An Analysis of the Saturation Assumption and Application to Adaptive Finite Elements

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Outline of Talk

Our Saturation Assumption

$$|u - \hat{\mathcal{I}}_p u|_{1,t} \leq \beta |u - \mathcal{I}_p u|_{1,t}$$

- t is a single shape regular simplex in $\mathcal{R}^d$, $1 \leq d \leq 3$, of size h.
- $|u|_{1,t} \equiv \|\nabla u\|_{L_2(t)}$ is the $\mathcal{H}^1(t)$ semi-norm.
- $\mathcal{S}_p$ is the space of polynomials of degree p.
- $\mathcal{I}_p u \in \mathcal{S}_p$ is the usual Lagrange interpolant of function u.
- $\hat{\mathcal{I}}_p u$ corresponds to h or p refinement of t. For p refinement $\hat{\mathcal{I}}_p \equiv \mathcal{I}_{p+1}$. For h refinement $\hat{\mathcal{I}}_p$ is $\mathcal{I}_p$ applied to child elements.
- $\beta \equiv \beta(u) < 1$.

Saturation Assumptions are typically global, refer to finite element solutions, and allow p refinement only.

Preliminaries

Assume a priori error estimates

$$\|(1-\mathcal{I}_p)u\|_{H^r(t)} \leq C(q,r)h^{q-r}|u|_{H^q(t)}$$

for $0 \leq r < q \leq p+1$ hold. Assume $\beta_0 < 1$ where

$$\beta_0 \equiv \max_{\nu \in \mathcal{S}_{p+1}} \frac{|\nu - \hat{\mathcal{I}}_p \nu|_{1,t}}{|\nu - \mathcal{I}_p \nu|_{1,t}}.$$

$\text{Null}(1-\mathcal{I}_p) \equiv \mathcal{S}_p \subset \text{Null}(1-\hat{\mathcal{I}}_p)$. For p refinement $\beta_0 = 0$. For h refinement we assume number of nodes for $\hat{\mathcal{I}}_p \geq$ number of nodes for $\mathcal{I}_{p+1}$ (else $\beta_0 = 1$). This may require several levels of h refinement to define $\hat{\mathcal{I}}_p$. Note that β_0^2 is largest generalized eigenvalue of two positive semidefinite matrices, thus independent of ν.

The Main Theorem

Theorem

Let $u \in H^{p+2}(t)$ satisfy a priori estimates, $u \notin \mathcal{S}_p$, and β_0 defined as above. Then there is a constant C depending on u, the degree p, the shape of element t, but not on its diameter h, such that

$$\frac{|u - \hat{\mathcal{I}}_p u|_{1,t}}{|u - \mathcal{I}_p u|_{1,t}} \leq \beta_0 + Ch.$$

Note this implies

$$\beta_0 \leq \frac{|u - \hat{\mathcal{I}}_p u|_{1,t}}{|u - \mathcal{I}_p u|_{1,t}} \leq \beta_0 + Ch$$

showing that (asymptotically) the saturation assumption holds.

Proof

Using a priori estimates, β_0, and the triangle inequality

$$\begin{aligned}|u - \hat{\mathcal{I}}_p u|_{1,t} &\leq |(1 - \hat{\mathcal{I}}_p)\mathcal{I}_{p+1} u|_{1,t} + |(1 - \hat{\mathcal{I}}_p)(u - \mathcal{I}_{p+1} u)|_{1,t} \\ &\leq \beta_0 |(1 - \mathcal{I}_p)\mathcal{I}_{p+1} u|_{1,t} + |(1 - \hat{\mathcal{I}}_p)(u - \mathcal{I}_{p+1} u)|_{1,t} \\ &\leq \beta_0 |u - \mathcal{I}_p u|_{1,t} + \beta_0 |(1 - \mathcal{I}_p)(u - \mathcal{I}_{p+1} u)|_{1,t} \\ &\qquad + |(1 - \hat{\mathcal{I}}_p)(u - \mathcal{I}_{p+1} u)|_{1,t}\end{aligned}$$

If $u \in \mathcal{S}_p$, $|u - \mathcal{I}_p u|_{1,t} = 0$, then the saturation assumption is trivially satisfied for any choice of β, so we exclude $\mathrm{Null}(1 - \mathcal{I}_p) \equiv \mathcal{S}_p$. Then as shown by Lin, Xie, and Xu

$$|u - \mathcal{I}_p u|_{1,t} \geq C_0(u) h^p > 0.$$

Proof (Continued)

We bound the last two terms as follows

$$
\begin{aligned}
|(1-\mathcal{I}_p)(u-\mathcal{I}_{p+1}u)|_{1,t} &\le C_1 h|u-\mathcal{I}_{p+1}u|_{2,t} \\
&\le C_2 h^{p+1}|u|_{p+2,t} \\
&\le \left(\frac{C_2|u|_{p+2,t}}{C_0(u)}\right) h|u-\mathcal{I}_p u|_{1,t} \\
&\equiv C_3(u)h|u-\mathcal{I}_p u|_{1,t}
\end{aligned}
$$

for functions $u \in H^{p+2}(t)$. A similar argument shows

$$|(1-\hat{\mathcal{I}}_p)(u-\mathcal{I}_{p+1}u)|_{1,t} \le \hat{C}_3(u)h|u-\mathcal{I}_p u|_{1,t}.$$

The theorem now follows with $C(u) = \beta_0 C_3(u) + \hat{C}_3(u)$.

