INRIA Gudhi Workshop 2014

Computational Homology via Discrete Morse Theory

Shaun R. Harker
Rutgers University
INRIA Gudhi Workshop 2014

Outline

Harker, Shaun, Konstantin Mischaikow, Marian Mrozek, and Vidit Nanda. "Discrete Morse theoretic algorithms for computing homology of complexes and maps." *Foundations of Computational Mathematics* 14, no. 1 (2014): 151-184.

- Discrete Morse Theory
- Induced Homology on Maps (if time permits)

Part 1. Discrete Morse Theory

Introduction

- We can use Discrete Morse theory, originally developed by Forman, to compute homology.
- We generate discrete Morse functions by a generalization of the coreductions approach of Mrozek and Batko.
- We give an efficient way to compute the boundary in the reduced complex.

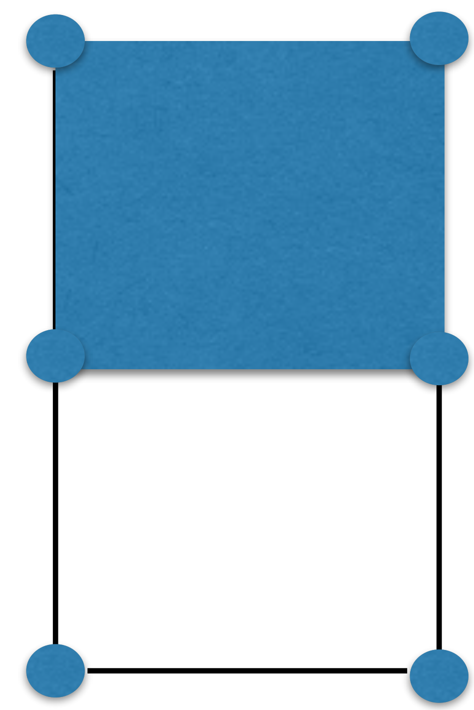
Chain Complex Definition

A principal ideal domain $\mathcal{R}$

A finite set of cells $\mathcal{S}_k$, $\mathrm{k} = 0, 1, \cdots, d$

Chain modules $\mathcal{C}_k = \mathcal{R}(\mathcal{S}_k)$

Boundary maps $\partial_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$



satisfying: $\partial^2 = 0$

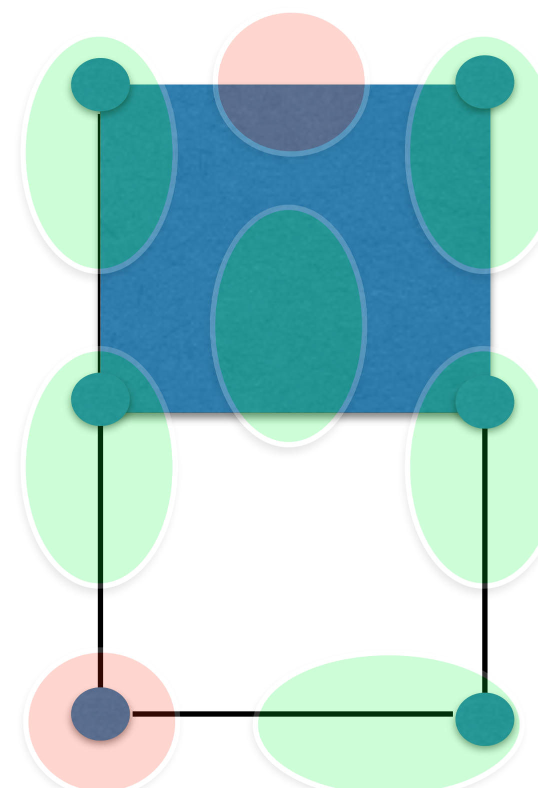
Discrete Gradient Vector Field (AKQ Decomposition)

- Captures information in a discrete Morse function up to equivalence
- Essentially, a partitioning into three classes called Aces, Kings, and Queens.
- Kings and Queens are paired up 1-1

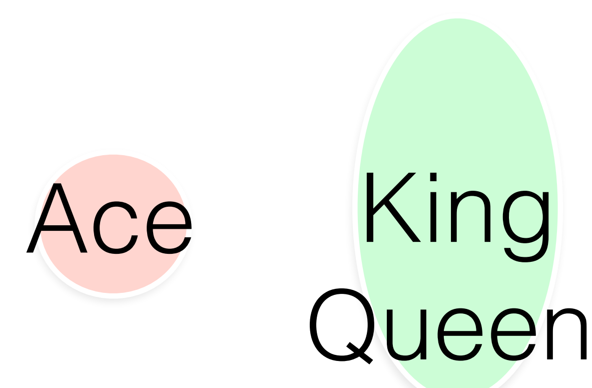
Discrete Gradient Vector Field (AKQ Decomposition)

- Captures information in a discrete Morse function up to equivalence
- Essentially, a partitioning into three classes called Aces, Kings, and Queens.
- Kings and Queens are paired up 1-1

example:



legend:



Rules on AKQ Decomposition

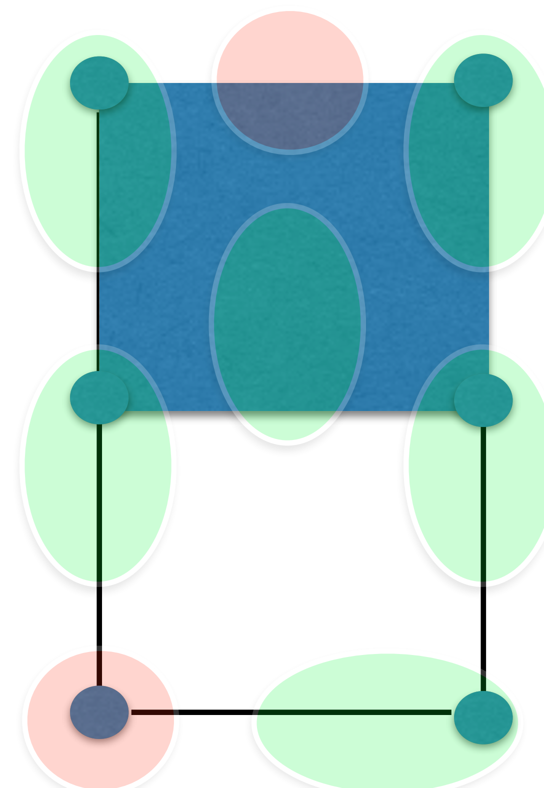
- A queen cell must be in the boundary of its associated king cell, so that dim K = dim Q + 1

- The incidence number between King and Queen pairs must be a unit -- that is, multiplicatively invertible in the ring

- Also, there is an acyclicity condition we will discuss.

Rules on AKQ Decomposition

- A queen cell must be in the boundary of its associated king cell, so that dim K = dim Q + 1
- The incidence number between King and Queen pairs must be a unit -- that is, multiplicatively invertible in the ring
- Also, there is an acyclicity condition we will discuss.



Comments on Notation

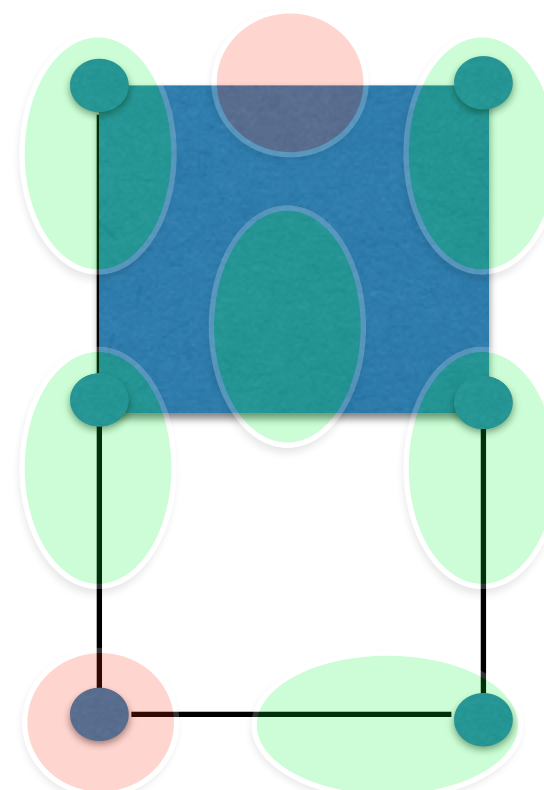
- An “Ace-King-Queen Decomposition” is just another way of expressing the discrete gradient vector field

- “Aces” are Critical Cells

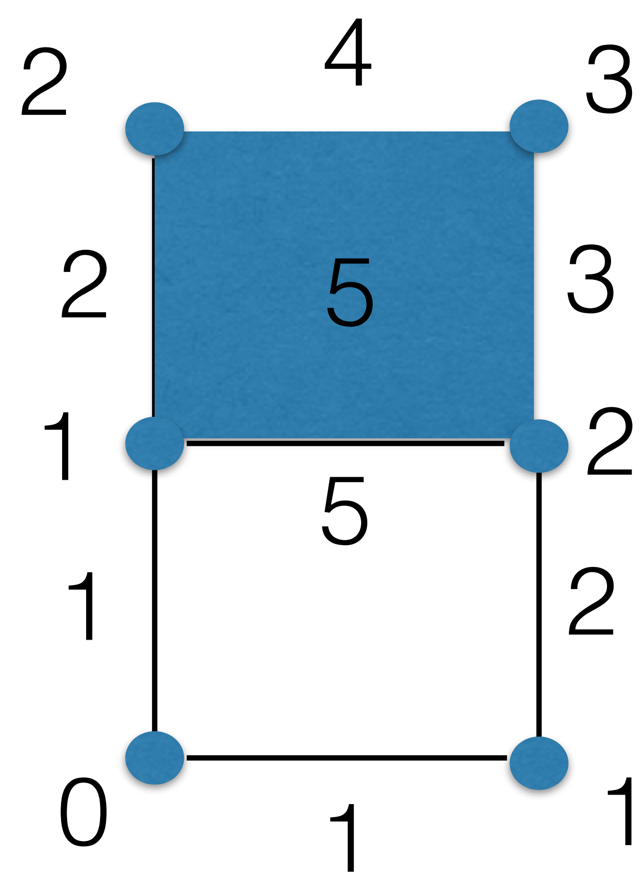
- “King-Queen” pairs give the “arrows” of the discrete gradient vector field

Comments on Notation

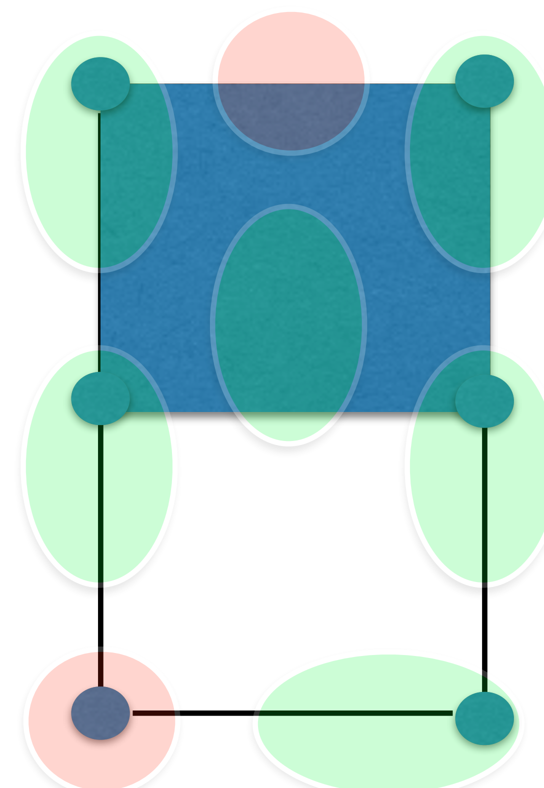
- An "Ace-King-Queen Decomposition" is just another way of expressing the discrete gradient vector field
- "Aces" are Critical Cells
- "King-Queen" pairs give the "arrows" of the discrete gradient vector field



Discrete Morse functions compared to AKQ decomposition

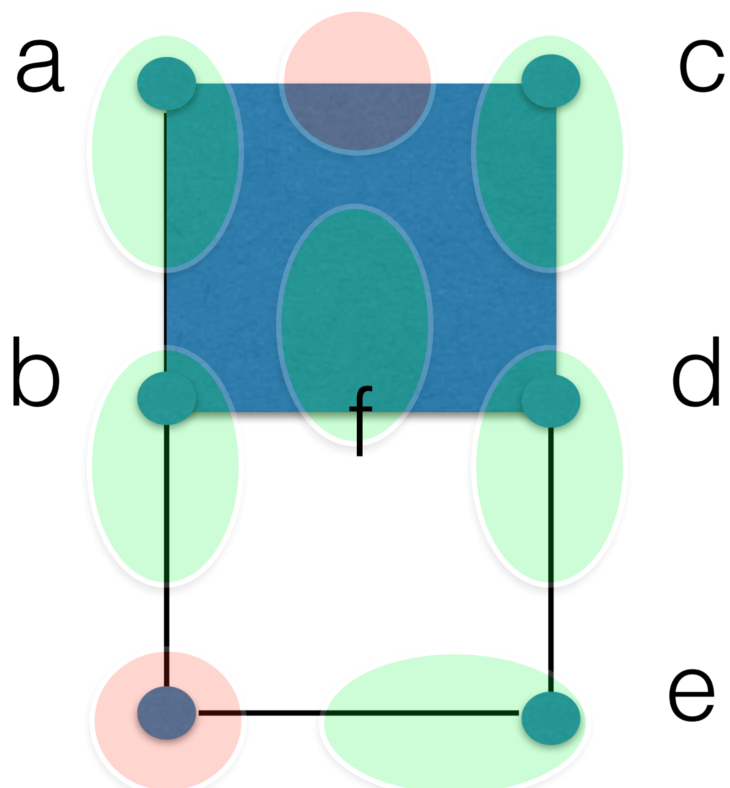


discrete Morse function



induced discrete gradient vector field

Acyclicity Assumption



- What does “no loops” mean?
- There is a partial order induced on the Queens by the generating relationship of

