda_i^\ell \rho_i^\ell \right) \right\|^2 \\
&= \left\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_i^\ell \rho_i^\ell \right\|^2 - 2 \lambda_i^L \\
&= \left\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_i^\ell \rho_i^\ell \right\|^2 - (\lambda_i^L \|\rho_i^L\|)^2 = \ldots = \|u_L - u_L^i\|^2 - \sum_{\ell=0}^{L} (\lambda_i^\ell \|\rho_i^\ell\|)^2
\end{align*}
\]
Proposition (Pythagorean error representation of one solver step)

For $u_L^i \in V_L^p$, let $u_L^{i+1} \in V_L^p$ be the next iterate constructed from $u_L^i$ by our solver.

\[
\|u_L - u_L^{i+1}\|^2 = \|u_J - u_J^i\|^2 - \sum_{j=0}^{J} (\lambda_k^i \|\rho_k^i\|)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes $\lambda^i_\ell := \frac{(f,\rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda_k^i \rho_k^i, \rho^i_\ell \rangle}{\|\rho^i_\ell\|^2}$:

\[
\|u_L - u_L^{i+1}\|^2 = \|u_L - (u_L^i + \sum_{\ell=0}^L \lambda^i_\ell \rho^i_\ell)\|^2
\]

\[
= \|u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell\|^2 - 2\lambda^i_L \left(\langle u_L^i, \rho^i_L \rangle - \langle u^i_L + \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell, \rho^i_L \rangle\right) + (\lambda^i_L \|\rho^i_L\|)^2
\]

\[
= \|u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell\|^2 - (\lambda^i_L \|\rho^i_L\|)^2 = \ldots = \|u_L - u_L^i\|^2 - \sum_{\ell=0}^{L} (\lambda^i_\ell \|\rho^i_\ell\|)^2
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in \mathbb{V}_L^p \), let \( u^{i+1}_L \in \mathbb{V}_L^p \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^J \left( \lambda^i_j \| \rho^i_\ell \| \right)^2.
\]

\( \Rightarrow \) \( \| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^J \left( \lambda^i_j \| \rho^i_\ell \| \right)^2. \)

\( \sum_{j=0}^J \left( \lambda^i_j \| \rho^i_\ell \| \right)^2 = \left( \eta^i_{\text{alg}} \right)^2 \) computable error decrease

**Proof:** From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f, \rho^i_\ell) - \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k, \rho^i_\ell}{\| \rho^i_\ell \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \left\| u_L - \left( u^i_L + \sum_{\ell=0}^L \lambda^i_\ell \rho^i_\ell \right) \right\|^2
\]

\[
= \left\| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \right\|^2 - 2\lambda^i_L \left( (f, \rho^i_L) - \left\langle u^i_L + \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell, \rho^i_L \right\rangle \right) + \left( \lambda^i_L \| \rho^i_L \| \right)^2
\]

\[
= \left\| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \right\|^2 - \left( \lambda^i_L \| \rho^i_L \| \right)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^L \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in \mathbb{V}_L^p \), let \( u^{i+1}_L \in \mathbb{V}_L^p \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\Rightarrow \quad \| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^J \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2.
\]

**Proof:** From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f, \rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k, \rho^i_\ell \rangle}{\| \rho^i_\ell \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \| u_L - \left( u^i_L + \sum_{\ell=0}^L \lambda^i_\ell \rho^i_\ell \right) \|^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - 2 \lambda^i_L \left( (f, \rho^i_L) - \langle u^i_L + \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell, \rho^i_L \rangle \right) + \left( \lambda^i_L \| \rho^i_L \| \right)^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - \left( \lambda^i_L \| \rho^i_L \| \right)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^L \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2
\]
Proposition (Pythagorean error representation of one solver step)

For \( u_L^i \in \mathbb{V}_L^p \), let \( u_L^{i+1} \in \mathbb{V}_L^p \) be the next iterate constructed from \( u_L^i \) by our solver. 

\[
\Rightarrow \quad \| u_L - u_L^{i+1} \|^2 = \| u_J - u_J^i \|^2 - \sum_{j=0}^J \left( \lambda_j^i \| \rho_j^i \| \right)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda_j^i := \frac{(f, \rho_j^i) - \left\langle u_L^i + \sum_{k=0}^{L-1} \lambda_k^i \rho_k^i, \rho_j^i \right\rangle}{\| \rho_j^i \|^2} \):

\[
\| u_L - u_L^{i+1} \|^2 = \| u_L - \left( u_L^i + \sum_{\ell=0}^L \lambda_{\ell}^i \rho_{\ell}^i \right) \|^2
\]

\[
= \| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_{\ell}^i \rho_{\ell}^i \|^2 - 2 \lambda_L^i \left( (f, \rho_L^i) - \left\langle u_L^i + \sum_{\ell=0}^{L-1} \lambda_{\ell}^i \rho_{\ell}^i, \rho_L^i \right\rangle \right) + \left( \lambda_L^i \| \rho_L^i \| \right)^2
\]

\[
= \| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_{\ell}^i \rho_{\ell}^i \|^2 - (\lambda_L^i \| \rho_L^i \|)^2 = \cdots = \| u_L - u_L^i \|^2 - \sum_{\ell=0}^L \left( \lambda_{\ell}^i \| \rho_{\ell}^i \| \right)^2.
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in \mathbb{V}^p_L \), let \( u^{i+1}_L \in \mathbb{V}^p_L \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^{J} (\lambda^i_j \| \rho^i_j \|)^2.
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f, \rho^i_\ell)}{\| \rho^i_\ell \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \| u_L - (u^i_L + \sum_{\ell=0}^{L} \lambda^i_\ell \rho^i_\ell) \|^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - 2 \lambda^i_L \left( \langle f, \rho^i_L \rangle - \langle \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell, \rho^i_L \rangle \right) + (\lambda^i_L \| \rho^i_L \|)^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - (\lambda^i_L \| \rho^i_L \|)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^{L} (\lambda^i_\ell \| \rho^i_\ell \|)^2
\]

\[
= \| u_L - u^i_L \|^2 - (\eta^i_{\text{alg}})^2
\]
Proposition (Pythagorean error representation of one solver step)

