Elastic Rods: Tying connections between Graphics & Mechanics

Bernhard Thomaszewski & Pedro Reis
Our goal is to embrace mechanical instabilities of thin (soft) structures, guided primarily through precision model experiments, towards understanding and exploiting novel functional mechanisms over a wide range of length scales.

Plenty of opportunities for discovery & innovation
Welcome

Welcome to the Computational Design Group at Université de Montréal. Our research lies at the intersection of computer graphics, computational mechanics, and digital fabrication. Specific focus areas include computational design of mechanisms, structured materials, and physical surfaces, as well as visual simulation. See here for an overview.

Research Themes

Physical Surface     Structured Materials     Mechanism Design    Robotics     Visual Simulation
Elastic Rods
Filamentary Structures...

... in nature and engineering.
Elastic Kirchhoff Rods

“... the classical theory of thin rods as developed by Kirchhoff [1859, 1876] and Clebsch [1862], and presented by Love [1892, 1906].”

Elastic Kirchhoff Rods

Represent rod as an adapted framed curve \( \Gamma(s) = \{\gamma(s); F(s)\} \)

- \( \gamma(s): \mathbb{R} \rightarrow \mathbb{R}^3 \) and arc-length parameterized curve describing the rod’s centerline
- \( F(s) = \{t(s), m_1(s), m_2(s)\} \) an orthonormal material frame adapted to centerline, i.e., \( t = \gamma' \)

Bending:

\[
\begin{align*}
   b_1 &= t' \cdot m_1 \\
   b_2 &= t' \cdot m_2 \\
   t &= m_1' \cdot m_2
\end{align*}
\]

Twist:

\[
E(\Gamma) = \int_0^l k_b [b_1(s)^2 + b_2(s)^2] + k_t t(s)^2 \, ds
\]

Stored energy*:

* for an initially-straight, isotropic rod
Rod Models from Graphics

[Bertails et al. ‘06]

[Bertails ‘09]

[Casati & Bertails-Descoubes ‘13]

[Spillmann & Teschner ‘07]

[Spillmann & Teschner ‘08]

[Umetani et al. ‘13]

[Pai ‘02]

[Gregoire & Schoemer ‘07]

[Bergou et al. ‘08]

[Bergou et al. ‘10]
Thin Rods: Experiments

Fabrication of soft filaments

[A. Lazarus, J.T. Miller and P.M. Reis, JMPS 2013]
[A. Lazarus, J.T. Miller and P.M. Reis, Soft Matter 2013]

Suspended shapes of naturally curved rods

[Vary Natural curvature]
\[ \kappa = \frac{1}{R} \]

[k=0.089 cm^{-1}, k=0.041 cm^{-1}, k=0 cm^{-1}, k=0.62 cm^{-1}, k=0.56 cm^{-1}, k=0.45 cm^{-1}, k=0.23 cm^{-1}]

[A. Lazarus, J.T. Miller, B. Audoly and P.M. Reis, PRL 2014]

Thin Rods: Numerics

Discrete Elastic Rods

\[ \gamma(s) \text{ Twist-free frame} \]

Simulation of \( \sim100,000 \) hairs using DER
Example 1:
The elastic sewing machine & bacterial locomotion
Deployment of pipelines/cables onto seabed

Internet under-sea cable network

Deployment of subsea pipelines

Deployment of an elastic rod onto a moving substrate

Excellent agreement between experiments (PDEs) and simulations (DER) with no fitting parameters!

[PNAS 2014]
[Extreme Mech. Lett. 2015]
Excellent agreement between experiments (PDEs) and simulations (DER) with no fitting parameters!
Dynamics of rotating helical rods in viscous fluids

**Bacterial locomotion**

- **Vibrio cholerae**
  - R. Stocker (MIT)

- **R. Sphaeroides**
  - H. Berg (Harvard)

95% of bacteria in the ocean locomotive themselves by rotation of a single flexible helical flagellum.

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**Helical filament:**
- Length, \( L \)
- Pitch, \( \lambda \)
- Radius, \( R_n, \lambda \)

---

**Couple DER code for rods with Lighthill's Slender Body Theory:**


- **Velocity** \( \mathbf{u}(s) \) at flagellum vs. force \( f(s) \) exerted by fluid.
  - local
  - non-local

\[
\mathbf{u}(s) = \frac{\mathbf{f}_\perp}{4\pi\mu} + \int_{r(s',s) > \delta} \mathbf{f}(s') \cdot \mathbf{J}(r) ds'
\]

\[
\mathbf{f}_\perp = \mathbf{f} \cdot \left( \mathbf{I} - \mathbf{t}\mathbf{t}^T \right)
\]

\[
\mathbf{J}(r) = \frac{1}{8\pi\mu} \left( \frac{\mathbf{I}}{|r|} + \frac{rr^T}{|r|^3} \right)
\]

Cutoff:

\[
\delta = \frac{1}{2} r_0 \sqrt{\epsilon}
\]
Dynamics of rotating helical rods in viscous fluids

Bacterial locomotion

Could some bacteria be exploiting this buckling mechanism for turning?
Example 2:
Structured Sheet Materials
Natural Network Materials

- Muscle tissue (actin filaments)
- Cancellous bone (trabeculae)
- Vein structure (chitin, resilin)
Digital Network Materials

[Carbon]

[Schumacher et al. 2018]

[Zhou et al. 2015]

