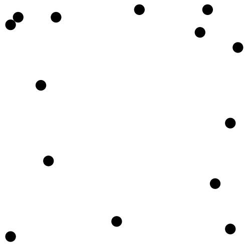
Efficient and Robust Persistent Homology for Measures

Mickaël Buchet

joint work with F. Chazal, S. Oudot and D. Sheehy

Topological inference



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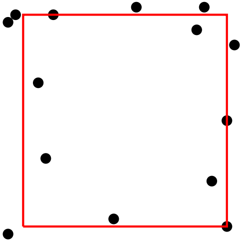
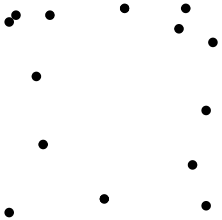


Persistence



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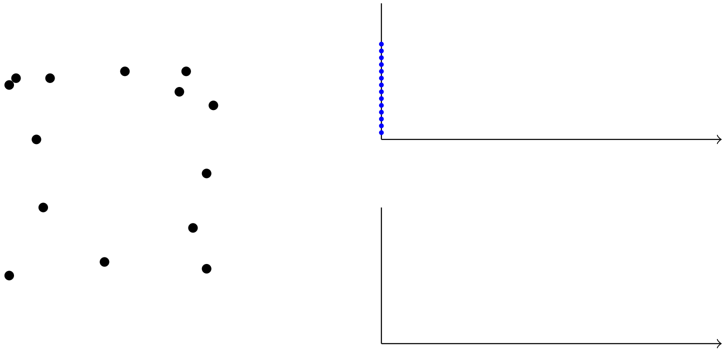

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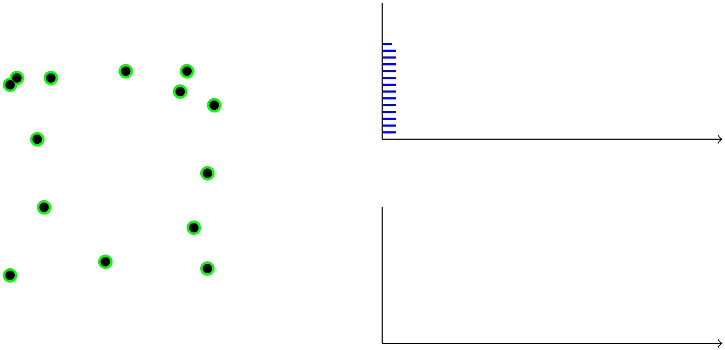

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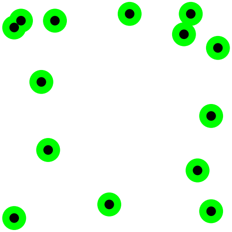

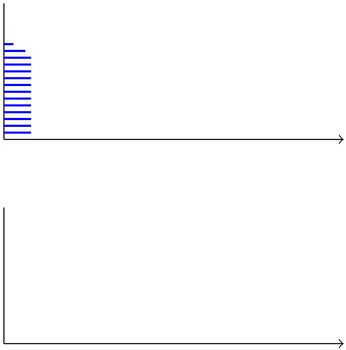

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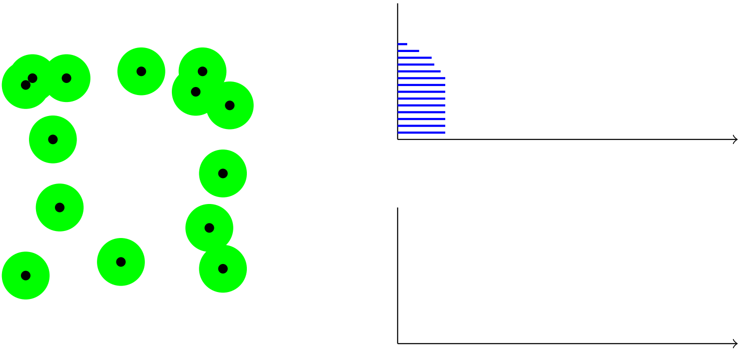