For \( u^i_L \in V^p_L \), let \( u^{i+1}_L \in V^p_L \) be the next iterate constructed from \( u^i_L \) by our solver.

\[
\| u_L - u^{i+1}_L \|^2 = \| u_J - u^i_J \|^2 - \sum_{j=0}^J \left( \lambda^i_j \| \rho^i_j \| \right)^2.
\]

\[
= \left( \eta^i_{\text{alg}} \right)^2 \text{ computable error decrease}
\]

Proof: From finest to coarsest level and by the optimal step-sizes \( \lambda^i_\ell := \frac{(f, \rho^i_\ell) - \langle u^i_L + \sum_{k=0}^{\ell-1} \lambda^i_k \rho^i_k, \rho^i_\ell \rangle}{\| \rho^i_\ell \|^2} \):

\[
\| u_L - u^{i+1}_L \|^2 = \| u_L - (u^i_L + \sum_{\ell=0}^L \lambda^i_\ell \rho^i_\ell) \|^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - 2 \lambda^i_L \left( (f, \rho^i_L) - \langle u^i_L + \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell, \rho^i_L \rangle \right) + \left( \lambda^i_L \| \rho^i_L \| \right)^2
\]

\[
= \| u_L - u^i_L - \sum_{\ell=0}^{L-1} \lambda^i_\ell \rho^i_\ell \|^2 - \left( \lambda^i_L \| \rho^i_L \| \right)^2 = \ldots = \| u_L - u^i_L \|^2 - \sum_{\ell=0}^L \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2
\]

\[
= \| u_L - u^i_L \|^2 - \left( \eta^i_{\text{alg}} \right)^2
\]
Main results

### Theorem ($p$-robust reliable and efficient bound on the algebraic error)

Let $u^i_L \in \mathbb{V}^p_L$ be arbitrary. Let $\eta^i_{\text{alg}}$ be the associated estimator of the algebraic error.

$$\implies \|u_L - u^i_L\| \geq \eta^i_{\text{alg}} \quad \text{and} \quad \eta^i_{\text{alg}} \geq \beta \|u_L - u^i_L\| \quad \text{with} \quad 0 < \beta(\kappa_T, L, d, K) < 1$$

### Theorem ($p$-robust error contraction of the multilevel solver)

For $u^i_L \in \mathbb{V}^p_L$, let $u^{i+1}_\ell \in \mathbb{V}^p_L$ be constructed from $u^i_L$ using one step of the solver.

$$\implies \|u_L - u^{i+1}_\ell\| \leq \alpha \|u_L - u^i_L\| \quad \text{with} \quad \alpha = \sqrt{1 - \beta^2}$$

**Remark:**
- $\beta$ is independent of the polynomial degree $p$
- The dependence on $L$ is at most linear under minimal $H^1$-regularity
- Complete independence from $L$ is obtained in $H^2$-regularity setting
- $\|u_L - u^{i+1}_L\|^2 = \|u_L - u^i_L\|^2 - (\eta^i_{\text{alg}})^2$
Main results

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Main results

Theorem \((p\text{-robust reliable and efficient bound on the algebraic error})\)

Let \(u^i_L \in \nabla^p_L\) be arbitrary. Let \(\eta^i_{\text{alg}}\) be the associated estimator of the algebraic error.

\[
\Rightarrow \|u_L - u^i_L\| \geq \eta^i_{\text{alg}} \quad \text{and} \quad \eta^i_{\text{alg}} \geq \beta \|u_L - u^i_L\| \quad \text{with} \quad 0 < \beta(\kappa_T, L, d, K) < 1
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Theorem \((p\text{-robust error contraction of the multilevel solver})\)

For \(u^i_L \in \nabla^p_L\), let \(u^{i+1}_\ell \in \nabla^p_L\) be constructed from \(u^i_L\) using one step of the solver.

\[
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\]

Remark:

- \(\beta\) is independent of the polynomial degree \(p\)
- The dependence on \(L\) is at most \textit{linear} under minimal \(H^1\)-regularity
- Complete \textit{independence} from \(L\) is obtained in \(H^2\)-regularity setting
- \(\|u_L - u^{i+1}_L\|^2 = \|u_L - u^i_L\|^2 - (\eta^i_{\text{alg}})^2\)
Stopping criterion:
\[
\frac{\|F_L - A_L U_L^{i_s}\|}{\|F_L\|} \leq 10^{-5} \frac{\|F_L - A_L U_L^0\|}{\|F_L\|}
\]

The mesh hierarchies here are obtained from \( L \) uniform refinements of an initial Delaunay mesh \( T_0 \)

<table>
<thead>
<tr>
<th>( L )</th>
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</table>
**Stopping criterion:**

\[
\frac{\|F_L - A_L U_{L}^{i_{s}}\|}{\|F_L\|} \leq 10^{-5} \frac{\|F_L - A_L U_{L}^{0}\|}{\|F_L\|}.
\]

The mesh hierarchies here are obtained from \(L\) **uniform refinements** of an initial Delaunay mesh \(T_0\).