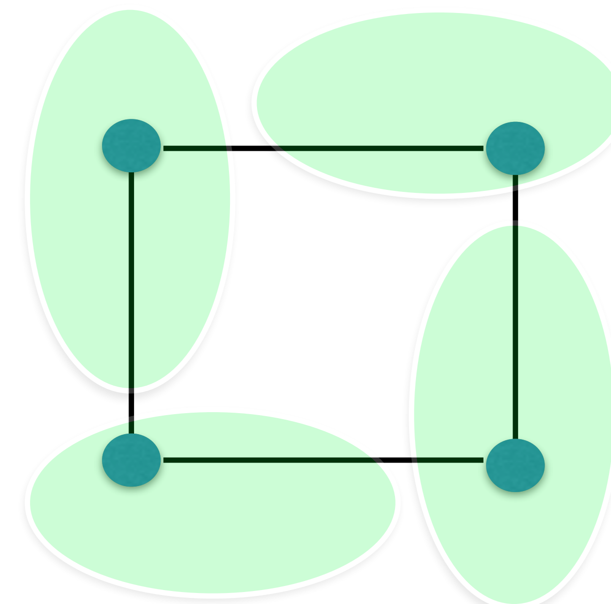
$$\langle Q', \partial K \rangle \neq 0 \text{ and } Q = K^* \text{ implies } Q' \leq Q$$

a > b

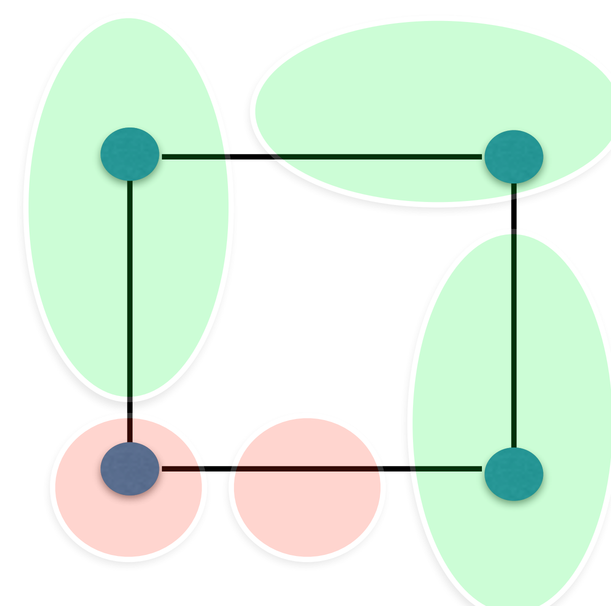
c > d > e

Acyclicity Examples:

Not an AKQ decomposition since it violates acyclicity condition



However, this is okay:



How do we generate AKQ decompositions?

One method is an extension of the coreduction technique of Mrozek and Batko.

In this method, a reduced complex was found by first excising one vertex from each connected component of the original complex, and then performing free coface collapses until they were exhausted. The vertices are then replaced.

This procedure is linear time.

Coreduction Based AKQ Decomposition Algorithm

n <- 0

Loop until complex is empty

If there is a free coface collapse pair (K, Q)

K is now a King, Q is now a Queen, K* := Q.

v(K) := v(Q) := n, and n <- n+1.

Excise K and Q from the complex.

If there is no free coface collapse,

Choose a cell A such that dA = 0.

A is now an Ace.

v(A) := n, and n <- n+1.

Excise A from the complex.