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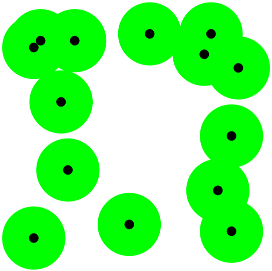

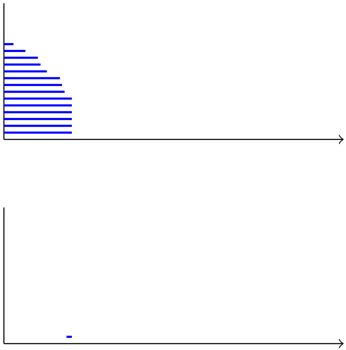

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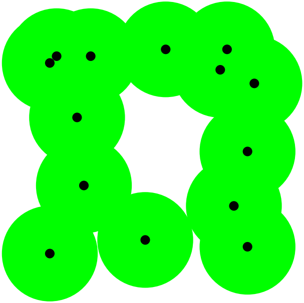

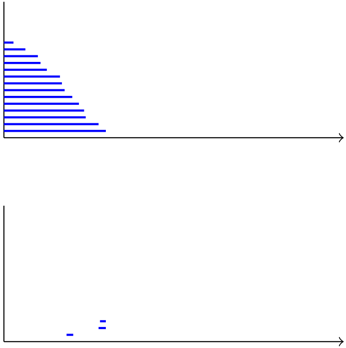

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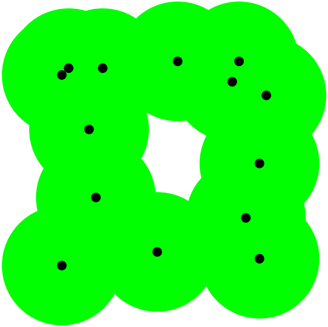
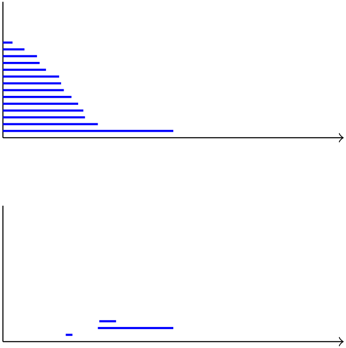
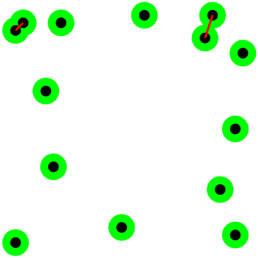




Practical application



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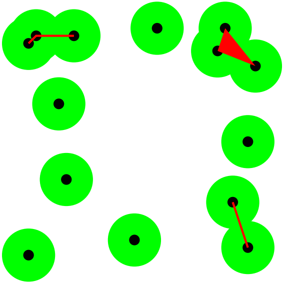

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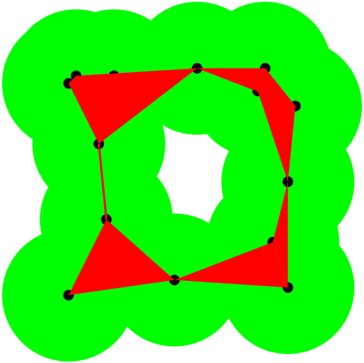

Complexity

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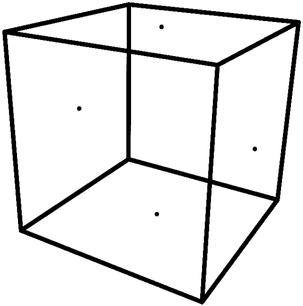
- Unusable in high dimensions.