**Numerical \(K\)- and \(L\)-robustness is observed even in low-regularity cases**
Stopping criterion:
\[
\frac{\|F_L - A_L U_i^L\|}{\|F_L\|} \leq 10^{-5} \frac{\|F_L - A_L U_0^L\|}{\|F_L\|}.
\]

The mesh hierarchies here are obtained from \textit{L uniform refinements} of an initial Delaunay mesh \(T_0\).
**Stopping criterion:**

\[
\frac{\| F_L - A_L U_L^i \|}{\| F_L \|} \leq 10^{-5} \frac{\| F_L - A_L U_L^0 \|}{\| F_L \|}.
\]

The mesh hierarchies here are obtained from \( L \) uniform refinements of an initial Delaunay mesh \( \mathcal{T}_0 \).

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<th>( \text{Peak} )</th>
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Numerical \( K \)- and \( L \)-robustness is observed even in low-regularity cases.
$p$-robustness: iteration numbers for graded meshes

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**Peak, $1,p \rightarrow p$**

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**L-shape, $K=I, 1,p \rightarrow p$**

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**Checkerboard, $J(K) = O(10^6), 1,p \rightarrow p$**

Low-regularity tests: indicate linear $L$-dependence in accordance with the theory.
$p$-robustness: iteration numbers for graded meshes

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L-shape, $K=I$, $1,p \to p$

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Checkerboard, $J(K) = O(10^6)$, $1,p \to p$

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Low-regularity tests: indicate linear $L$-dependence in accordance with the theory
We compare our methods with solvers from literature in terms of the number of iterations (and CPU times).

Comparison with other multilevel solvers

We compare our methods with solvers from literature in terms of the number of iterations (and CPU times).

<table>
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<th>MG(1,1)-PCG(iChol)</th>
<th>MG(0,1)-bGS</th>
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We compare our methods with solvers from literature in terms of the number of iterations (and CPU times).

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\( \sim \text{MG}(0,1)-\text{bJ} \), \( \sim \text{MG}(0,\text{adapt})-\text{bJ} (\text{wRAS}) \), \( \text{PCG(MG(3,3)-bJ)} \), \( \text{MG(1,1)-PCG(iChol)} \), \( \text{MG(0,1)-bGS} \), \( \text{MG(3,3)-GS} \)

\( p \rightarrow p \), \( 1 \rightarrow p \)

\( \not\text{p-robust} \)

---

Adaptivity in a-posteriori-steered solvers
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 \]

1. localization by levels

\[ \| P_0 \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2 \]

2. localization by patches

- Adaptive number of post-smoothing steps
- Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[ \| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2 \]

1. localization by levels
2. localization by patches

\[ \lambda_{\ell}^i \| \rho_{\ell}^i \| \]

\[ \| u_L - u_L^i \|^2 \]

\[ (\eta_{\text{alg}}^i)^2 \]

\[ \sum_{\ell=0}^{L} (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 \]

\[ \| \rho_0^i \|^2 \]

\[ \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2 \]
Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda^i_\ell \| \rho^i_\ell \|)^2 = \| \rho^i_0 \|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in V_\ell} \| \rho_{\ell,a} \|^2 \]

1. localization by levels

2. localization by patches
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} \left( \lambda^i_{\ell} \| \rho^i_{\ell} \| \right)^2 = \| \rho^i_0 \|^2 + \sum_{\ell=1}^{L} \lambda^i_{\ell} \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2 \omega^a_{\ell} \]

1. Localization by levels
2. Localization by patches

① Adaptive number of post-smoothing steps
② Adaptive local smoothing
Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[\|u_L - u_L^i\|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_i^\ell \|\rho^i_{\ell}\|)^2 = \|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_i^\ell \sum_{a \in V_\ell} \|\rho_{\ell,a}\|^2 \omega_a^\ell\]

1. localization by **levels**
2. localization by **patches**

1. Adaptive number of post-smoothing steps
2. Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[
\| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda^i_\ell \| \rho^i_\ell \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in V_\ell} \| \rho_{\ell,a} \|^2_{\omega_a} \]

1. **Adaptive number of post-smoothing steps**

2. **Adaptive local smoothing**

---

\[\eta_{\text{alg}}^i = \sum_{\ell=0}^{L} (\lambda^i_\ell \| \rho^i_\ell \|)\]

---

\[\| \rho_{\ell,a} \|^2_{\omega_a} = \sum_{a \in V_\ell} \| \rho_{\ell,a} \|^2_{\omega_a}\]
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[
\| u_L - u_L^i \| \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} \left( \lambda_{\ell} \| \rho_{\ell}^i \| \right)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell} \sum_{a \in \mathcal{V}_\ell} \| \rho_{\ell,a} \|_{\omega_a}^2
\]

1. **Localization by levels**
2. **Localization by patches**

1. **Adaptive number of post-smoothing steps**

2. **Adaptive local smoothing**

![Diagram](image-url)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} \left( \lambda_{i}^\ell \| \rho_{\ell}^i \| \right)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{i}^\ell \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|_{\omega_{a}}^2 \]

1) localization by levels

2) localization by patches

---

\begin{itemize}
  \item [1] Adaptive number of post-smoothing steps
  \item [2] Adaptive local smoothing
\end{itemize}
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} \left( \lambda^i_\ell \| \rho^i_\ell \| \right)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in \mathcal{V}_\ell} \| \rho_{\ell,a} \|^2 \]

1. Localization by levels
2. Localization by patches

1. Adaptive number of post-smoothing steps

Non-adaptive

Adaptive

Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[
\|u_L - u^i_L\|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} \left( \lambda^i_\ell \|\rho^i_\ell\| \right)^2 = \|\rho^i_0\|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in V_\ell} \|\rho_{\ell,a}\|^2 \omega^a_\ell
\]

1. **localization by levels**

2. **localization by patches**

1. **Adaptive number of post-smoothing steps**

2. **Adaptive local smoothing**
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2 \omega_{\ell}^a \]