End Loop

Illustration of the coreduction based decomposition algorithm

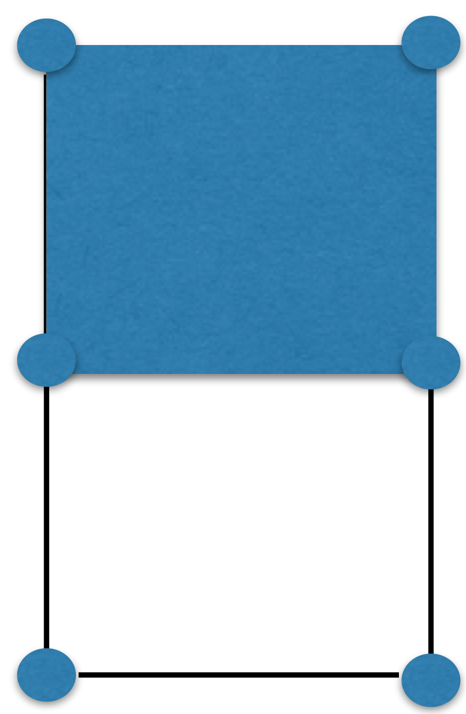


Illustration of the coreduction based decomposition algorithm

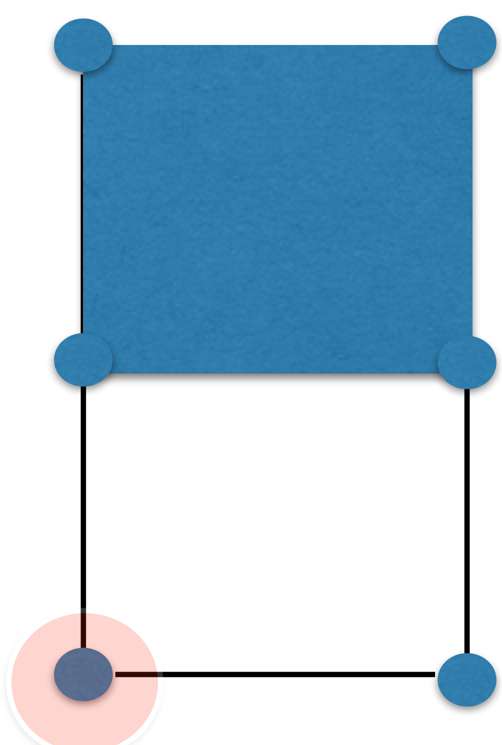


Illustration of the coreduction based decomposition algorithm

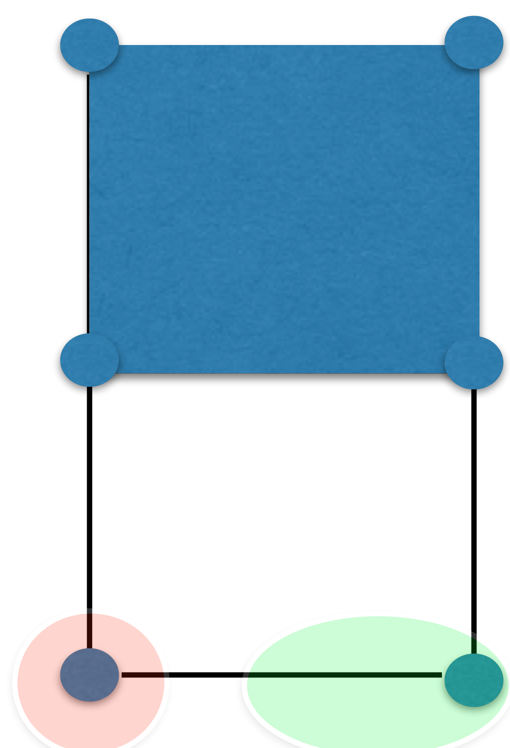


Illustration of the coreduction based decomposition algorithm

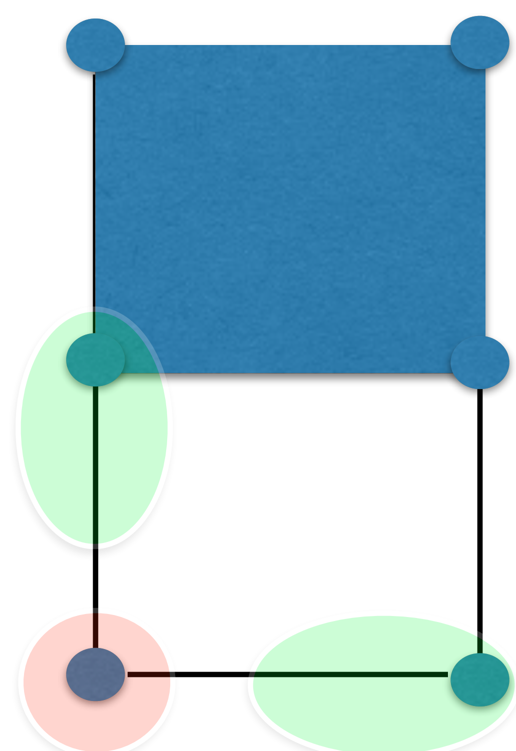


Illustration of the coreduction based decomposition algorithm

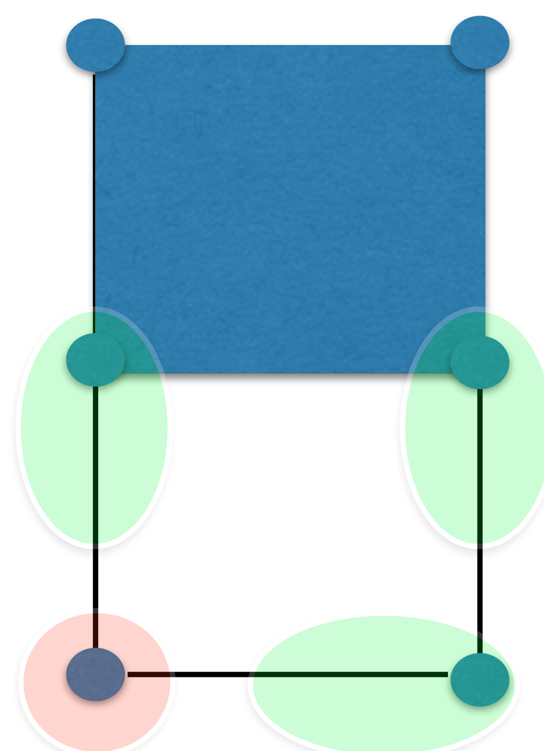