Remarks (Details in Bank, Xu, Yserentant)

$\beta_0 = 0$ for p refinement. For h refinement we compute $\beta_0 = 2^{-p}$ for regular/red refinement of simplicies in $\mathcal{R}^d$, $1 \leq d \leq 3$, and for newest node bisection (two levels) for $d = 2$.

Let $a \in W^1_\infty(t)$ and $a(x) > a_0 > 0$

$$a(u, v)_t = \int_t a(x) \nabla u \cdot \nabla v \, dx$$

$$|u|^2_{a,t} = a(u, u)_t$$

$$\beta_0 \equiv \max_{\nu \in \mathcal{S}_{p+1}} \frac{|\nu - \hat{\mathcal{I}}_p \nu|_{a,t}}{|\nu - \mathcal{I}_p \nu|_{a,t}}.$$

We show

$$\beta_0 \leq \frac{|u - \hat{\mathcal{I}}_p u|_{a,t}}{|u - \mathcal{I}_p u|_{a,t}} \leq \beta \leq \beta_0 + Ch$$

Regularity

Regularity $u \in H^{p+2}(t)$ can be relaxed to $u \in H^{p+1+\alpha}(t)$, $0 < \alpha \leq 1$, with result $\beta \leq \beta_0 + Ch^\alpha$.

Let $d = 1$, $p = 1$, h refinement. Then $\hat{\mathcal{I}}_p u$ is the exact finite element solution of the Dirichlet problem $-u'' = f$ on $(0, h)$ Galerkin Orthogonality shows

$$|u - \mathcal{I}_1 u|^2_{1,t} = |u - \hat{\mathcal{I}}_1 u|^2_{1,t} + |\mathcal{I}_1 u - \hat{\mathcal{I}}_1 u|^2_{1,t}.$$

If both $|\mathcal{I}_1 u - \hat{\mathcal{I}}_1 u|_{1,t}$ and $|u - \mathcal{I}_1 u|_{1,t}$ are positive

$$\beta^2 = 1 - \frac{|\mathcal{I}_1 u - \hat{\mathcal{I}}_1 u|^2_{1,t}}{|u - \mathcal{I}_1 u|^2_{1,t}} < 1.$$

without a strong regularity assumption.

A Posteriori Error Estimate

The saturation assumption is used to show, for all $\chi \in \mathcal{S}_p$

$$|u - \mathcal{I}_p u|_{1,t} \leq C|u - \chi|_{1,t}$$

Let u_h the finite element solution of a PDE on domain Ω. Then

$$C_1|u - \mathcal{I}_p u|_{1,\Omega} \leq |u - u_h|_{1,\Omega} \leq C_2|u - \mathcal{I}_p u|_{1,\Omega}$$

using our local lower bound and classical upper bound.
Thus interpolation error is a reliable and efficient a posteriori error estimator (assuming it could be computed). Let $|e_h|$ be a reliable and efficient a posteriori error estimate. Then

$$c_1|u - \mathcal{I}_p u|_{1,\Omega} \leq |e_h|_{1,\Omega} \leq c_2|u - \mathcal{I}_p u|_{1,\Omega}$$

Reference Adaptive Method

The reference adaptive method creates a sequence $\mathcal{S}_h^k$ of (possibly discontinuous) piecewise polynomial spaces.

- Assume a shape regular triangulation of Ω with $h_t \leq h_0 < 1$. Let $\mathcal{S}_h^0$ be the initial space with local degree p (e.g. $p = 1$) in each element. Assume local regularity and a priori estimates hold on each element.
- $\mathcal{S}_h^k \to \mathcal{S}_h^{k+1}$ as follows: pick an element t with largest interpolation error and refine it (h or p). Local regularity and a priori estimates must remain the same or improve.
- If both h and p refinement are possible, chose one with the largest error reduction.
- Let $u_k \in S_h^k$ be the (possibly discontinuous) piecewise polynomial interpolant of u generated in the k-th step of this process.

Reference Adaptive Method

Since the Saturation Assumption only holds asymptotically, could it impede convergence of the reference adaptive method?

Lemma

Assume that the sequence $\{u_k \in S_h^k\}$ *is generated as described above. Then*

$$\lim_{k\to\infty} |u - u_k|_{1,\Omega} = 0.$$

The proof is by the method of contradiction. If the adaptive procedure did not converge, there must be one or more elements with maximum positive error that was not reduced even after multiple (infinite) refinement steps. This contradicts the a priori error estimate.

Test Framework for Adaptive Feedback Loops

We consider adaptive feedback loops of the form

Classic (pltmg) solve $\rightarrow$ estimate $\rightarrow$ refine

Dörfler ($\theta = 1/2$) solve $\rightarrow$ estimate $\rightarrow$ mark $\rightarrow$ refine

- Pick known functions in the class of interest. Skip solve.
- Replace estimate with interpolation error.
- Prove reference adaptive scheme is optimal for the chosen class of functions, using all known information.
- Feedback loops can only use information available to them in the actual PDE application.