1. localization by levels
2. localization by patches

1. Adaptive number of post-smoothing steps

Non-adaptive

Adaptive

Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda_\ell^i \| \rho^i_\ell \|^2) = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_\ell} \| \rho^i_{\ell,a} \|^2 \]

1. localization by levels

2. localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

$$\| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell} \| \rho_{\ell}^i \|)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell} \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2$$

1. Localization by levels
2. Localization by patches

1. Adaptive number of post-smoothing steps
2. Adaptive local smoothing

Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[
\| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda^i_{\ell} \| \rho^i_{\ell} \|)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \lambda^i_{\ell} \sum_{a \in V_{\ell}} \| \rho_{\ell,a} \|^2_{\omega^a}
\]

① localization by **levels**

② localization by **patches**

### ① Adaptive number of post-smoothing steps

Non-adaptive | Adaptive
---|---
\[ u^i_L \rightarrow u^{i+1}_L \rightarrow u^{i+1}_L \rightarrow \cdots \]

### ② Adaptive local smoothing

Ani Miraçi (TU Wien)
Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} \left( \lambda^i_{\ell} \| \rho^i_{\ell} \| \right)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \sum_{a \in V^\ell} \| \rho_{\ell,a} \|^2 \omega^a_{\ell} \]

1. Localization by levels

2. Localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \|u_L - u_L^*\|^2 \approx (\eta_{alg}^*)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell}^i \|\rho_{\ell}^i\|)^2 = \|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \|\rho_{\ell,a}\|_{\omega_{\ell}^a}^2 \]

1. localization by levels
2. localization by patches

**1. Adaptive number of post-smoothing steps**

Non-adaptive

Adaptive

**2. Adaptive local smoothing**

Ani Miraçi (TU Wien)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda^i_{\ell} \| \rho^i_{\ell} \|)^2 \]

1 localization by levels

\[ \| \rho^i_{0} \|^2 + \sum_{\ell=1}^{L} \lambda^i_{\ell} \sum_{a \in V_{\ell}} \| \rho^i_{\ell, a} \|_{\omega^a_{\ell}}^2 \]

2 localization by patches

1 Adaptive number of post-smoothing steps

Adaptive local smoothing

# Adaptable number of post-smoothing steps

<table>
<thead>
<tr>
<th>Non-adaptive</th>
<th>Adaptive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^i_{\ell}$</td>
<td>$v^i_{\ell}$</td>
</tr>
<tr>
<td>$\rho^i_{\ell}$</td>
<td>$\rho^i_{\ell}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$v^i_{\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>L-1</td>
<td>1</td>
</tr>
<tr>
<td>L</td>
<td>0</td>
</tr>
</tbody>
</table>

\( (\eta^i_{\text{alg}})^2 \)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[
\|u_L - u^i_L\|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda^i_\ell \|\rho^i_\ell\|)^2 = \|\rho^i_0\|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in V^\ell} \|\rho^i_\ell, a\|_{\omega^a}^2
\]

Localization by levels

Localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u_L^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} \left( \lambda_{\ell} \| \rho_{\ell}^i \| \right)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell} \sum_{a \in \mathcal{V}_\ell} \| \rho_{\ell},a \|_{\omega_{\ell}} \]

1. localization by levels
2. localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing

Ani Miraşcu (TU Wien)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} (\lambda^i_{\ell} \| \rho^i_{\ell} \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda^i_{\ell} \sum_{a \in V_{\ell}} \| \rho^i_{\ell,a} \|^2_{\omega^a} \]

\( \text{① localization by levels} \)

\( \text{② localization by patches} \)

① Adaptive number of post-smoothing steps

② Adaptive local smoothing

Ani Miraçi (TU Wien)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell} \| \rho_{\ell}^i \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2 \]

\begin{enumerate}
  \item localization by levels
  \item localization by patches
\end{enumerate}

\[ \llbracket \frac{\lambda_{\ell}}{\rho_{\ell}^i} \| \rho_{\ell}^i \| \rrbracket = \frac{\| \rho_{\ell}^0 \|}{\| \rho_{\ell}^i \|} + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2 \]

\[ \begin{align*}
\| u_L - u^i_L \| &= \sum_{\ell=0}^{L} (\lambda_{\ell} \rho_{\ell}^i)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2 \\
&= \llbracket \frac{\lambda_{\ell}}{\rho_{\ell}^i} \| \rho_{\ell}^i \| \rrbracket = \frac{\| \rho_{\ell}^0 \|}{\| \rho_{\ell}^i \|} + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2
\end{align*} \]

1. Adaptive number of post-smoothing steps

\begin{align*}
\text{Non-adaptive} & \quad \text{Adaptive} \\
\| u_L - u^i_L \|^2 &= \sum_{\ell=0}^{L} (\lambda_{\ell} \rho_{\ell}^i)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2 \\
&= \llbracket \frac{\lambda_{\ell}}{\rho_{\ell}^i} \| \rho_{\ell}^i \| \rrbracket = \frac{\| \rho_{\ell}^0 \|}{\| \rho_{\ell}^i \|} + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2
\end{align*}

2. Adaptive local smoothing

\[ \begin{align*}
\text{Non-adaptive} & \quad \text{Adaptive} \\
\| u_L - u^i_L \|^2 &= \sum_{\ell=0}^{L} (\lambda_{\ell} \rho_{\ell}^i)^2 = \| \rho_0 \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2 \\
&= \llbracket \frac{\lambda_{\ell}}{\rho_{\ell}^i} \| \rho_{\ell}^i \| \rrbracket = \frac{\| \rho_{\ell}^0 \|}{\| \rho_{\ell}^i \|} + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_{\ell}} \| \rho_{\ell}, a \|_{\omega_{\ell}^a}^2
\end{align*} \]
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u_i^L \|^2 \approx (\eta_{alg}^i)^2 = \sum_{\ell=0}^{L} (\lambda_\ell^i \| \rho^i_\ell \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_\ell} \| \rho_{\ell,a} \|_{\omega_{\ell,a}}^2 \]