Illustration of the coreduction based decomposition algorithm

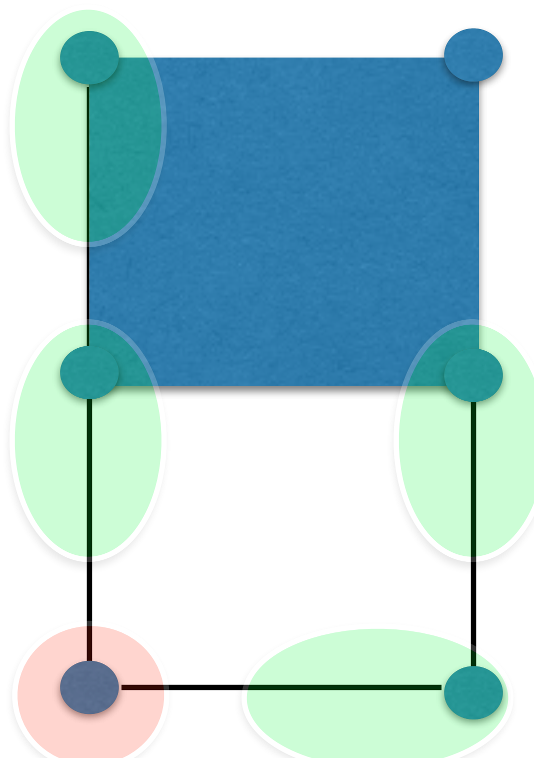


Illustration of the coreduction based decomposition algorithm

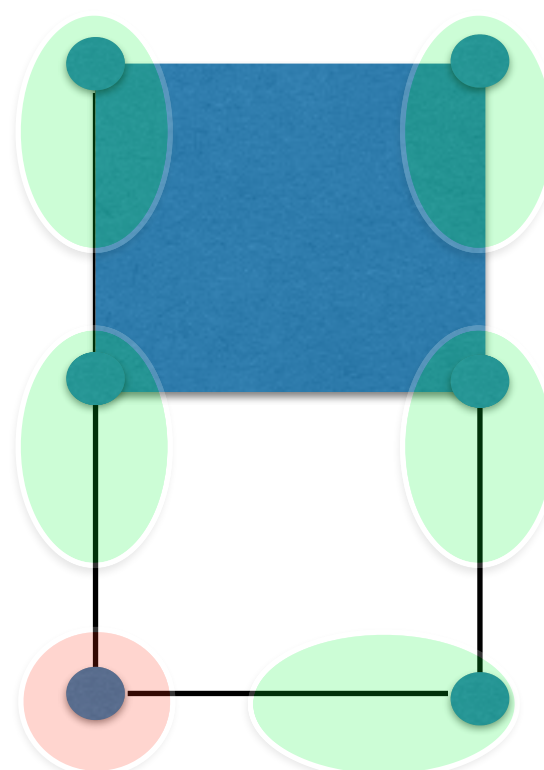


Illustration of the coreduction based decomposition algorithm

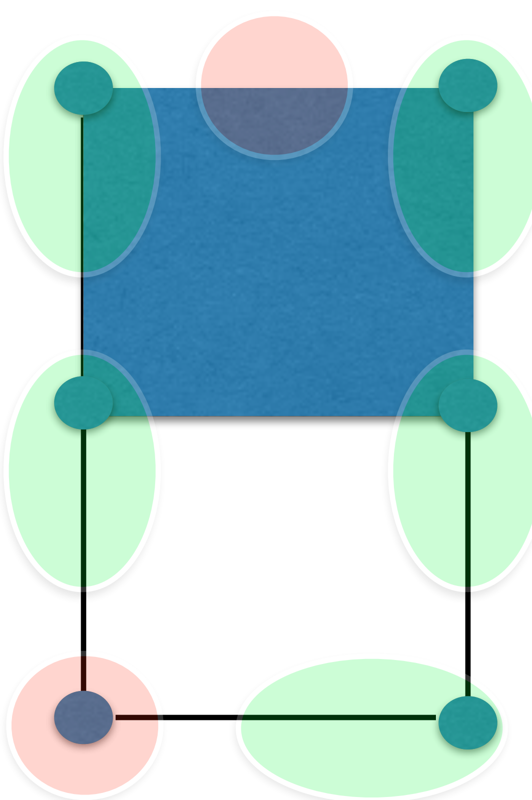
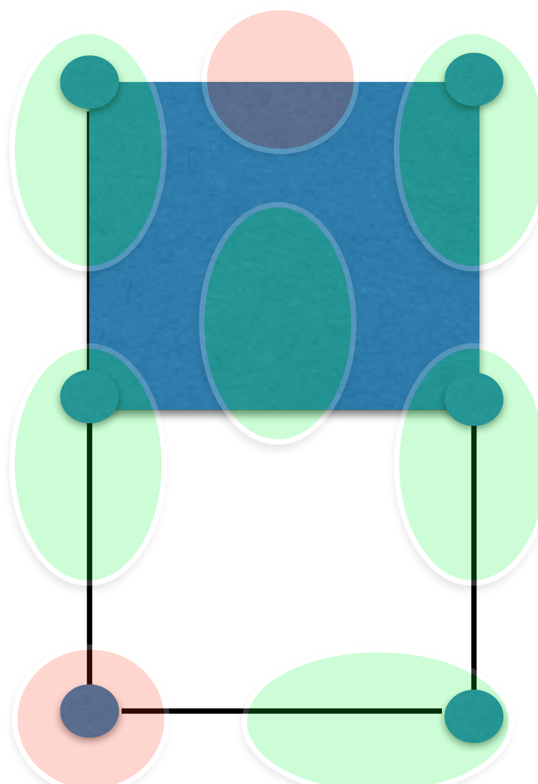
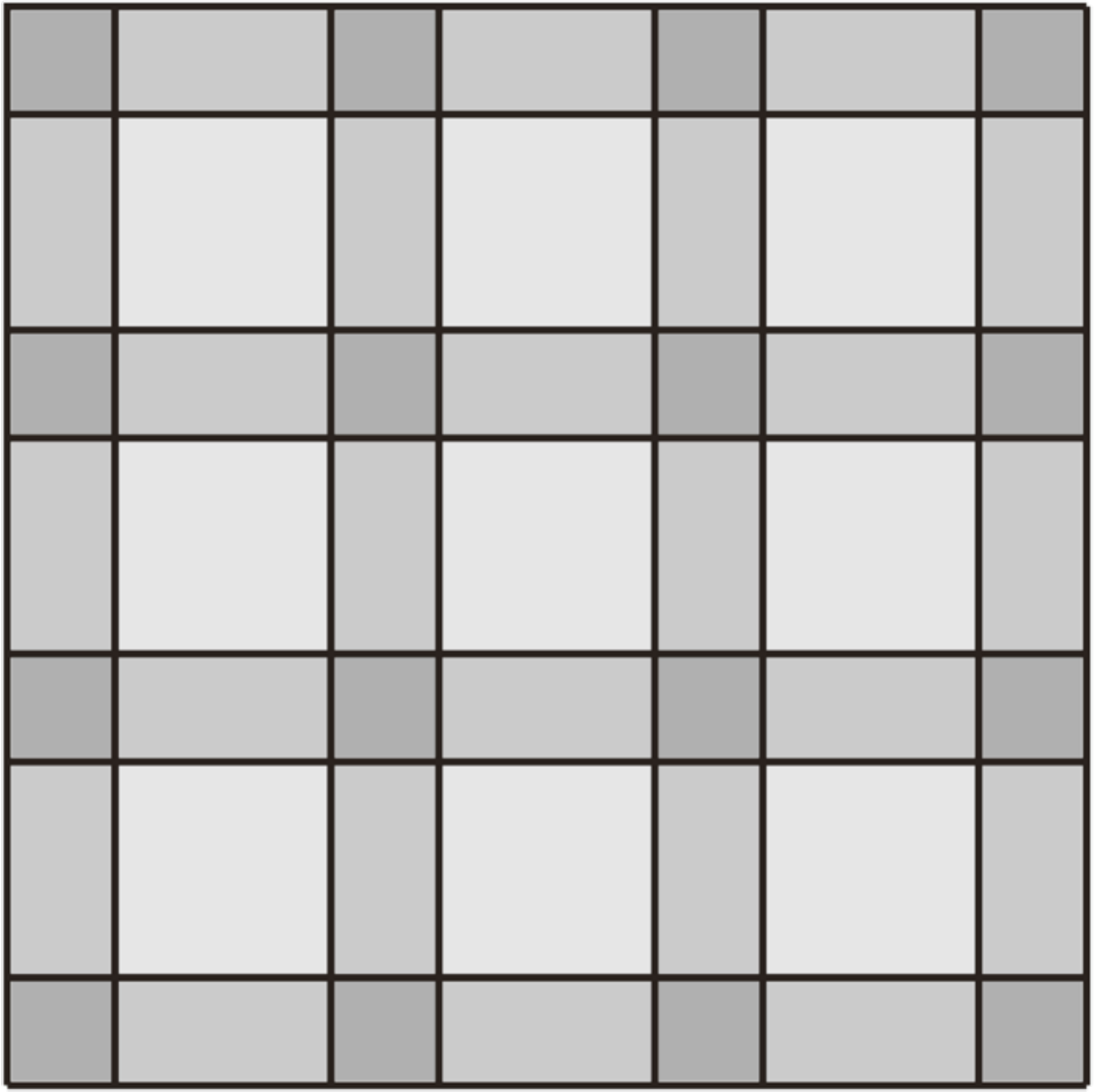
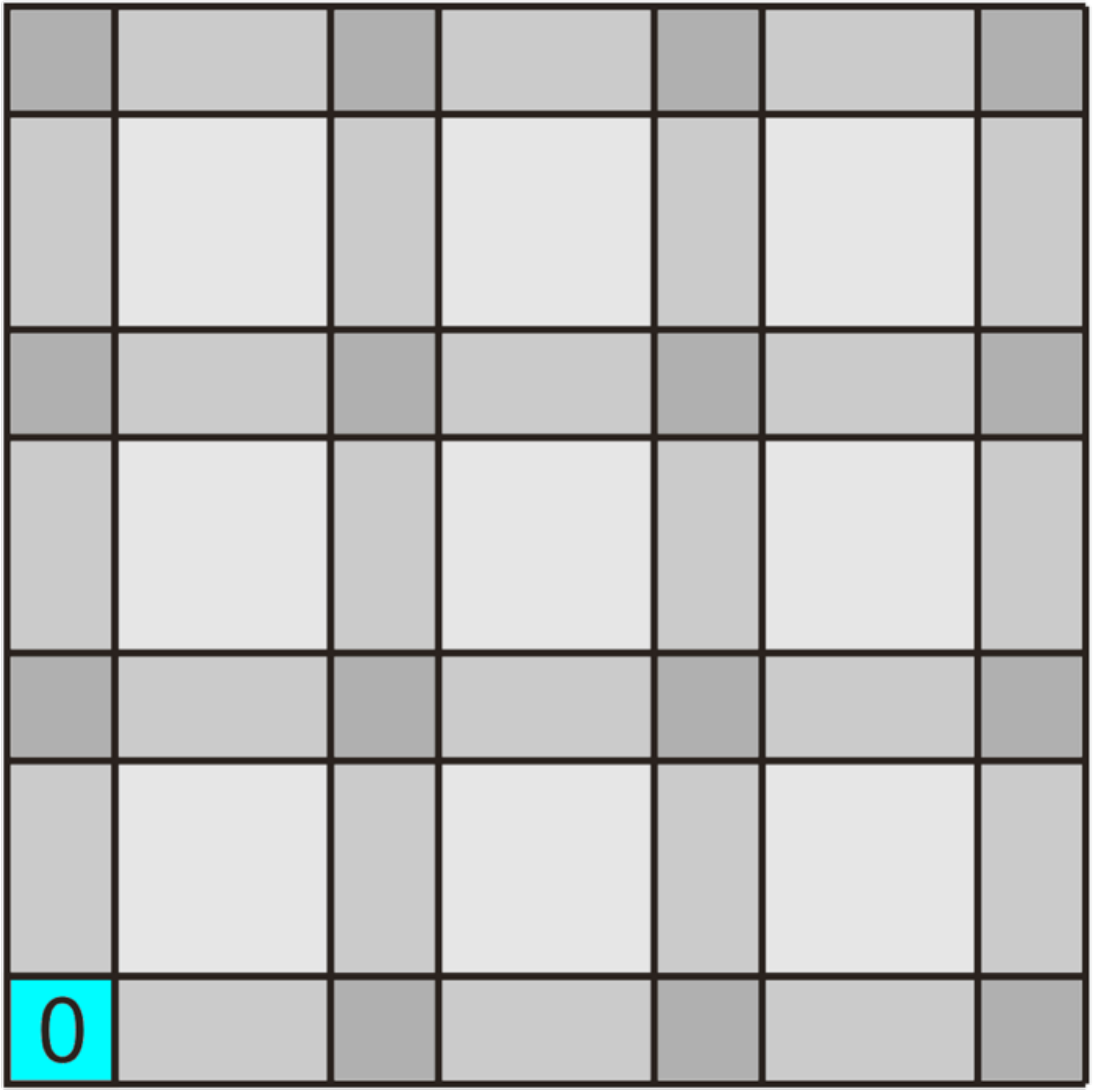
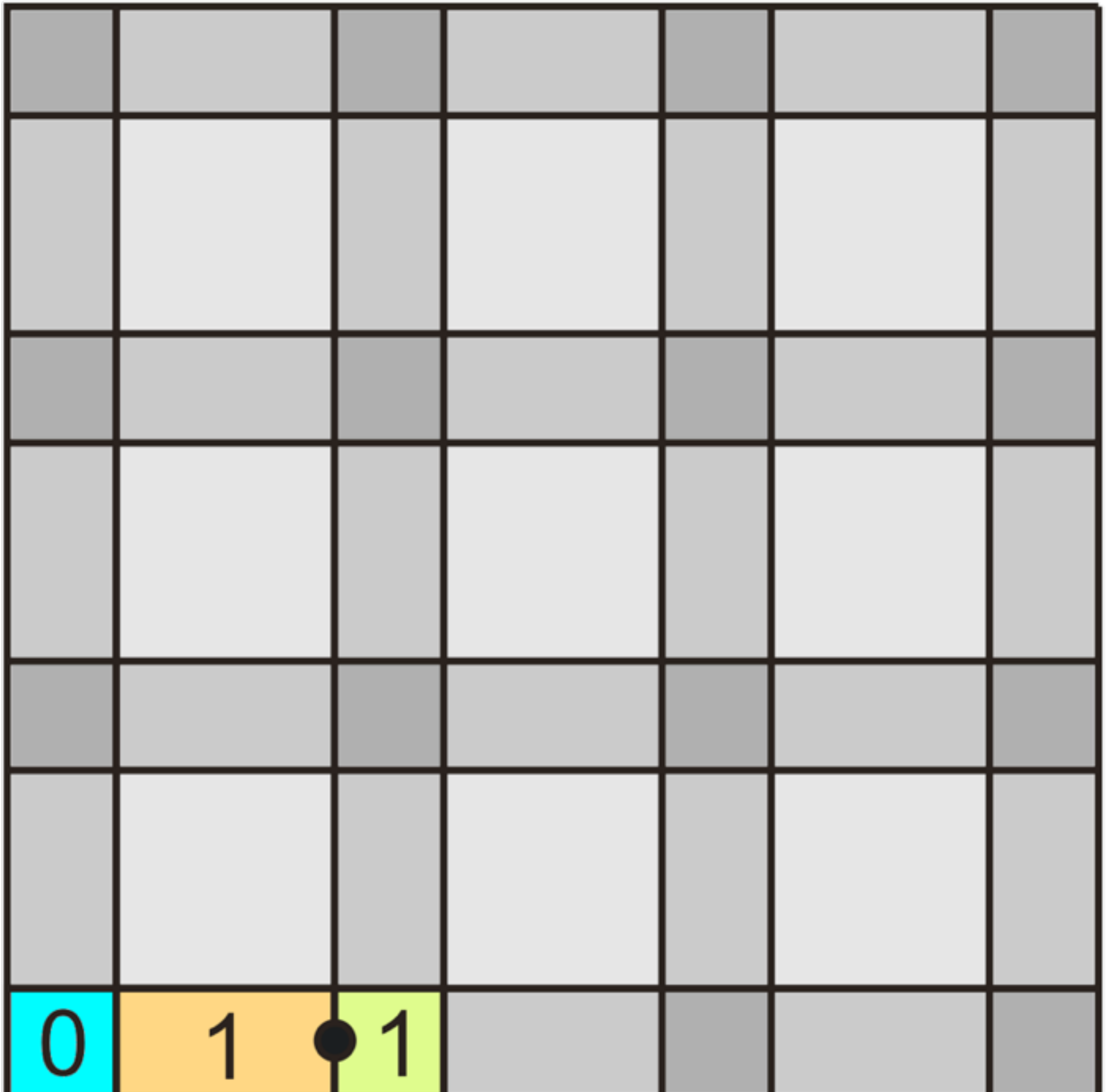


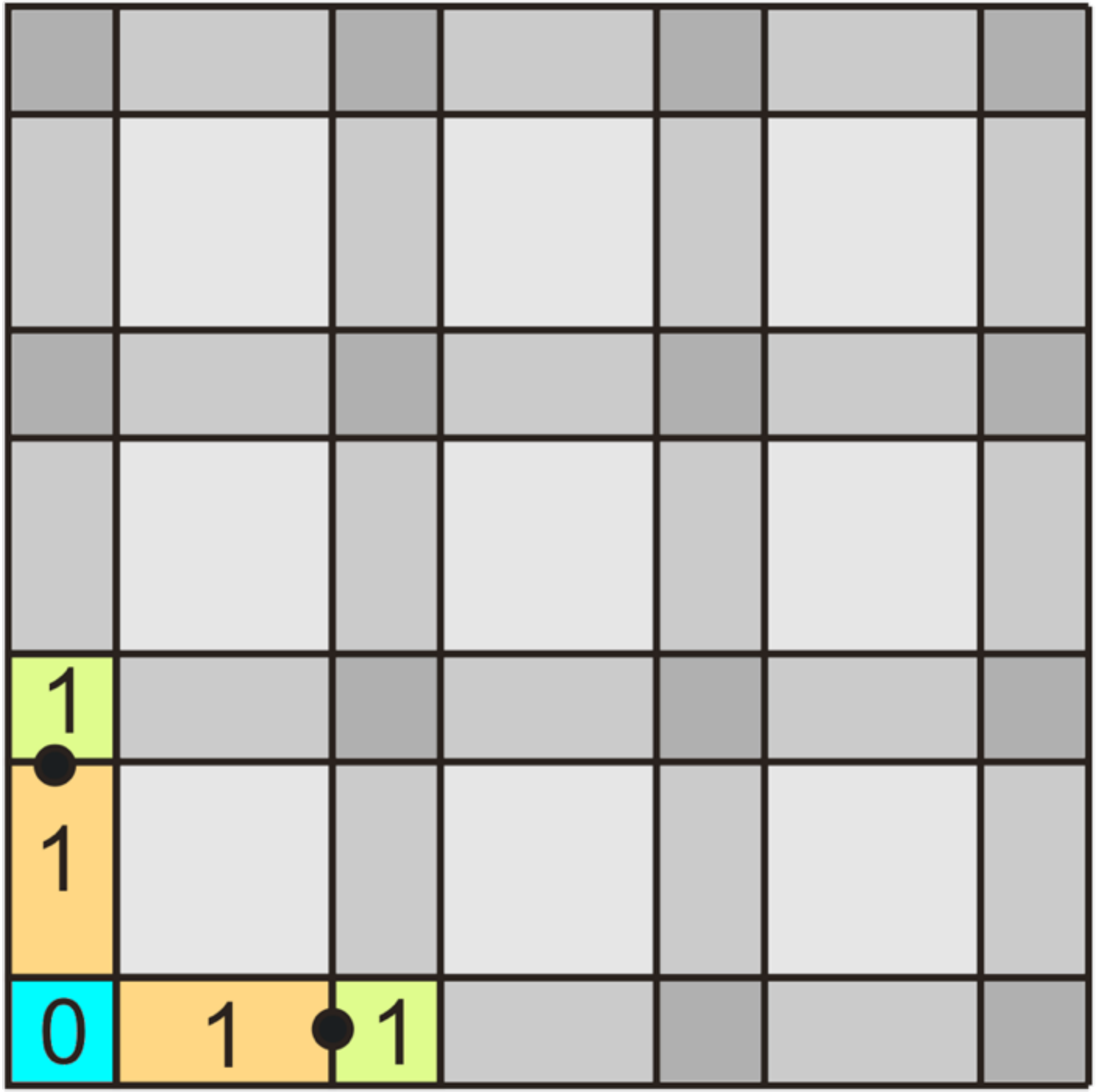
Illustration of the coreduction based decomposition algorithm