Outliers



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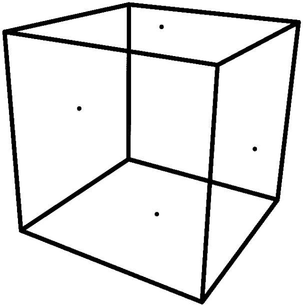

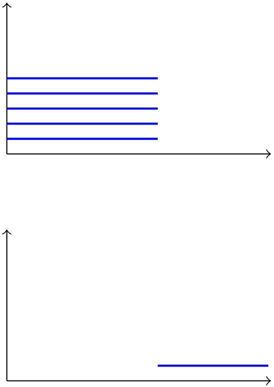

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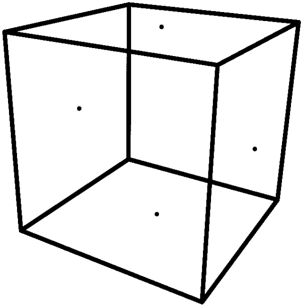

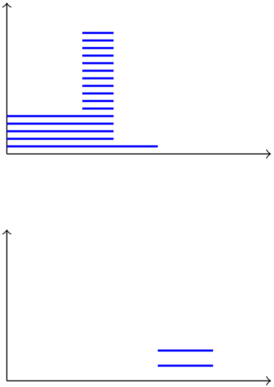

Computation of persistence diagrams

Distance to the point cloud and Vietoris-Rips filtration:

- Complexity: $O(n^{d+1})$
- Robustness to outliers: none

Distance to a measure

Definition (Chazal, Cohen-Steiner, Mérigot, 2011)

Let μ be a measure and $m \in]0,1[$, then

$$d_{\mu,m}(x) = \frac{1}{\sqrt{m}} \inf_{\nu \in \mathrm{Sub}_m(\mu)} W_2(m\delta_x, \nu)$$

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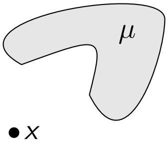

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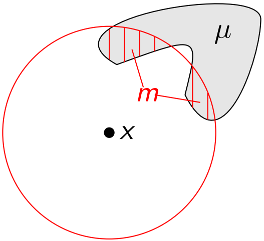


Case of an empirical measure

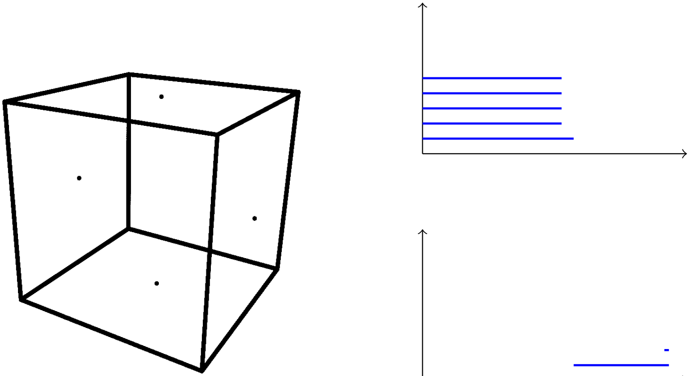
Theorem

Let μ be an empirical P and $k = m|P|$ is an integer then:

$$d_{\mu,m}(x) = \sqrt{\frac{1}{k}\sum_{i=1}^{k} d_{\mathbb{X}}(x, p_i(x))^2}$$

where $p_i(x)$ is the i^{th}-neighbour of x in P.

Results



Barycentric decomposition

Theorem

In Euclidean spaces, the distance to an empirical measure is a power distance.

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where $bar(x) = \sum_{i=1}^{k} p_i(x)$

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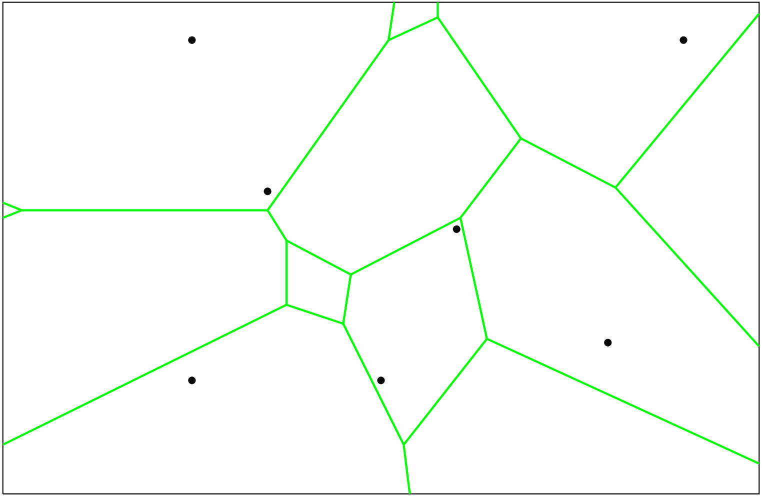
&= ||x - bar(x)||^2 + w_{bar(x)}^2 \\
&= \min_{b \in B} \left(||x - b||^2 + w_b^2 \right)
\end{aligned}
$$

where $bar(x) = \sum_{i=1}^{k} p_i(x)$ and B is the set of all barycentres of k points.