1. Localization by levels
2. Localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing

Adaptive local smoothing
Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta^i_{\text{alg}})^2 = \sum_{\ell=0}^{L} \left( \lambda^i_{\ell} \| \rho^i_{\ell} \| \right)^2 = \| \rho^i_0 \|^2 + \sum_{\ell=1}^{L} \lambda^i_{\ell} \sum_{a \in V_\ell} \| \rho^i_{\ell,a} \|^2 \]

1. localization by levels
2. localization by patches

1. Adaptive number of post-smoothing steps

Adaptive local smoothing

Non-adaptive

Adaptive

Ani Miraçi (TU Wien)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx \left( \eta_{\text{alg}}^i \right)^2 = \sum_{\ell=0}^{L} \left( \lambda_{\ell}^i \| \rho_{\ell}^i \| \right)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \sum_{a \in \mathcal{V}_\ell} \lambda_{\ell}^i \| \rho_{\ell,a} \|_{\omega_{\ell}}^2 \]

1. localization by levels
2. localization by patches

1. Adaptive number of post-smoothing steps
2. Adaptive local smoothing

Adaptive local smoothing

Non-adaptive

Adaptive

Adaptive local smoothing

full-smoothing substep

Non-adaptive

Adaptive

estimated distribution of \( \| u_L - u^i_L \| \) on level \( \ell = 1 \)

Adaptive local smoothing

non-adaptive

Adaptive

estimated distribution of \( \| u_L - u^i_L \| \) on level \( \ell = 1 \)
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

$$||u_L - u^i_L||^2 \approx (\eta_{alg}^i)^2 = \sum_{\ell=0}^{L} (\lambda^i_\ell ||\rho^i_\ell||)^2 = ||\rho_0^i||^2 + \sum_{\ell=1}^{L} \sum_{a \in V_\ell} \lambda^i_\ell ||\rho_\ell,a||^2_{\omega_\ell}$$

1. localization by levels

2. localization by patches

1. Adaptive number of post-smoothing steps

2. Adaptive local smoothing

Adaptive local smoothing

Marking the problematic regions

Non-adaptive

Adaptive
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

\[
\|u_L - u_i^L\|^2 \approx \left(\eta_{\text{alg}}^i\right)^2 = \sum_{\ell=0}^{L} \left(\lambda_{\ell}^i \|\rho_{\ell}^i\|\right)^2 = \|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_{\ell}} \|\rho_{\ell,a}\|^2 \omega_{\ell,a}
\]

1. localization by **levels**
2. localization by **patches**

1. **Adaptive number of post-smoothing steps**

2. **Adaptive local smoothing**
Adaptivity in a-posteriori-steered solvers

Starting point: equivalence of the algebraic error with a localized a posteriori estimate

\[ \| u_L - u^i_L \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^L (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^L \sum_{a \in V_{\ell}} \lambda_{\ell}^i \| \rho_{\ell,a} \|^2 \omega_{\ell}^a \]

1. localization by levels
2. localization by patches

1. Adaptive number of post-smoothing steps
2. Adaptive local smoothing

Non-adaptive
Adaptive

Adaptive local smoothing

Ani Miraçi (TU Wien)
Adaptive finite element setting
Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost for contractive solvers.

Algorithm

**Input** $\mathcal{T}_0, u^0_0, 0 < \theta \leq 1$

For each $L = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE** For $i = 1, 2, \ldots, i_s$, repeat
  - compute $u^i_L, \eta^i_{\text{alg}} =: \eta_{\text{alg}}(u^i_L)$
  - compute $\eta_{\text{disc}}(T, u^i_L)$ for all $T \in \mathcal{T}_L$
  
  until $\eta_{\text{alg}}(u^i_L) \leq \mu \eta_{\text{disc}}(u^i_L)$ $\rightarrow$ idea: equilibrate algebraic and discretization errors

- **MARK** choose $\mathcal{M}_L \subseteq \mathcal{T}_L$ such that
  
  $\theta \sum_{T \in \mathcal{T}_L} \eta_{\text{disc}}(T, u^i_L)^2 \leq \sum_{T \in \mathcal{M}_L} \eta_{\text{disc}}(T, u^i_L)^2$

- **REFINE** $\mathcal{T}_{L+1} := \text{refine}(\mathcal{T}_L, \mathcal{M}_L), \quad u^0_{L+1} := u^i_L$

**Output** Discrete solutions $u^i_L$ and corresponding estimators $\eta_{\text{alg}}(u^i_L), \eta_{\text{disc}}(u^i_L)$

---


Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost for contractive solvers.

**Algorithm**

**Input** $\mathcal{T}_0, u_0^0$, $0 < \theta \leq 1$ sufficiently small, $\mu > 0$ sufficiently small

For each $L = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE** For $i = 1, 2, \ldots, i_s$, repeat
  - compute $u^i_L$, $\eta^i_{\text{alg}} =: \eta_{\text{alg}}(u^i_L)$
  - compute $\eta_{\text{disc}}(T, u^i_L)$ for all $T \in \mathcal{T}_L$

  until $\eta_{\text{alg}}(u^i_s) \leq \mu \eta_{\text{disc}}(u^i_s)$ \quad idea: equilibrate algebraic and discretization errors

- **MARK** choose $\mathcal{M}_L \subset \mathcal{T}_L$ such that
  \[ \theta \sum_{T \in \mathcal{T}_L} \eta_{\text{disc}}(T, u^i_s)^2 \leq \sum_{T \in \mathcal{M}_L} \eta_{\text{disc}}(T, u^i_s)^2 \]

- **REFINE** $\mathcal{T}_{L+1} := \text{refine}(\mathcal{T}_L, \mathcal{M}_L)$, $u^0_{L+1} := u^i_s$ \quad nested iterations with error control on all $u^i_L$ except $u^0_0$

**Output** Discrete solutions $u^i_s$ and corresponding estimators $\eta_{\text{alg}}(u^i_s), \eta_{\text{disc}}(u^i_s)$

---


Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost for contractive solvers.