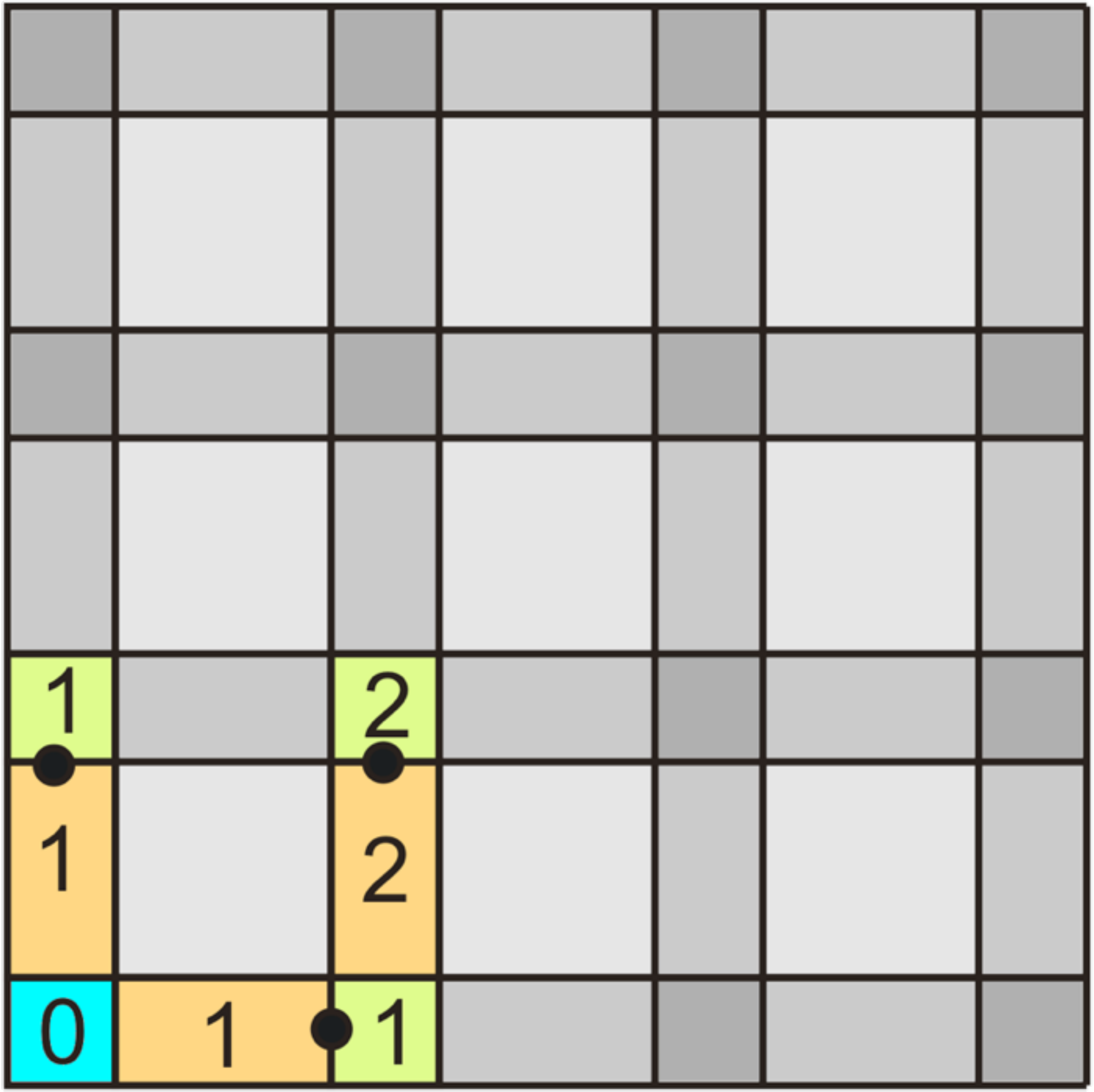


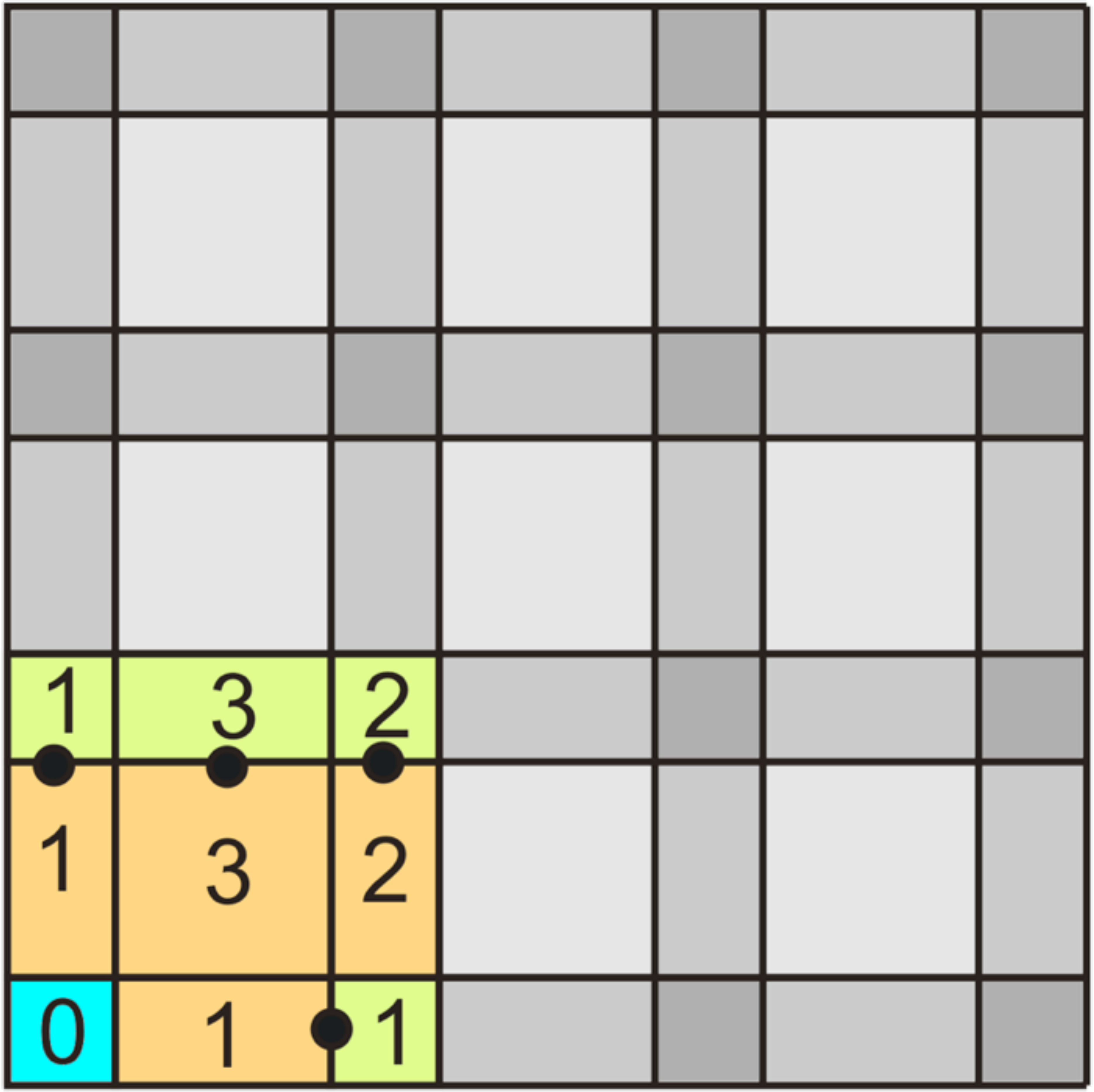


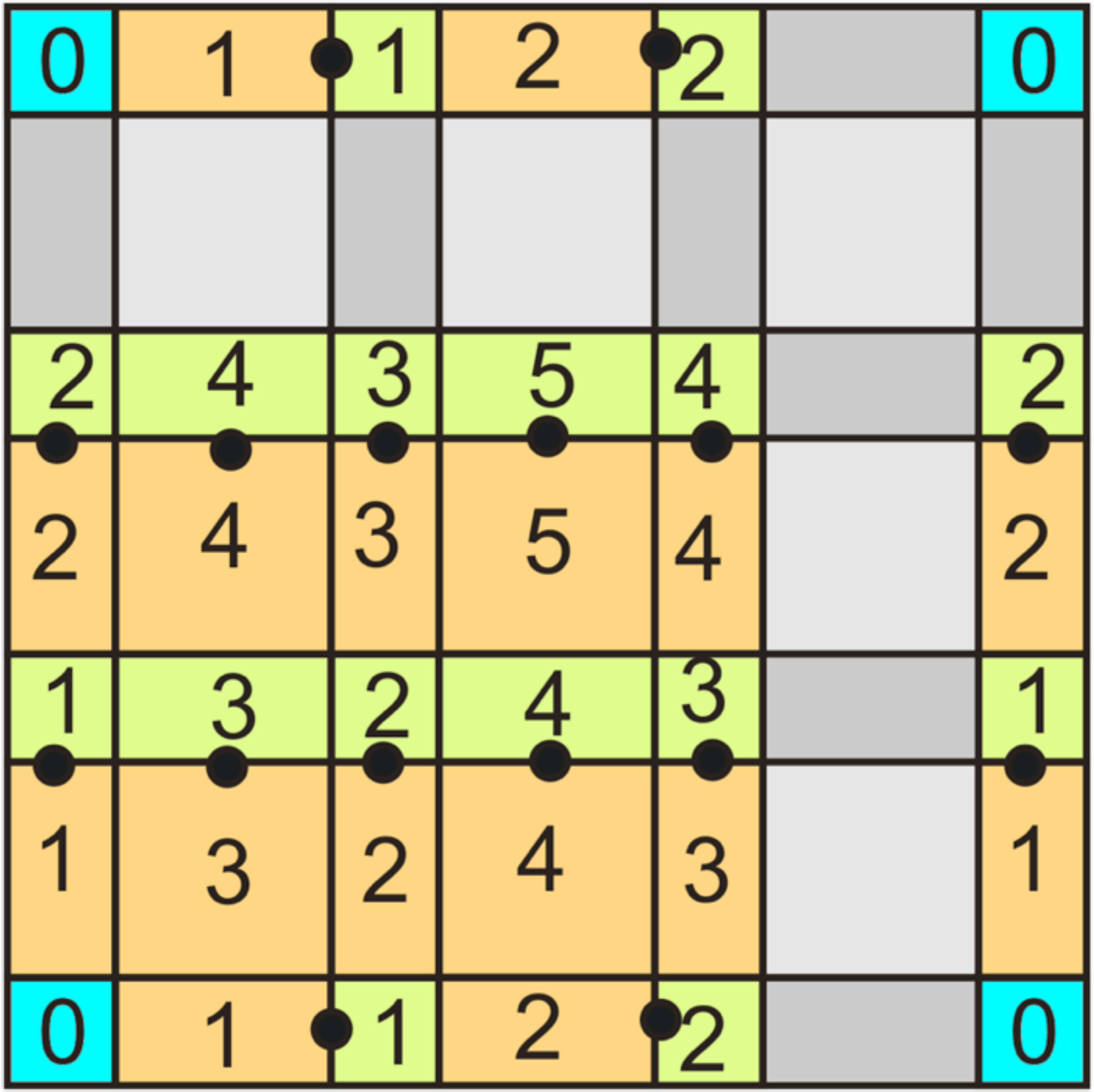

0


0
1
1


1
1
0
1
1

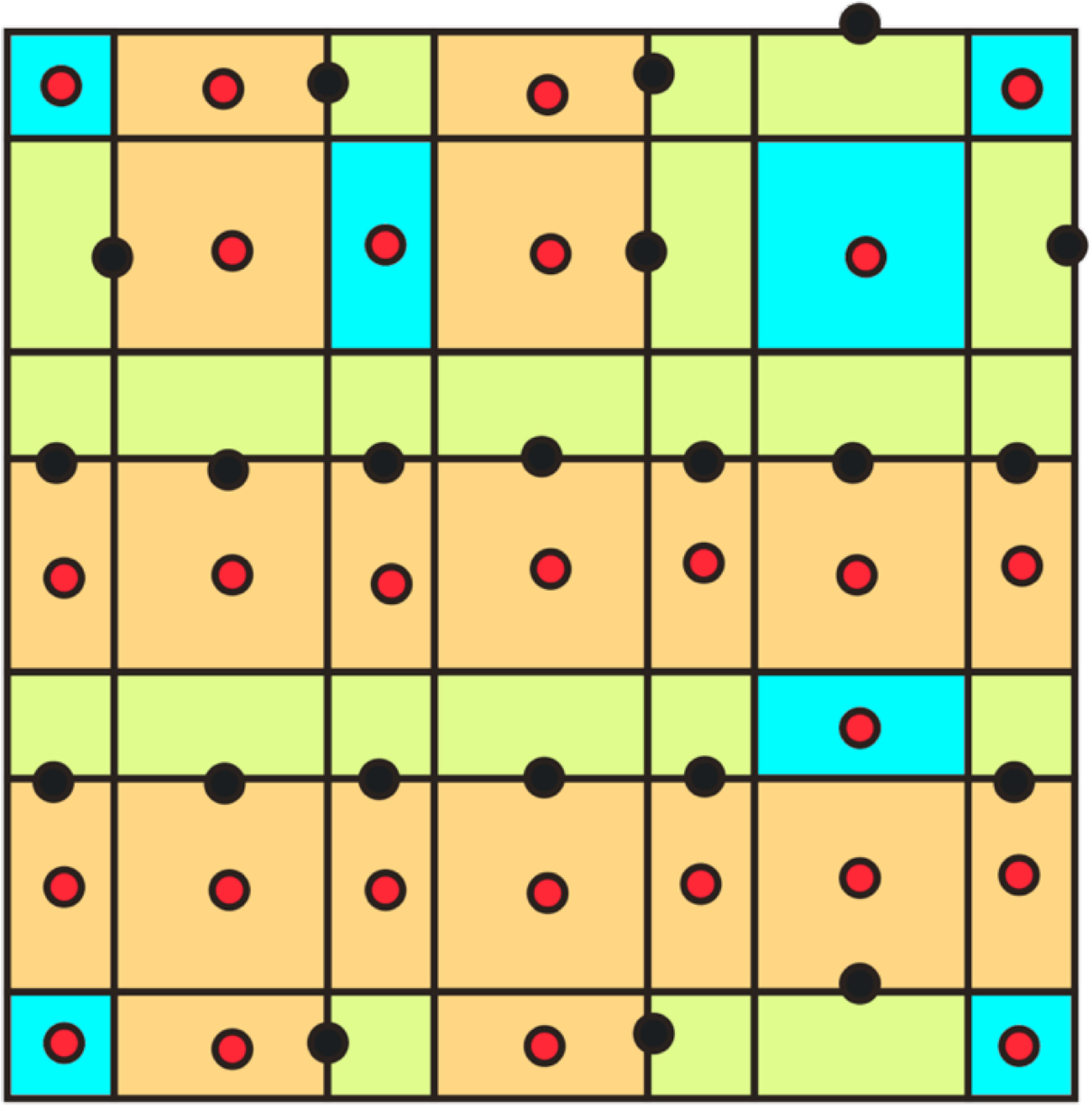

1
2
1
2
0
1
1


1
3
2
1
3
2
0
1
1


0 1 1 2 2 0
2 4 3 5 4 2
2 4 3 5 4 2
1 3 2 4 3 1
1 3 2 4 3 1
0 1 1 2 2 0


0 1 1 2 2 0
2 4 3 5 4 2
2 4 3 5 4 2
1 3 2 4 3 4 1
1 3 2 4 3 1
0 1 1 2 2 0

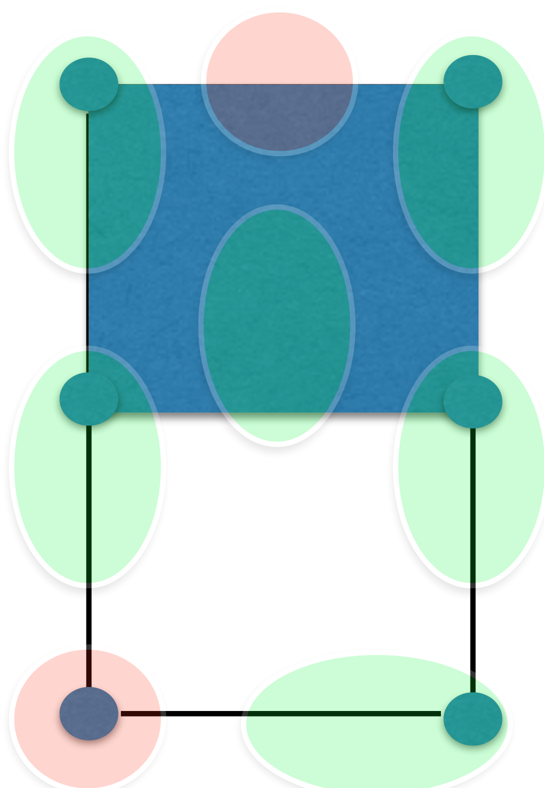
0 1 1 2 2 5 0
4
2 4 3 5 4 5 2
2 4 3 5 4 5 2
1 3 2 4 3 4 1
1 3 2 4 3 5 1
0 1 1 2 2 5 0



The Morse Complex

- The point of making the decomposition is to obtain a reduced complex with isomorphic homology groups.
- The “Morse Complex” is formed by taking $\mathcal{M}_k = \mathcal{R}(\mathcal{A}_k)$
- We must also specify the boundary homomorphisms Δ_k

That is, we keep only the critical cells
("aces"):



1 dim

0 dim

and we need to know
the boundary here

Forman's Formula: Connecting Orbits

- Consider the set of all "V-paths" from one critical cell to another of one less dimension
- For each path, determine the multiplicity
- Sum the multiplicity over all possible paths
- This yields the incidence numbers
- Direct computation does not yield a tractable procedure (combinatorial explosion in # of paths)

Another solution: Deformation

- Deform chains under a flow given by discrete gradient vector field

- More generally, consider equivalence classes of chains modulo the boundaries of ``King'' cells.

- These equivalence classes have a canonical representative which can be used to determine the Morse Boundary operator

- Takes linear time to deform a chain

The key to the Morse Boundary problem: Canonicalization

Definition: A king chain is formal combination of king cells.

Definition: A canonical chain is a formal combination of ace and king cells.

Lemma. Consider the equivalence classes of chains modulo the boundaries of king chains. In each such class, there is a unique canonical representative. We call this representative the canonicalization of any chain in that class.

How do we canonicalize a chain in practice?

Canonicalization Algorithm

Loop until c is canonical

Choose a maximal queen Q in c

Let K = Q*.

$$c \leftarrow c - \frac{\langle Q, c \rangle}{\langle Q, \partial K \rangle} \partial K$$

End Loop

Note: By choosing the maximal queen, we never process the same queen twice, via acyclicity assumption

Notation:

Given a chain c, the unique king chain k such that c + dk is canonical is called $\gamma(c)$

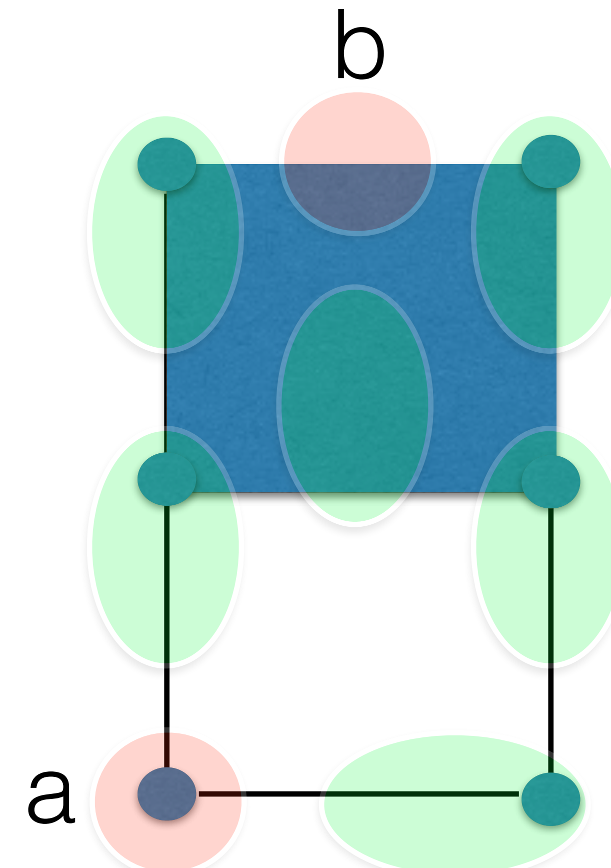
Hence $\alpha := \mathrm{id} + \partial \circ \gamma$ canonicalizes.

We also define $\beta := \mathrm{id} + \gamma \circ \partial$ which we call completion (it canonicalizes the boundary of a chain).

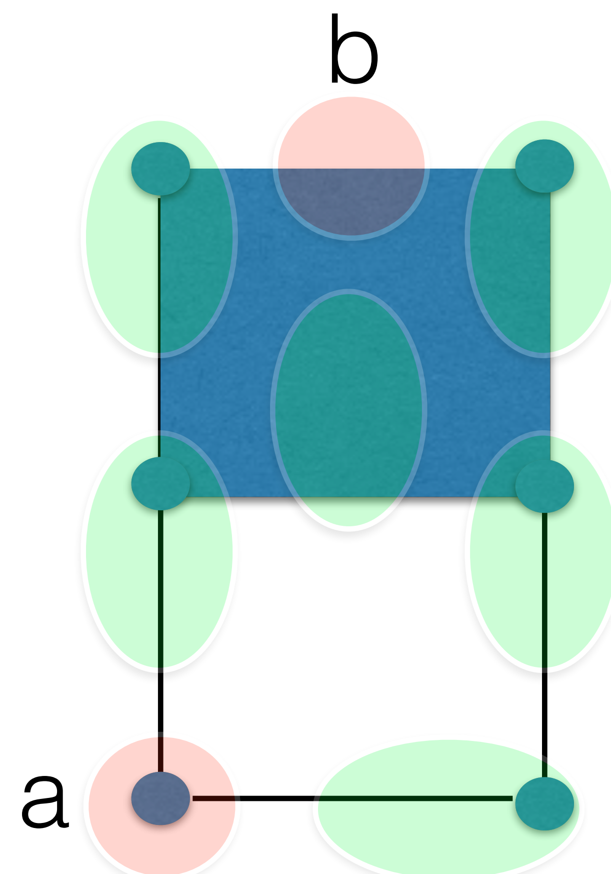
There are chain maps i and j between the original complex and the Morse complex naturally induced from inclusion and projection.