k^{th}-order Voronoi diagram



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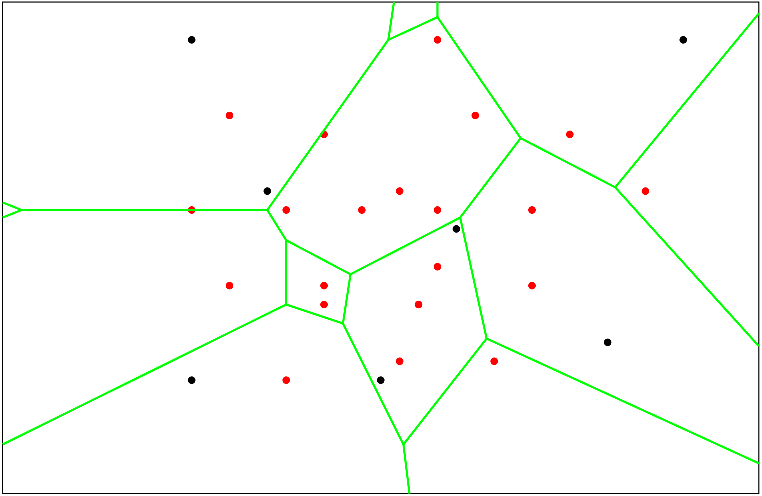

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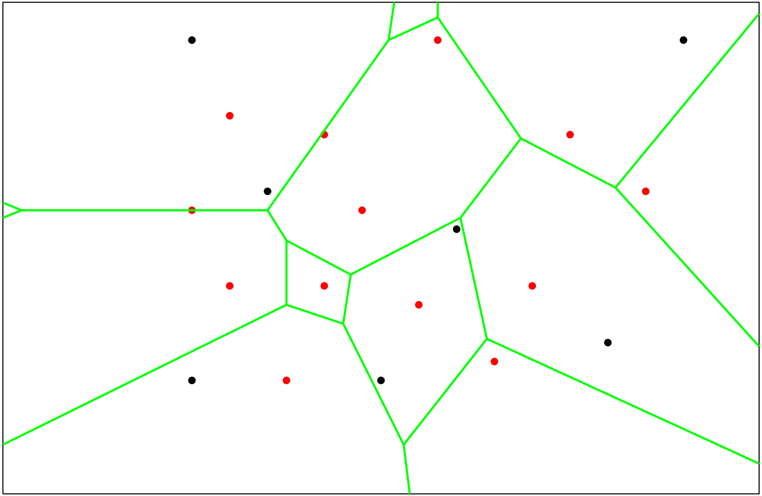

Size of k^{th}-order Voronoi diagram

Sub-level sets of $d_{\mu,m}$ are unions of balls.

$$d_{\mu,m}^{-1}(]-\infty,\alpha]) = \bigcup_{b \in B} \bar{B}(b, \sqrt{\alpha^2 - w_b^2})$$

Theorem (Clarkson, Shor, 1989)

The number of non-empty cells can be as large as:

$$O\left(n^{\lfloor \frac{d+1}{2} \rfloor} k^{\lceil \frac{d+1}{2} \rceil}\right).$$

Computation of persistence diagrams

Distance to the empirical measure and weighted Rips filtration:

- Complexity: $O(n^{\left\lfloor \frac{3(d+1)}{2} \right\rfloor} k^{\left\lceil \frac{d+1}{2} \right\rceil})$
- Robustness to outliers: yes

Witnessed k-distance

Approximation by sampling barycentres.