Algorithm

Input $\mathcal{T}_0$, $u^0_0$, $0 < \theta \leq 1$ sufficiently small, $\mu > 0$ sufficiently small

For each $L = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE**  For $i = 1, 2, \ldots, i_s$, repeat
  - compute $u^i_L$, $\eta^i_{alg} =: \eta_{alg}(u^i_L)$
  - compute $\eta_{disc}(T, u^i_L)$ for all $T \in \mathcal{T}_\ell$

  until $\eta_{alg}(u^{i_s}_L) \leq \mu \eta_{disc}(u^{i_s}_L)$  \(\rightarrow\) idea: equilibrate algebraic and discretization errors

- **MARK** choose $\mathcal{M}_L \subseteq \mathcal{T}_L$ such that
  \[ \theta \sum_{T \in \mathcal{T}_L} \eta_{disc}(T, u^{i_s}_L)^2 \leq \sum_{T \in \mathcal{M}_L} \eta_{disc}(T, u^{i_s}_L)^2 \]

- **REFINE** $\mathcal{T}_{L+1} := \text{refine}(\mathcal{T}_L, \mathcal{M}_L)$,  $u^0_{L+1} := u^{i_s}_L$  \(\rightarrow\) nested iterations with error control on all $u^1_L$ except $u^0_0$

Output Discrete solutions $u^{i_s}_L$ and corresponding estimators $\eta_{alg}(u^{i_s}_L), \eta_{disc}(u^{i_s}_L)$

References:

Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost for contractive solvers.

Algorithm

Input $\mathcal{T}_0$, $u^0_0$, $0 < \theta \leq 1$ sufficiently small, $\mu > 0$ sufficiently small

For each $L = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE** For $i = 1, 2, \ldots, i_s$, repeat
  - compute $u^i_L$, $\eta^i_{\text{alg}} =: \eta_{\text{alg}}(u^i_L)$
  - compute $\eta_{\text{disc}}(T, u^i_L)$ for all $T \in \mathcal{T}_L$

until $\eta_{\text{alg}}(u^i_{i_s}) \leq \mu \eta_{\text{disc}}(u^i_{i_s})$ $\rightarrow$ idea: equilibrate algebraic and discretization errors

- **MARK** choose $\mathcal{M}_L \subseteq \mathcal{T}_L$ such that

$$\theta \sum_{T \in \mathcal{T}_L} \eta_{\text{disc}}(T, u^i_{i_s})^2 \leq \sum_{T \in \mathcal{M}_L} \eta_{\text{disc}}(T, u^i_{i_s})^2$$

- **REFINE** $\mathcal{T}_{L+1} := \text{refine}(\mathcal{T}_L, \mathcal{M}_L)$, $u^0_{L+1} := u^i_{i_s}$ $\rightarrow$ nested iterations with error control on all $u^i_L$ except $u^0_0$

Output Discrete solutions $u^i_{i_s}$ and corresponding estimators $\eta_{\text{alg}}(u^i_{i_s}), \eta_{\text{disc}}(u^i_{i_s})$

---


Gantner, Haberl, Praetorius, Schimanko. Math. Comp. 2021


Key to obtaining $L$-robustness

Remark: From now on, consider $p_0 = \ldots = p_{\ell-1} = 1$ and $p_L = p$.

For intermediate levels $\ell \in \{1, \ldots, L - 1\}$:

- smoothing on all patches

For the finest level $L$: smoothing on all patches when $p > 1$.

Takeaway message:
- $L$-robustness by local smoothing on lowest-order levels
- $p$-robustness by smoothing on all patches of the high-order level
- the new construction guarantees linear cost of the solver step
Remark: From now on, consider $p_0 = \ldots = p_{\ell-1} = 1$ and $p_L = p$.

For intermediate levels $\ell \in \{1, \ldots, L-1\}$:

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**Main results**

**Theorem (\(h\)- and \(p\)-robust reliable and efficient bound on the algebraic error)**

Let \(u^i_L \in \nabla^p_L\) be arbitrary. Let \(\eta^i_{\text{alg}}\) be the associated estimator on the algebraic error.

\[
\Rightarrow \quad \|u^i_L - u^i_L\| \geq \eta^i_{\text{alg}} \quad \text{and} \quad \eta^i_{\text{alg}} \geq \beta \|u^i_L - u^i_L\| \quad \text{with} \quad 0 < \beta(\kappa_T, d, K) < 1
\]

**Theorem (\(h\)- and \(p\)-robust error contraction of the multilevel solver)**

For \(u^i_L \in \nabla^p_L\), let \(u^{i+1}_L \in \nabla^p_L\) be constructed from \(u^i_L\) using one step of the solver.

\[
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\]

**Remark:** Complete independence from \(L\) is obtained even under minimal \(H^1\)-regularity.
**Main results**

**Theorem (\(h\)- and \(p\)-robust reliable and efficient bound on the algebraic error)**

Let \(u^i_L \in V^p_L\) be arbitrary. Let \(\eta^i_{\text{alg}}\) be the associated estimator on the algebraic error.