And now the Morse boundary: $\Delta = j \circ \alpha \circ \partial \circ i$

Let's use this formula to compute Δb

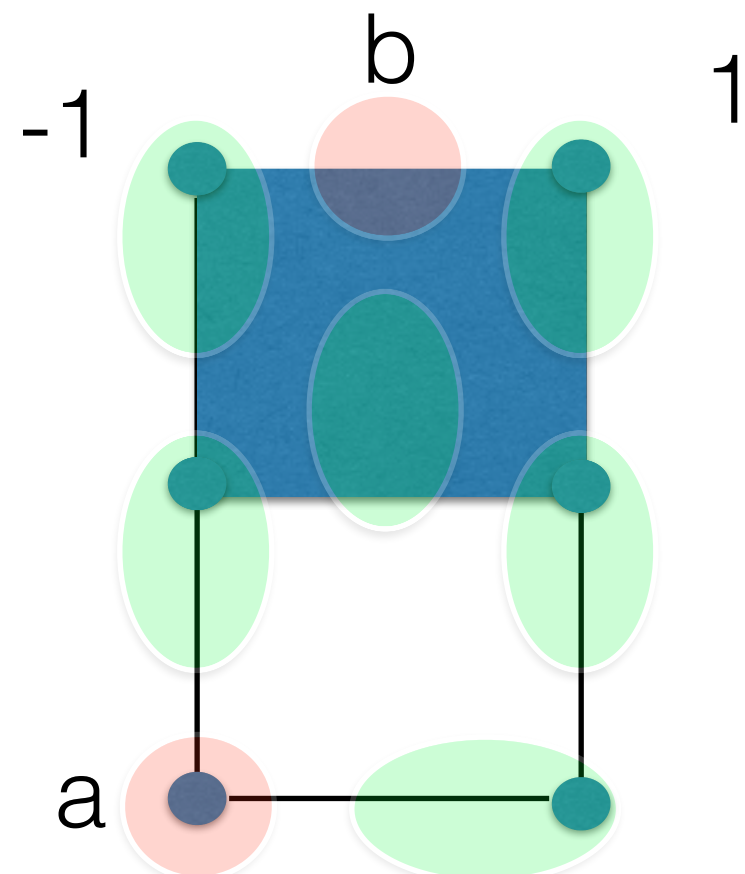


Let's use this formula to compute Δb



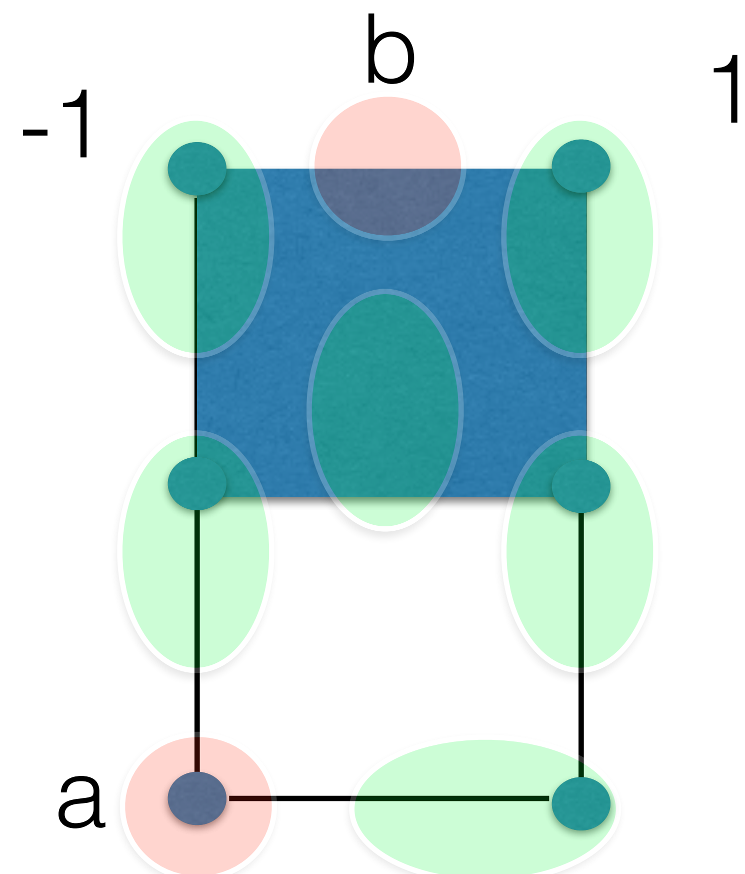
∂b

Let's use this formula to compute Δb



First compute ∂b

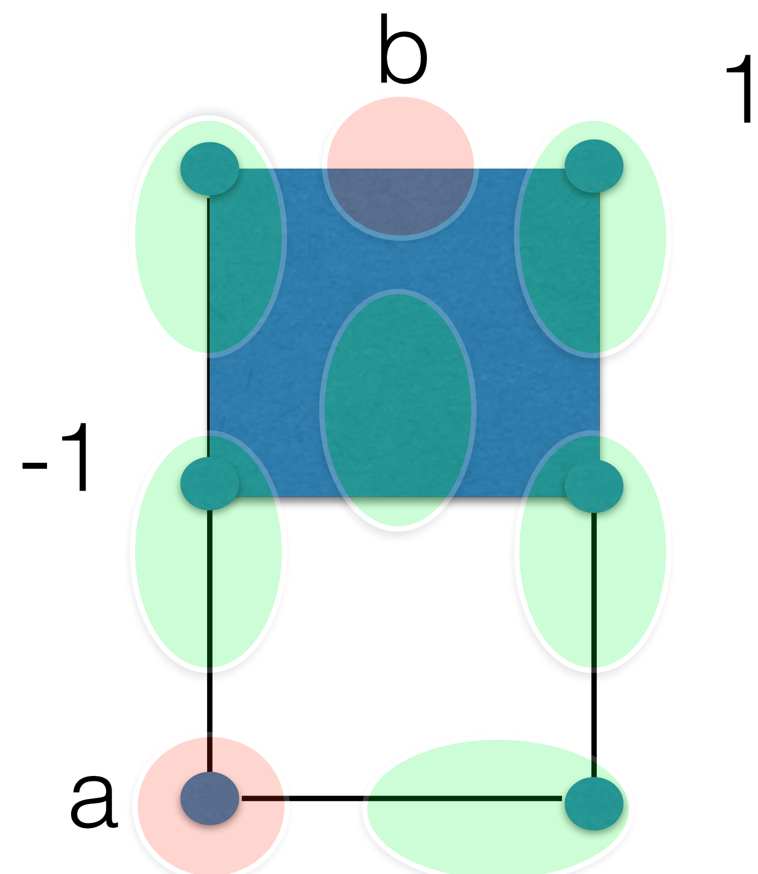
Let’s use this formula to
compute Δb



First compute ∂b

Now canonicalize.

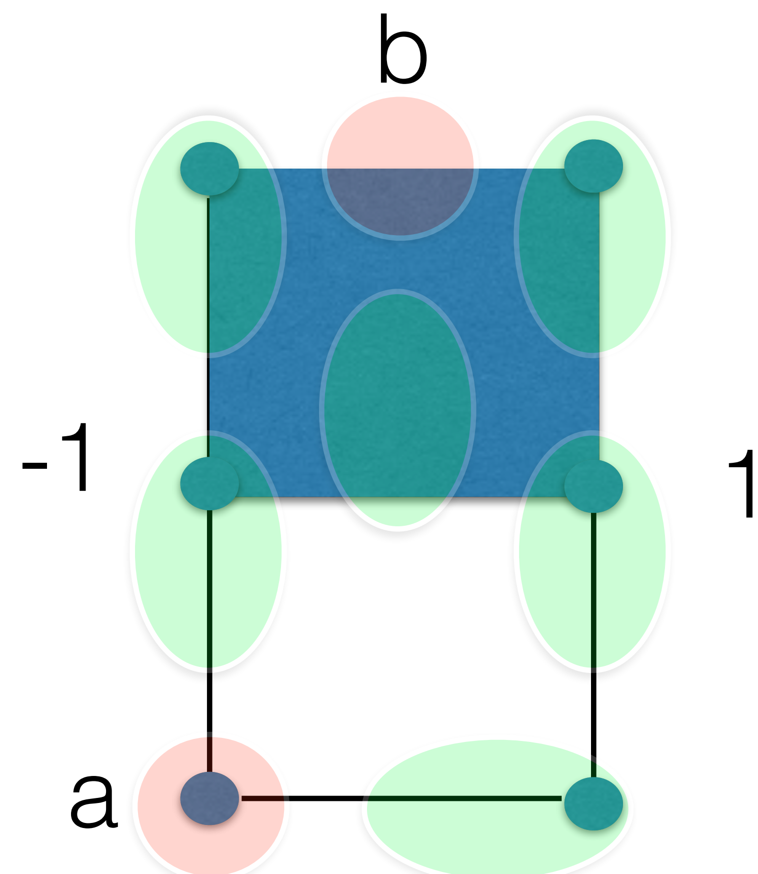
Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

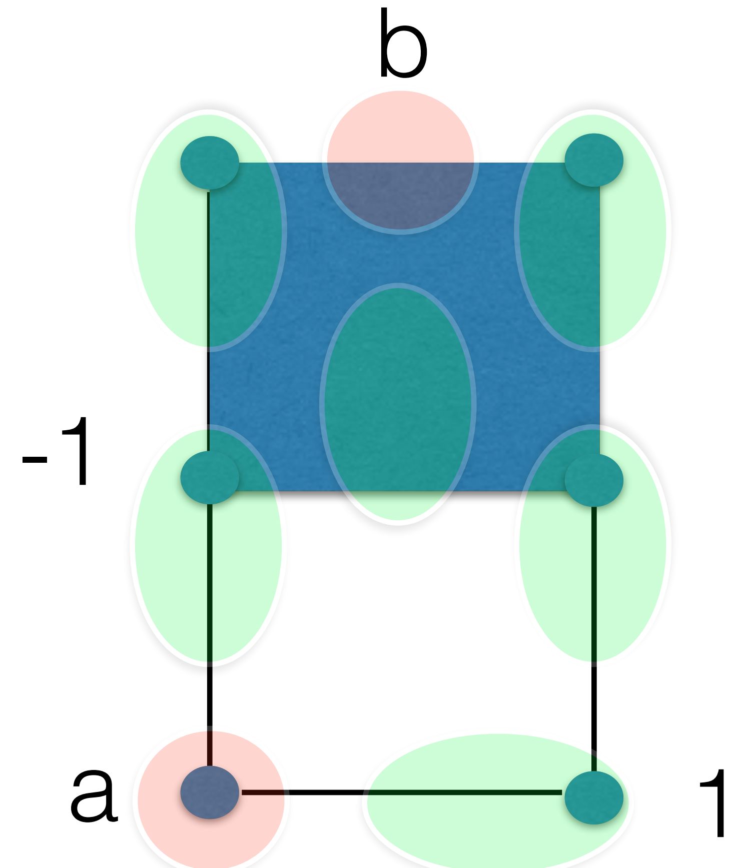
Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

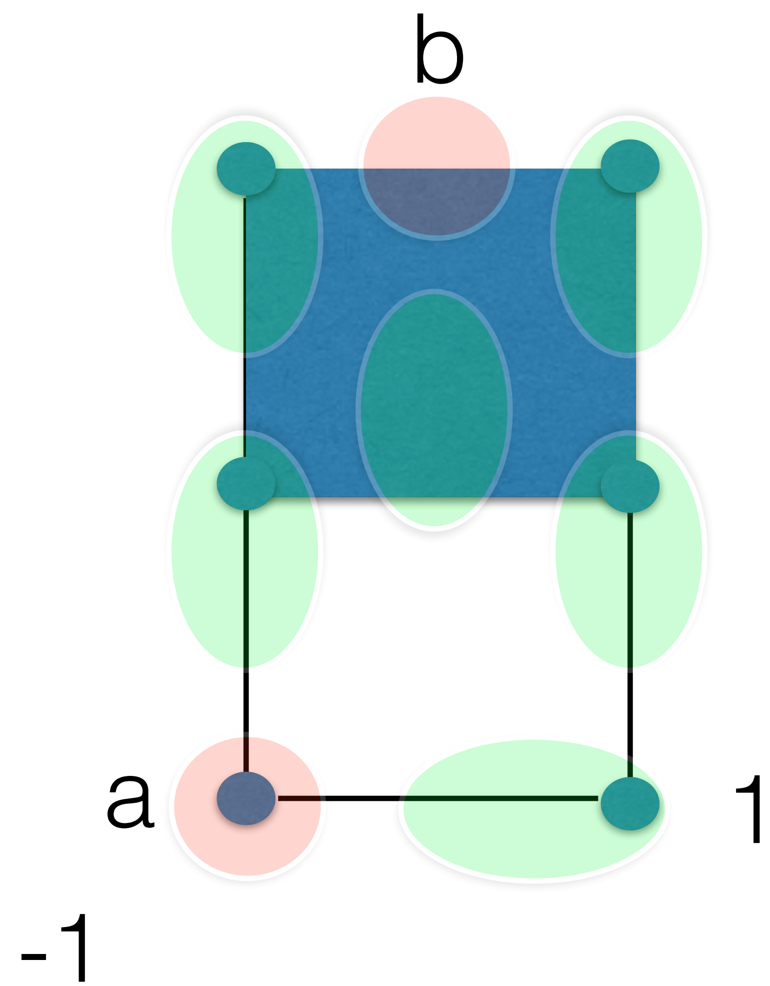
Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

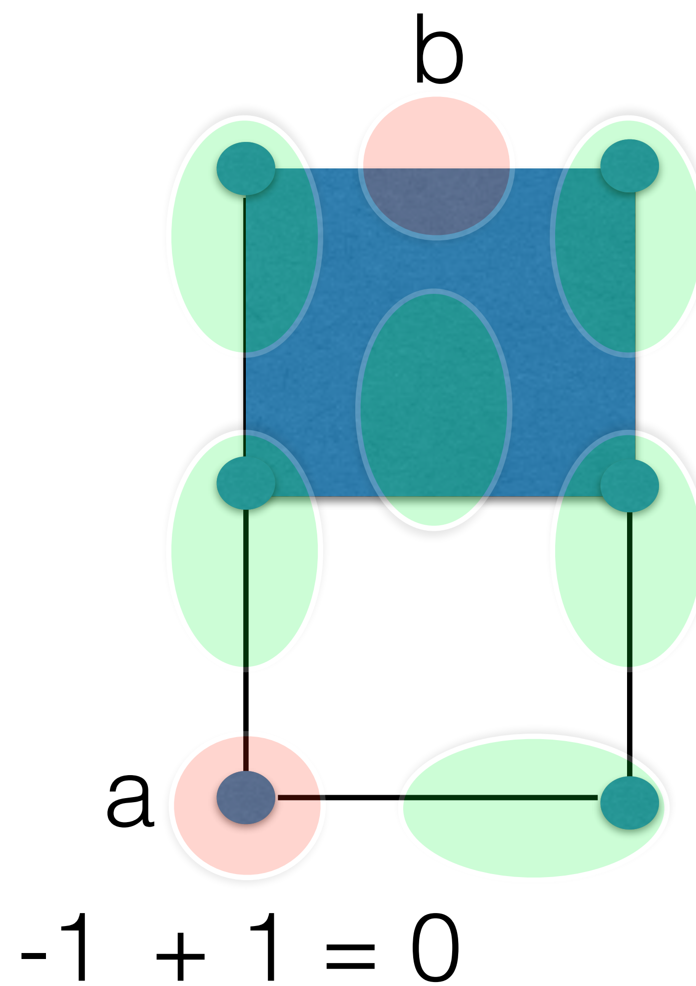
Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

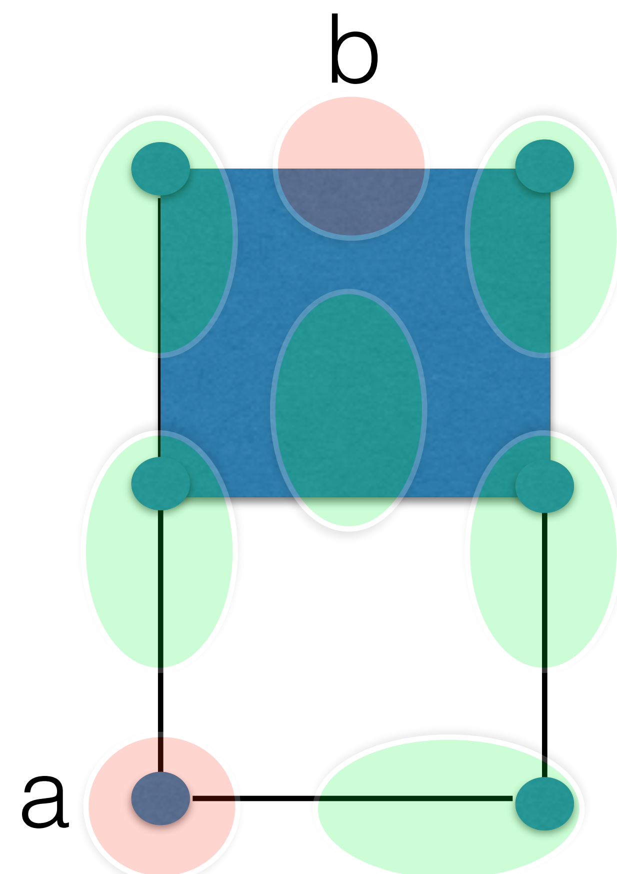
Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

Let's use this formula to compute Δb



First compute ∂b

Now canonicalize.