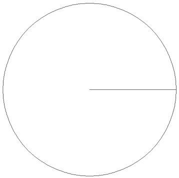
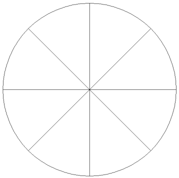
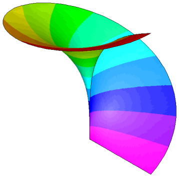
This provides a standardized framework to evaluate feedback loops, eliminating much white noise (e.g. approximate solution, details of a posteriori error estimates), allowing focus on the feedback loop itself.

Point Singularity Example

We consider the case $r = 1$ singular point $u \in \mathcal{H}^{1+\alpha}$ for $0 < \alpha < 1$, and use for numerical illustration the Circle Problem:

$$-\Delta u = 0 \quad \text{in } \Omega, \text{ with}$$
$$u = g \quad \text{on } \partial\Omega_1, \text{ and} \quad u_n = 0 \quad \text{on } \partial\Omega_2,$$

Ω is a circle of radius one, with a crack along the positive x-axis $0 \le x \le 1$. The boundary $\partial\Omega_2$ is the bottom edge of the crack, and $\partial\Omega_1 = \partial\Omega \setminus \partial\Omega_2$. g is chosen such that the exact solution is $u = r^{1/4} \sin(\theta/4) \in \mathcal{H}^{5/4-\epsilon}$.

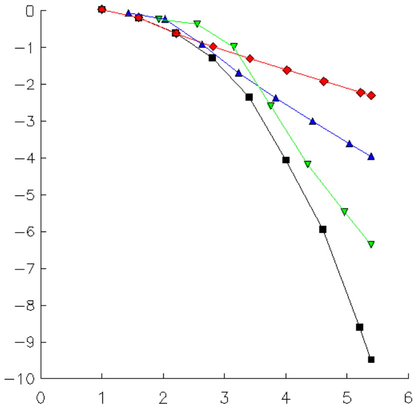
Notation

The table data is labeled as follows.

- Loops: number of adaptive feedback steps; $L =$ loops.
- Digits: $-\log(|u - \mathcal{I}_p u|_1/|u|_1)$ for $N_L \approx 250K$ DOFs.
- Order: least squares fit to $AN^{-Q/2}$; Q=Order.
- Exp: nonlinear least squares fit to $A\exp(-BN_k^Q)$; Q=Exp.
- The reference scheme has $N_k \approx \min(4N_{k-1}, 250K)$ to provide data points for graphs and tables.

The graphs are $\log(|u - \mathcal{I}_p u|_1/|u|_1)$ v $\log N$.

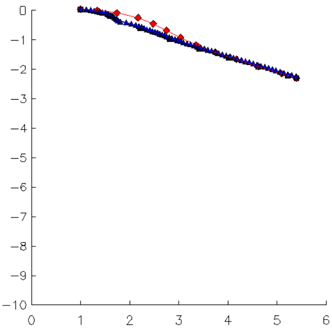
Reference Scheme



h refinement for $p = 1$, $p = 2$, $p = 4$, and *hp* refinement.

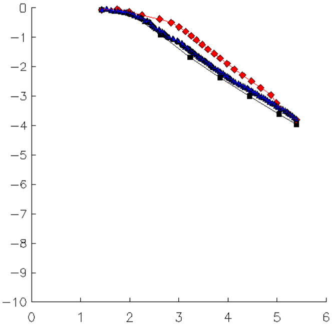
h Refinement $p = 1$

h-refinement, $p = 1$			
	Digits	Order	Loops
reference	2.30	1.09	9
pltmg	2.29	1.26	13
Dörfler	2.26	1.10	69



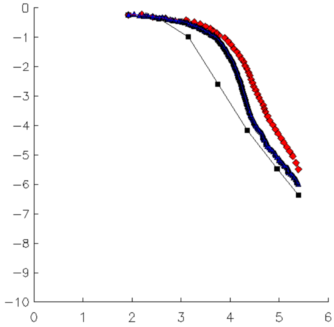
h Refinement $p = 2$

h-refinement, $p = 2$			
	Digits	Order	Loops
reference	3.95	2.23	8
pltmg	3.81	2.30	23
Dörfler	3.76	2.12	105



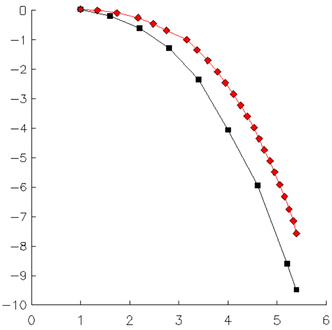
h Refinement $p = 4$

h-refinement, $p = 4$			
	Digits	Order	Loops
reference	6.36	4.45	7
pltmg	5.49	4.21	46
Dörfler	5.99	4.53	260



hp Refinement

hp-refinement			
	Digits	Exp	Loops
reference	9.48	0.24	9
pltmg	7.57	0.30	24



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