\[
\Rightarrow \quad \|u_L - u^i_L\| \geq \eta^i_{\text{alg}} \quad \text{and} \quad \eta^i_{\text{alg}} \geq \beta \|u_L - u^i_L\| \quad \text{with} \quad 0 < \beta(\kappa_T, d, K) < 1
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Theorem \((h\text{- and } p\text{-robust reliable and efficient bound on the algebraic error})\)

Let \(u^i_L \in \mathbb{V}^p_L\) be arbitrary. Let \(\eta_{\text{alg}}^i\) be the associated estimator on the algebraic error.

\[ \Rightarrow \quad \|u_L - u^i_L\| \geq \eta_{\text{alg}}^i \quad \text{and} \quad \eta_{\text{alg}}^i \geq \beta \|u_L - u^i_L\| \quad \text{with} \quad 0 < \beta(\kappa_T, d, K) < 1 \]

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L-shape problem

Innerberger, Praetorius. MooAFEM: An object oriented Matlab code for higher-order (nonlinear) adaptive FEM.

May 30, 2022
Conclusion
We presented:

- A $p$-robustly efficient a posteriori algebraic error estimator
- A $p$-robust contractive multigrid solver steered by the a posteriori estimator
- Optimal level-wise step-sizes in the error correction stage
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Miraçi, Praetorius, and Streitberger. Optimal local $p$-robust multigrid for FEM on graded bisection grids. *In preparation.*

Dr. Ani Miraçi

TU Wien
Institute of Analysis and Scientific Computing
NumPDEs group
ani.miraci@asc.tuwien.ac.at
https://www.asc.tuwien.ac.at/~amiraci
Proving the efficiency of the a posteriori estimator \( \eta_{\text{alg}}^i \) is equivalent to proving the solver contraction.

**Proof:** By using the link between solver and estimator given by the Pythagorean formula, there holds:

\[
(\eta_{\text{alg}}^i)^2 \geq \beta^2 \| u_L - u_{L}^i \|^2 \quad \text{(estimator efficiency)}
\]

\[
\Leftrightarrow \| u_L - u_{L}^i \|^2 - \| u_L - u_{L}^{i+1} \|^2 \geq \beta^2 \| u_L - u_{L}^i \|^2
\]

\[
\Leftrightarrow \| u_L - u_{L}^{i+1} \|^2 \leq (1 - \beta^2) \| u_L - u_{L}^i \|^2 \quad \text{(solver contraction)}.
\]

---

Let the assumptions of Theorem 2 hold. Then

\[
\| u_L - u_{L}^i \|^2 \approx (\eta_{\text{alg}}^i)^2 = \sum_{\ell=0}^{L} (\lambda_{\ell}^i \| \rho_{\ell}^i \|)^2 = \| \rho_0^i \|^2 + \sum_{\ell=1}^{L} \lambda_{\ell}^i \sum_{a \in V_\ell} \| \rho_{\ell,a} \|^2 \omega_{\ell}^a.
\]
**L-shape problem**, $L = 3$, and mesh hierarchy $p_\ell = 1$ (left) and $p_\ell = p$ (right), $\ell \in \{1, \ldots, L - 1\}$
Stopping criterion:

\[
\frac{\|F_L - A_L U^i_L\|}{\|F_L\|} \leq 10^{-5} \frac{\|F_L - A_L U^0_L\|}{\|F_L\|}.
\]

The mesh hierarchies here are obtained from \(L\) \textit{uniform refinements} of an initial Delaunay mesh \(\mathcal{T}_0\).

<table>
<thead>
<tr>
<th>(L)</th>
<th>(p)</th>
<th>DoF</th>
<th>Sine</th>
<th>L-shape</th>
<th>Skyscraper</th>
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<tr>
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<td>19</td>
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Numerical \(K\)- and \(L\)-robustness is observed even in low-regularity cases.
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\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
L & p & DoF & Sine & Peak & L-shape & Checkerboard & Skyscraper \\
\hline
3 & 1 & \text{2e}^4 & i_5 & i_5 & i_5 & i_5 & i_5 & i_5 & i_5 & i_5 & i_5 & i_5 \\
3 & 1 & \text{e}^5 & 19 & 19 & 19 & 19 & 21 & 21 & 18 & 18 & 18 & 18 \\
6 & 6 & \text{e}^5 & 30 & 13 & 28 & 14 & 29 & 11 & 27 & 11 & 28 & 11 \\
6 & 1 & \text{e}^6 & 31 & 14 & 30 & 14 & 26 & 9 & 24 & 9 & 25 & 10 \\
9 & 1 & \text{e}^6 & 31 & 14 & 30 & 14 & 23 & 9 & 23 & 9 & 26 & 10 \\
4 & 6 & \text{e}^4 & 21 & 21 & 20 & 20 & 21 & 21 & 19 & 19 & 19 & 19 \\
3 & 6 & \text{e}^5 & 29 & 13 & 28 & 14 & 28 & 11 & 26 & 11 & 27 & 11 \\
6 & 2 & \text{e}^6 & 31 & 13 & 30 & 14 & 25 & 9 & 24 & 9 & 24 & 9 \\
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\hline
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### \( H^2 \)-regular

<table>
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<th>Sine ( K = I )</th>
<th>Peak ( K = I )</th>
<th>L-shape ( K = I )</th>
<th>Checkerboard ( J(K) = O(10^6) )</th>
<th>Skyscraper ( J(K) = O(10^7) )</th>
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### \( H^1 \)-regular

<table>
<thead>
<tr>
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<th>( p )</th>
<th>( \text{DoF} )</th>
<th>( i_s )</th>
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<td>9</td>
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<td>23</td>
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</table>

Numerical \( K \)- and \( L \)-robustness is observed even in low-regularity cases.
Numerical tests in three space dimensions

**Test cases:** exact solution $u$ when available; $K = I$ except where explicitly specified, uniform mesh refinement, $p_\ell = 1$, $\ell \in \{1, \ldots, L\}$, and $L = 4$.