Complexity

- Worst case for time: $O(n)$ critical cells, $O(n^2)$ time to canonicalize — yields $O(n^3)$
- Best case for time: $O(n)$
- Space complexity: $O(n)$ + (size of Morse complex)

Chain Equivalences

- We know the Morse complex has isomorphic homology groups because there are chain equivalences to and fro:

$$\phi : (\mathcal{C}, \partial) \to (\mathcal{M}, \Delta)$$

$$\psi : (\mathcal{M}, \Delta) \to (\mathcal{C}, \partial)$$

- We can directly exhibit these chain equivalences using our language of canonicalization and completion:

$$\phi = j \circ \alpha$$

where

$$\psi = \beta \circ i$$

$$\Delta = \phi \circ \partial \circ \psi$$

Experimental Results

- Several Implementations:
- CHOMP-CR
- CHOMP-DMT
- CHOMP-CR+DMT
- REDHOM-CR
- REDHOM-CR+DMT

Computer Experiments with CHOMP version

Dimension	Size	CR Time	DMT Time
4	8.5 x 10^6	9.82 sec	2.54 sec
4	40.8 x 10^6	560 sec	13.59 sec
5	3.8 x 10^6	91.6 sec	1.98 sec
5	5.8 x 10^6	458 sec	4.11 sec
6	3.0 x 10^6	42.5 sec	3.26 sec
6	7.2 x 10^6	2346 sec	8.51 sec

Randomly Generated Cubical Complexes

Cubical Complexes for Selected Spaces

	$T \times S^1$	$(S^1)^3$	$S^1 \times K$	$T \times T$
dim	5	6	6	6
size in millions	0.07	0.10	0.40	2.36
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}^3$	0	$\mathbb{Z}^2 + \mathbb{Z}_2$	$\mathbb{Z}^4$
H_2	$\mathbb{Z}^3$	0	$\mathbb{Z} + \mathbb{Z}_2$	$\mathbb{Z}^6$
H_3	$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}^4$
H_4				$\mathbb{Z}$
Linbox::Smith	130	350	> 600	> 600
RedHom::Shave+**Linbox**::Smith	0.5	0.1	2.2	> 600
ChomP	1.3	1.7	10	56
RedHom::CR	0.03	0.04	0.26	2.5
ChomP::DMT	0.06	0.15	1.6	5.9
ChomP::CR+DMT	0.04	0.16	1.7	3
RedHom::CR+DMT	0.02	0.08	0.5	1.1

Cahn-Hillard Equation

	P0001	P0050	P0100
dim	3	3	3
size in millions	75.56	73.36	71.64
H_0	$\mathbb{Z}^7$	$\mathbb{Z}^2$	$\mathbb{Z}$
H_1	$\mathbb{Z}^{6554}$	$\mathbb{Z}^{2962}$	$\mathbb{Z}^{1057}$
H_2	$\mathbb{Z}^2$		
Linbox::Smith	> 600	> 600	> 600
RedHom::Shave+**Linbox**::Smith	> 600	> 600	> 600
ChomP	400	360	310
RedHom::CR	36	34	33
ChomP::DMT	110	110	100
ChomP::CR+DMT	45	43	42
RedHom::CR+DMT	26	25	24

Random Cubical Sets

	d4s8f50	d4s12f50	d4s16f50	d4s20f50
dim	4	4	4	4
size in millions	0.07	0.34	1.04	2.48
H_0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
H_1	$\mathbb{Z}^2$	$\mathbb{Z}^{17}$	$\mathbb{Z}^{30}$	$\mathbb{Z}^{51}$
H_2	$\mathbb{Z}^{174}$	$\mathbb{Z}^{1389}$	$\mathbb{Z}^{5510}$	$\mathbb{Z}^{15401}$
H_3	$\mathbb{Z}^2$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{71}$	$\mathbb{Z}^{179}$
Linbox::Smith	120	> 600	> 600	> 600
RedHom::Shave+**Linbox**::Smith	4	> 600	> 600	> 600
ChomP	1	8.3	41	170
RedHom::CR	0.08	1.4	15	140
ChomP::DMT	0.05	0.38	1.8	5.3
ChomP::CR+DMT	0.03	0.16	0.56	1.4
RedHom::CR+DMT	0.03	0.16	0.58	2.9

Simplicial Complexes

	random set	Björner set	S^5
dim	4	2	5
size in millions	4.8	1.9	4.3
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	$\mathbb{Z}^{39}$	0	0
H_2	$\mathbb{Z}^{84}$	$\mathbb{Z}$	0
H_3			0
H_4			$\mathbb{Z}$
ChomP	830	310	2100
RedHom::CR+DMT	65	11	100

Part 2. Induced Homology on maps

Practical uses of chain equivalences:

- We can not only compute homology using the reduced Morse complex, but we can lift homology generators via a chain equivalence.

- It is possible to solve the “preboundary” equation for c given b via an iterative approach:

$$\partial c = b$$
$$P_{\mathcal{C}} = \gamma + \psi \circ P_{\mathcal{M}} \circ \phi$$

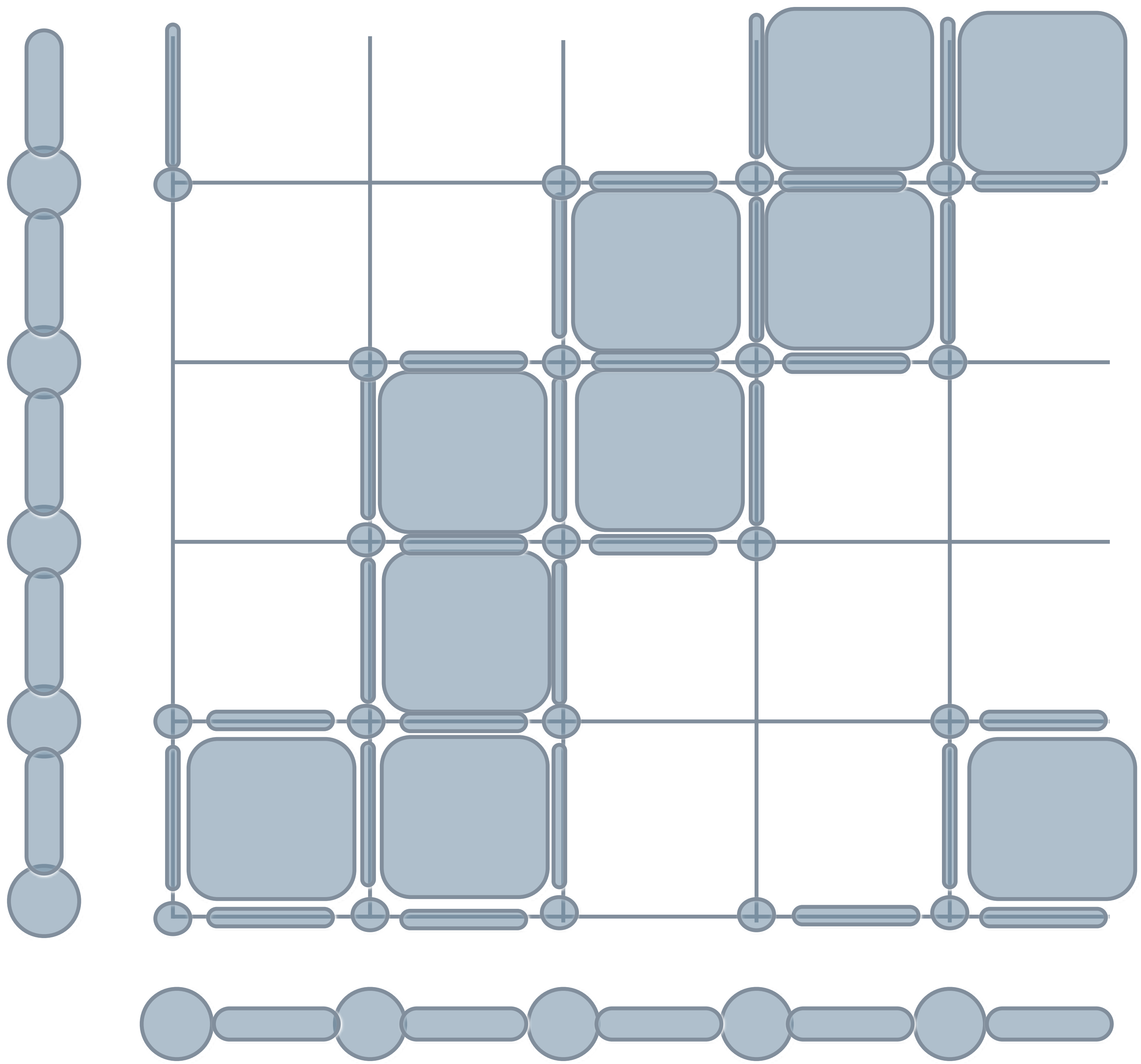
Induced Homology of Maps

- This allows us to use discrete Morse theory to help find induced homology on maps.

- The setting where this is accomplished involves a graph complex arising from an acyclic-valued combinatorial map equipped with projections to both a domain complex and a codomain complex. We define:

$$\text{For every cycle } c \in \Gamma,\ \mathcal{F}_*([\pi_X(c)]) = [\pi_Y(c)]$$

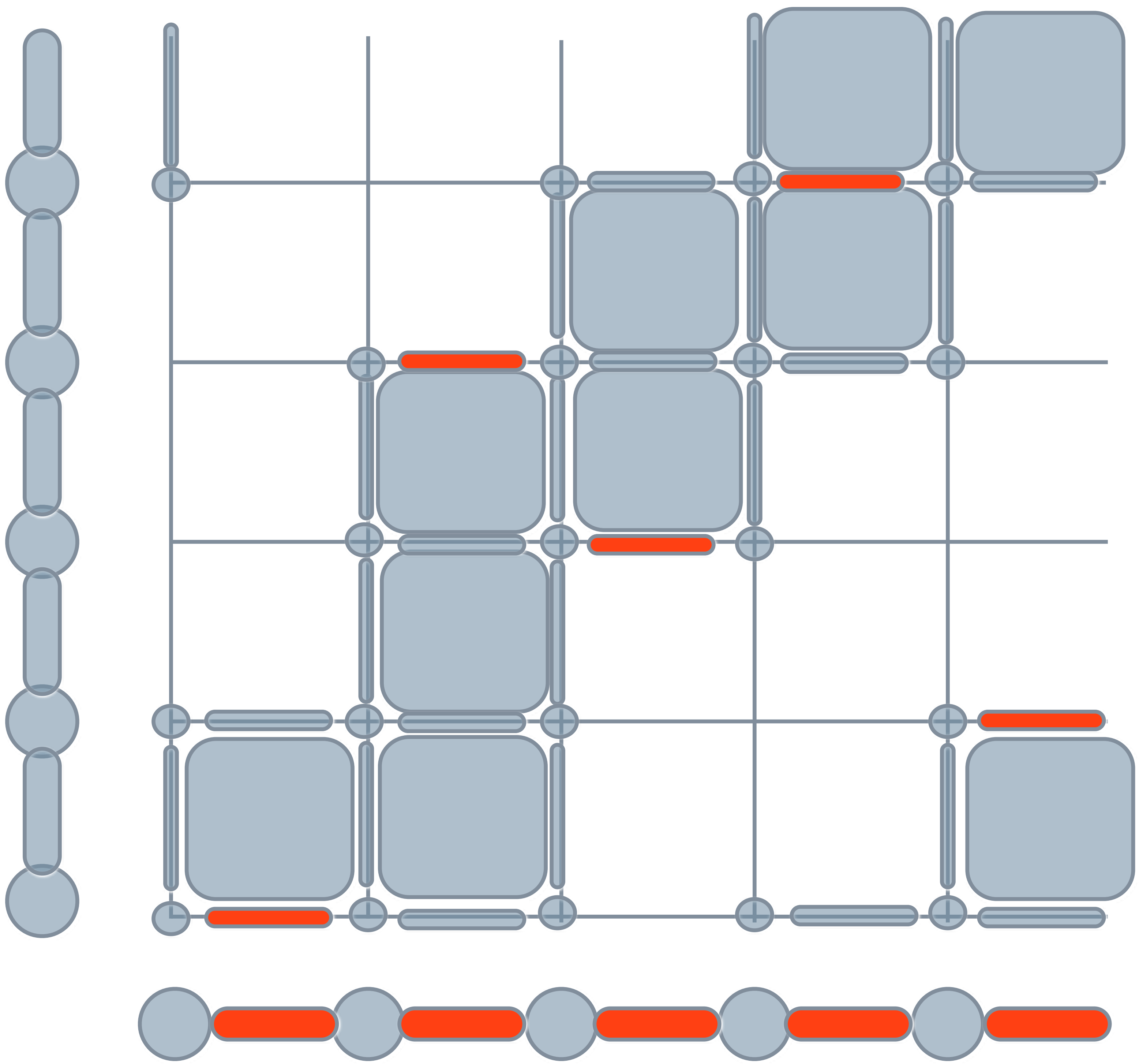
Acyclic Combinatorial Map



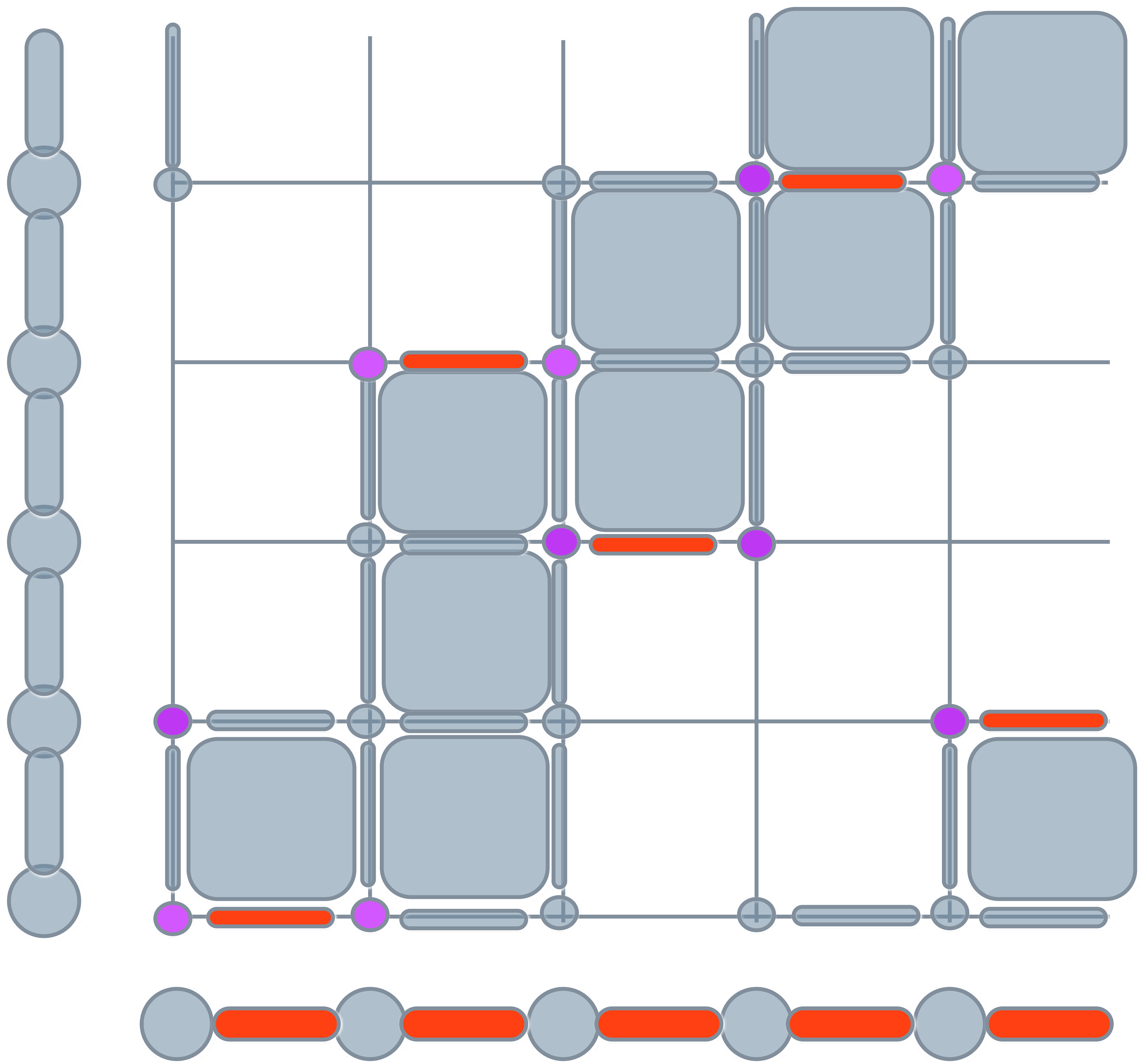
Algorithm

- Compute homology generators of X and Y
- Lift homology generators of X to cycles in graph by arbitrarily lifting to a chain and then iteratively solving the preboundary equation in the fibers (goal: get a cycle).
- Project the cycle into Y and re-express it in terms of the homology generators of Y.