[Renishaw]
3D-Printed Fabric

Danit Peleg
Elastic Isohedral Tilings

3D-Printed Tilings
Rod Network Mechanics
Simulation

Discrete Elastic Rods ([Bergou '08,'10])
+ Extension to Networks ([Perez ‘15], [Zehnder ‘16])
Mechanical Characterization
Mechanical Characterization

\[\epsilon_1, \sigma_1, \kappa_1, W_1\]

\[\epsilon_2, \sigma_2, \kappa_2, W_2\]

\[\epsilon_n, \sigma_n, \kappa_n, W_n\]

\[\epsilon, \sigma, \kappa, W\]
Macromechanical Model

Membrane
\[ \mathbb{C}(\epsilon_1, \sigma_1, \epsilon_2, \sigma_2, \ldots, \epsilon_n, \sigma_n) \]

Bending
\[ \mathbb{B}(\kappa_1, W_1, \kappa_2, W_2, \ldots, \kappa_n, W_n) \]

\[ W = \epsilon : \mathbb{C} : \epsilon + \kappa : \mathbb{B} : \kappa \]

Membrane Bending
Macromechanical Representation

- Young's modulus
- Poisson's ratio
- Bending stiffness

0.1% strain
0.1 m\(^{-1}\) curvature

10% strain
5 m\(^{-1}\) curvature

--- exact
--- fitted
Exploration

http://www.structuredsheets.com
Inverse Design

Young's modulus
Example 3: Elastic gridshells
Shaping through buckling in elastic gridshells

Forum Café Gridshell for Solidays Festival, Paris (2011)
Mannheim Multihalle, Germany

Frei Otto, 1975
Anatomy and Actuation of an Elastic Gridshell

Footprint (rest configuration)
* Quadrilateral grid

Rods:
* Nitinol
  * $E = 83 \text{ GPa}$
  * $d = 254 \mu\text{m}$

Joints:
* VPS
  * $E = 1.3 \text{ MPa}$
  * diameter $d = 3 \text{ mm}$
  * height $h = 5 \text{ mm}$

Boundary points:
* 3D printed ball joints
* Pinned B.C.s

Original boundary of footprint
Actuated boundary
Anatomy and Actuation of an Elastic Gridshell

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Given a target shape, what should be the footprint (inverse design)?

Given a footprint, what is the elastic gridshell generated?
Fabrication of a hemispherical gridshell
Physical Experiments v. DER

3D Digital Scanning (NextEngine)

- Excellent quantitative agreement between exps. & DER.
- Multiplicity of states for same input parameters.
Pafnuty Chebyshev (1821-94)
- Probability, Chebyshev polynomials, number theory
- How do textiles drape?

Chebyshev Net (1878):
- Maps 2D-to-3D:
  \[ r : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]
- Inextensible rods in \( u, v \) directions:
  \[ |r_u| = |r_v| = 1 \]
- (Shearing) angle changes of tangent vectors at a point deform the metric:
  \[ \omega(u, v) = \angle(r_u, r_v) \]

Gauss equation for Chebyshev nets:

\[ -\kappa(u, v) \sin \omega(u, v) = \omega_{uv}(u, v) \]

For \( \kappa = 1 \) [Chebyshev, 1878]
Spherical Elastic Gridshells

2D Footprint

Actuated hemispherical gridshell

3D Digital Scanning

Excellent quantitative agreement!
Max. deviation ~2%

Chebyshev Hemisphere
DER simulations
Experiments
Let’s cut a pasta strainer
Building with blocks: more complex gridshells

Quarter-sphere + Cylinder

Quarter-sphere + Saddle
Rigidity for a gridshell under point indentation?

For Hemispherical shell: 
\[ \frac{K^0}{Et} = \frac{4}{\sqrt{3(1-\nu^2)}} \left( \frac{t}{R} \right) \]

[Reissner, 1946]

For Euler-Bernoulli beam: 
\[ \frac{K/}{Et} \sim \left( \frac{t}{R} \right)^3 \]

Q: What is \( K \) for hemispherical gridshell?

Amazing collapse into 3 master curves!
Example 4: Self-deploying Surfaces
Self-Deploying Surfaces

Kovac et al. (2009)

NASA

Paik lab (RRL), EPFL
Self-Deploying Surfaces
Nonlinear Mechanics
Constrained Design Space

Minimal surfaces have zero mean curvature throughout,
\[ \frac{\kappa_1 + \kappa_2}{2} = 0 \]

Minimal surfaces are either
- locally flat
  \[ \kappa_1 = \kappa_2 = 0 \]
- or saddle-shaped
  \[ \kappa_1 = -\kappa_2 \neq 0 \]
Solution of the Kirchhoff–Plateau problem

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Abstract

The Kirchhoff–Plateau problem concerns the equilibrium shapes of a system in which a flexible filament in the form of a closed loop is spanned by a liquid film, with the filament being modeled as a Kirchhoff rod and the action of the spanning surface being solely due to surface tension. We establish the existence of an equilibrium shape that minimizes the total energy of the system under the physical constraint of non-interpenetration of matter, but allowing for points on the surface of the bounding loop to come into contact. In our treatment, the bounding loop retains a finite cross-sectional thickness and a nonvanishing volume, while the liquid film is represented by a set with finite two-dimensional Hausdorff measure. Moreover, the region where the liquid film...
Computational Model

Nonlinear Mechanics

Parameters

Simulation Mesh

CST finite elements with orthotropic material

Discrete Elastic Rods

Rod control points and cross sections

Coupled membrane and rod elements (collocation)
Forward Design

Deployed State
Inverse Design
Exploring Design Variations
Concluding thoughts...

- **Computer graphics** tools and **physically-based simulation** offer unprecedented opportunities as quantitatively predictive tools in **experimental mechanics**.

- Timely **engineering applications & physical scenarios** and novel **experimental mechanics** tools can challenge and push **physically-based simulation** into new grounds.

Let’s Keep Mixing