**Cube:** $\Omega := (0, 1)^3$, $u(x, y, z) = x(x-1)y(y-1)z(z-1)$.

**Nested cubes:** $\Omega := (-1, 1)^3$, unknown analytic solution, $K = 10^5 \ast I$ in $(-0.5, 0.5)^3$.

**Checkers cubes:** $\Omega := (0, 1)^3$, unknown analytic solution, $K = 10^6 \ast I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.
Test cases: exact solution $u$ when available; $K = I$ except where explicitly specified, uniform mesh refinement, $p_\ell = 1$, $\ell \in \{1, \ldots, L\}$, and $L = 4$.

Cube: $\Omega := (0,1)^3$, 

\[ u(x,y,z) = x(x-1)y(y-1)z(z-1). \]

Nested cubes: $\Omega := (-1,1)^3$, 

unknown analytic solution, 

\[ K = 10^5 \ast I \text{ in } (-0.5,0.5)^3. \]

Checkers cubes: $\Omega := (0,1)^3$, 

unknown analytic solution, 

\[ K = 10^6 \ast I \text{ in } (0,0.5)^3 \cup (0.5,1)^3. \]
Numerical advantages of optimal step-sizes

Level-wise optimal step-sizes determined by line search:

- **analytically**: Pythagorean formula for the algebraic error
- **numerically**: advantages of using even a single global step-size on level $L$

<table>
<thead>
<tr>
<th>$L$</th>
<th>$p$</th>
<th>Sine</th>
<th>Peak</th>
<th>L-shape</th>
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<td>MG(0,1)-J</td>
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<td>9</td>
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<td>-</td>
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</table>

For $p = 1$: wRAS and MG(0,1)-J only differ by the use of the global optimal step-size.
Number of post-smoothing steps: adaptive vs fixed

Checkerboard $O(10^6)$ problem, $J=3$, $p=6$, $p_i=[1666]$, $\theta=0.2$

<table>
<thead>
<tr>
<th>adapt</th>
<th>$\nu = 1$</th>
<th>$\nu = 3$</th>
<th>$\nu = 5$</th>
<th>$\nu = 10$</th>
</tr>
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<tr>
<td>nflops</td>
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<td>40</td>
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<tr>
<td>sync</td>
<td>3</td>
<td>84</td>
<td>93</td>
<td></td>
</tr>
</tbody>
</table>

Checkerboard $O(10^6)$ problem, $J=3$, $p=6$, $p_i=[1116]$, $\theta=0.2$

<table>
<thead>
<tr>
<th>adapt</th>
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<th>$\nu = 3$</th>
<th>$\nu = 5$</th>
<th>$\nu = 10$</th>
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<tr>
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<tr>
<td>sync</td>
<td>1</td>
<td>143</td>
<td>155</td>
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</table>

nflops := $\frac{|V_0|^3}{3} + \sum_{\ell=1}^{L} \sum_{a \in V_L} \frac{\text{ndof}(V^a_\ell)^3}{3} + \sum_{i=1}^{i_a} \left[ 2|V_0|^2 + \sum_{\ell=1}^{L} \nu^i_{\ell} \sum_{a \in V_L} 2\text{ndof}(V^a_\ell)^2 \right] + \sum_{i=1}^{i_a} \sum_{\ell=1}^{L} \left[ 2 \text{nnz}(I^L_{\ell-1}) + 2 \text{nnz}(I^L_{\ell-1}) + 2\nu^i_{\ell} \text{nnz}(A_L) + 3\nu^i_{\ell} (2 \text{size}(A_L)) \right]$;

sync := $i_s + \sum_{i=1}^{i_a} \sum_{\ell=1}^{L} \nu^i_{\ell}$.

Ani Miraçi (TU Wien)
Can we predict the distribution of the algebraic error?

Dörfler’s bulk-chasing criterion: \( \theta^2 \left( \| K^{1/2} \nabla \rho^0 \|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{a \in V_L} \| K^{1/2} \nabla \rho_{\ell,a} \|^2_{\omega^a_\ell} \right) \leq \sum_{\ell \in \mathcal{M}} \lambda^i_\ell \sum_{a \in \mathcal{M}_L} \| K^{1/2} \nabla \rho_{\ell,a} \|^2_{\omega^a_\ell} \).

**Hierarchy**: uniform refinement, \( L = 2, p_1 = p_2 = 3 \).

- local algebraic error indicators \( \| \rho_{\ell,a} \|_{\omega^a_\ell} \)
- local algebraic error distribution \( \| \tilde{\rho}^i_\ell \|_{\omega^a_\ell} \)

with \( \tilde{\rho}^i_0 = \rho^i_0 \) and \( \tilde{\rho}^i_\ell \in \nabla_{\omega^a_\ell} \), for \( \ell \in \{1, \ldots, L\} \), given by

\[
\langle \tilde{\rho}^i_\ell, v_\ell \rangle = (f, v_\ell) - (u_{i,L}, v_\ell) - \sum_{k=0}^{\ell-1} \langle \tilde{\rho}^i_k, v_\ell \rangle \quad \forall v_\ell \in \nabla_{\omega^a_\ell},
\]

so that \( \sum_{\ell=0}^{L} \tilde{\rho}^i_\ell = u_L - u_{i,L} \).
Does the adaptivity pay off?

Skyscraper $O(10^2)$ test case (non-adaptive 15 iterations)

Skyscraper $O(10^5)$ test case (non-adaptive 15 iterations)

Hierarchy: $L = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$