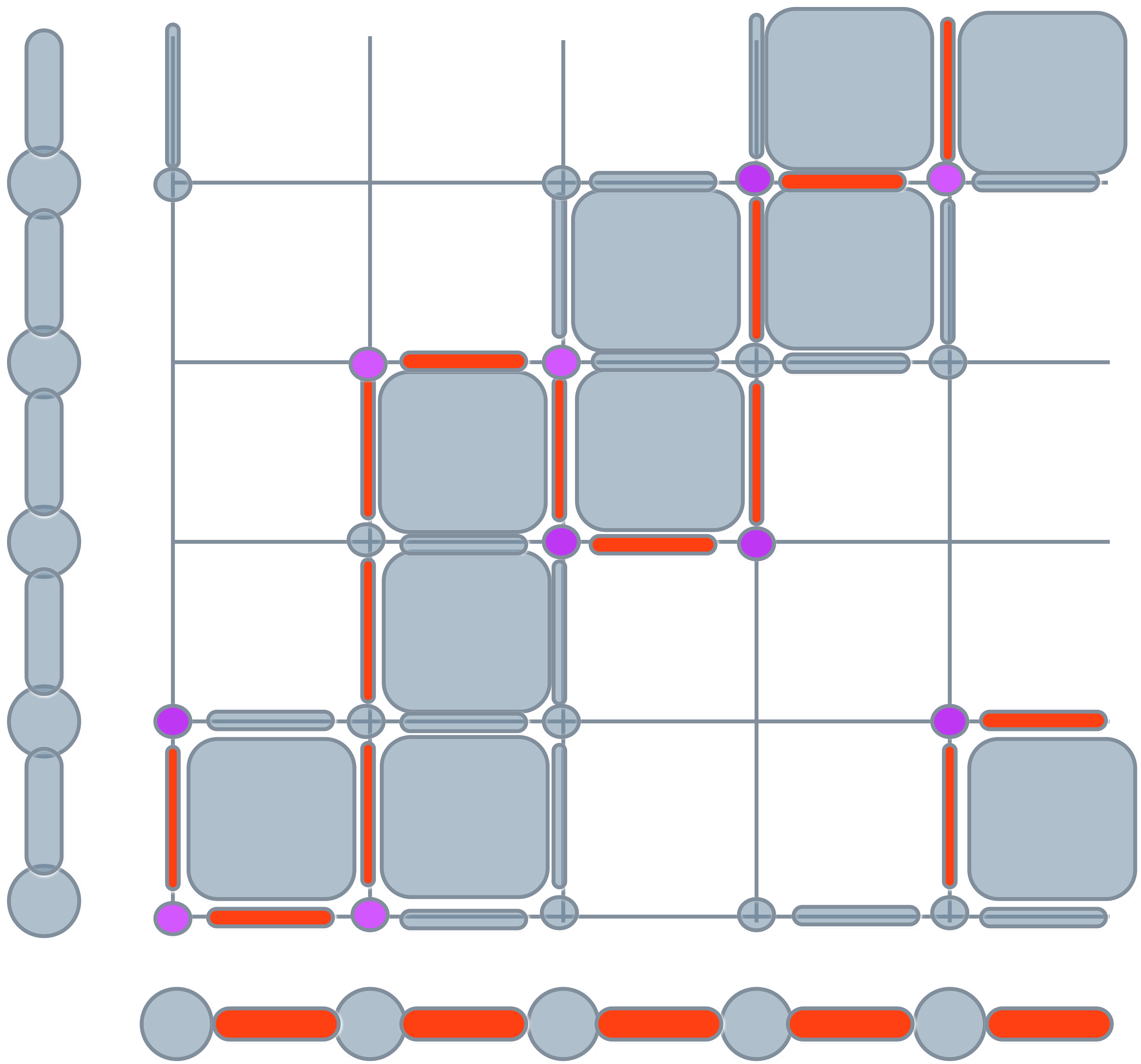
Chain Lift



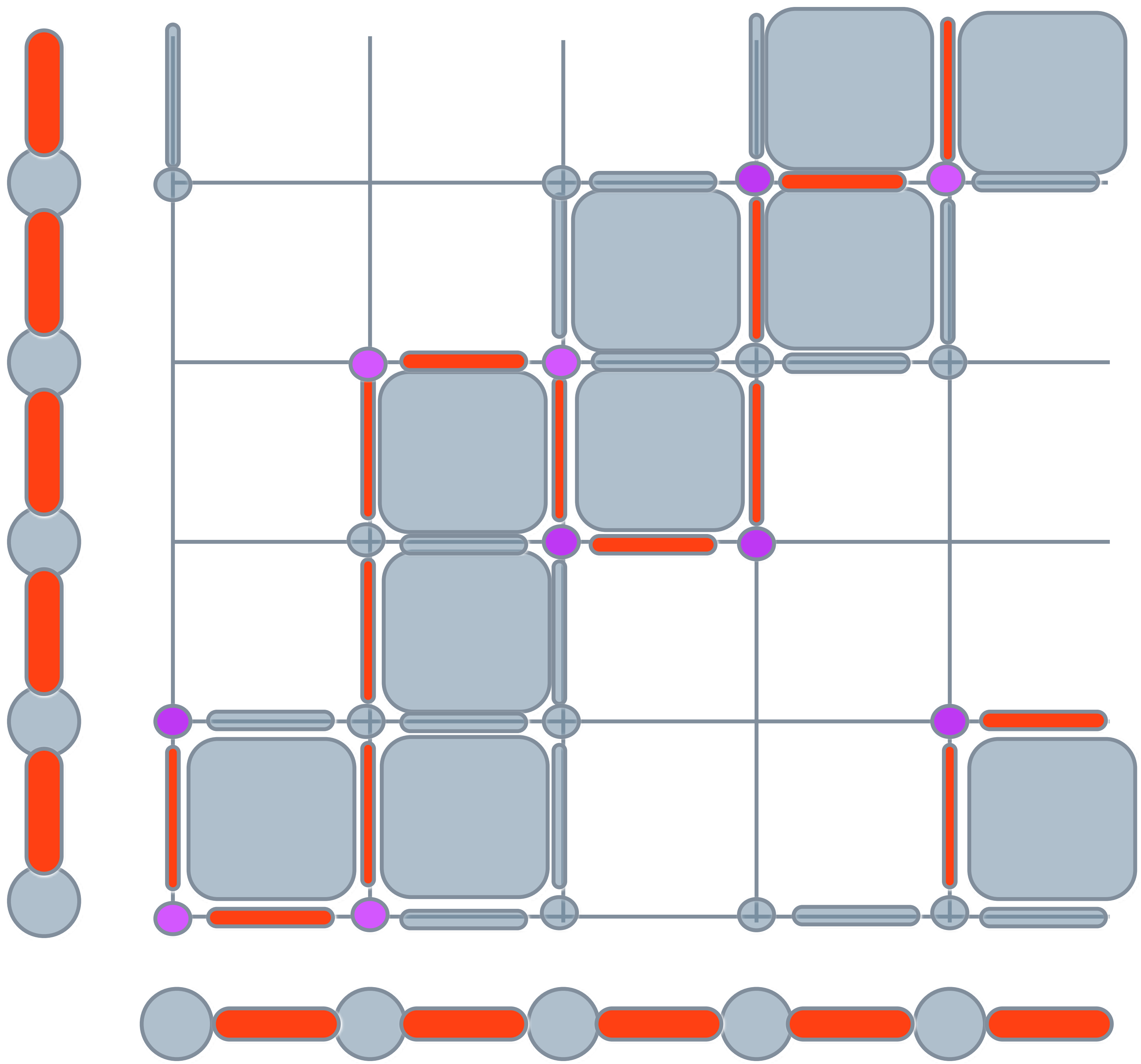
Not a cycle yet



Add Preboundaries in Fibers



Project to Codomain



Higher Dimensions

- The example given was from 1d to 1d. This makes it easy to draw, and the algorithm finishes after a single iteration.
- Higher dimensional problems have more steps; in fact, d of them.
- Let’s take a more careful look at how these steps progress

Notation

- For each fiber in the graph complex, declare its “dimension” to be the dimension of its X-projection.

- We say that a cell of dimension $j+k$ in a fiber of dimension j is a cell of type (j, k)

Higher Dimension, cont.

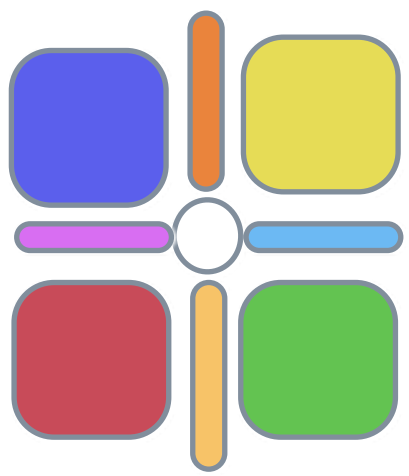
- Our initial “Chain Lift” of a d-dimensional homology generator in X gives us 0-cells in the fibers of dimension d. (Cells of type (d,0).)

- The k-th iteration adds in cells of dimension k in fibers of dimension d-k. (Cells of type (d-k, k))

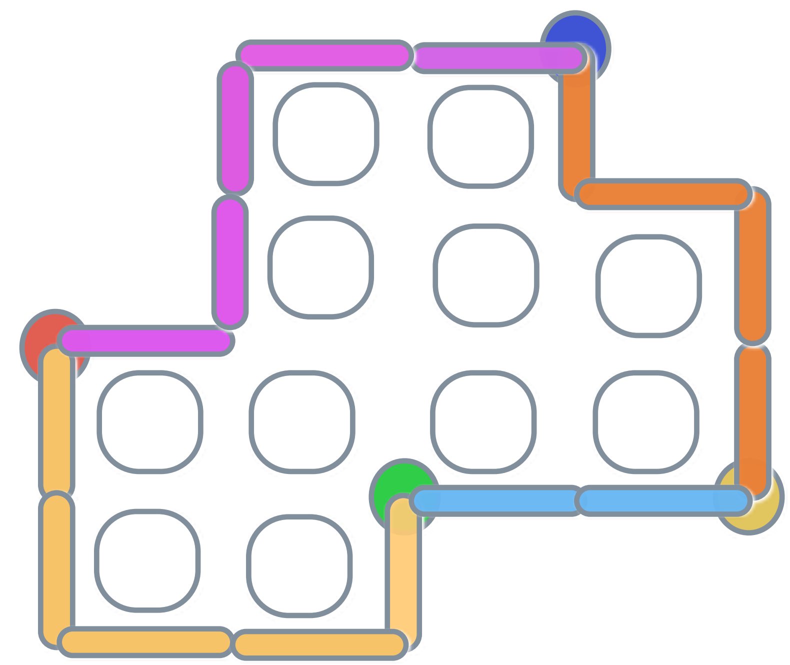
- The last step adds cells of dimension d in fibers of dimension 0. (Cells of type (0,d))

- These final cells are the only ones that get projected to Y for the final answer.

Higher Dimensional example



Domain Projection



Codomain Projection

Code

CHomP: Shaun Harker

http://code.google.com/p/chomp-rutgers/

https://github.com/sharker81/CHomP

REDHOM: Hubert Wagner, Mateusz Juda, Pawel Dlotko, Marian Mrozek

http://capd.ii.uj.edu.pl/

Perseus: Vidit Nanda. (nice, good for persistence)

http://www.sas.upenn.edu/~vnanda/perseus/index.html

References

1. Marian Mrozek and Bogdan Batko. *Coreduction Homology Algorithm.* Discrete and Computational Geometry. **41**(1) (2009) 96-118
2. Robin Forman. *Morse Theory for Cell Complexes.* Advances in Mathematics. **134** (1998) 90-145
3. Tomasz Kaczyński, Konstantin Mischaikow, Marian Mrozek. *Computational Homology*, Applied Mathematical Sciences, **157**, Springer 2004
4. Dmitry Kozlov. *Combinatorial Algebraic Topology.* Algorithms and Computation in Mathematics, **21**, Springer 2008

and...

Harker, Shaun, Konstantin Mischaikow, Marian Mrozek, and Vidit Nanda. "Discrete Morse theoretic algorithms for computing homology of complexes and maps." *Foundations of Computational Mathematics* 14, no. 1 (2014): 151-184.