Definition (Guibas, Mérigot, Morozov, 2011)

$$d^{W}_{\mu,m}(x) = \min_{p \in P} \sqrt{||x - bar(p)||^2 + w^2_{bar(p)}}$$

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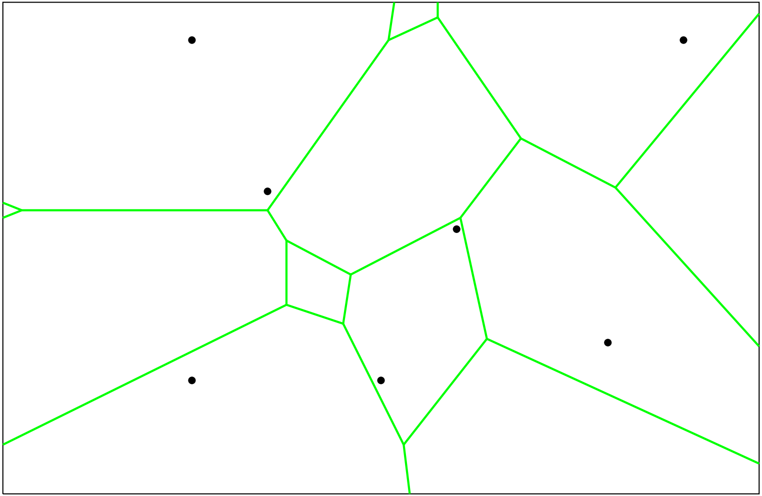
$$d_{\mu,m} \leq d^W_{\mu,m} \leq \sqrt{6} d_{\mu,m}$$

Building the witnessed k-distance



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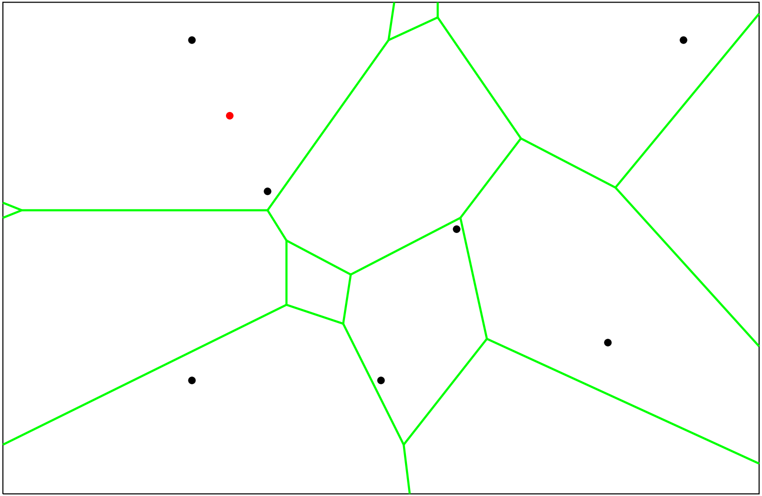

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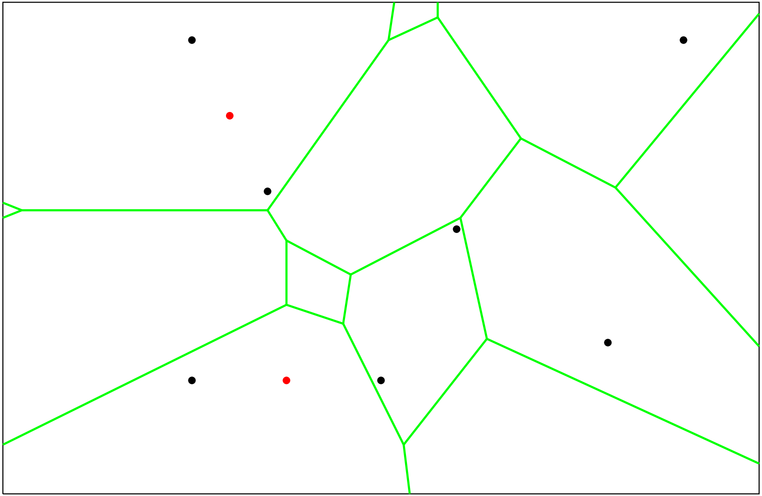

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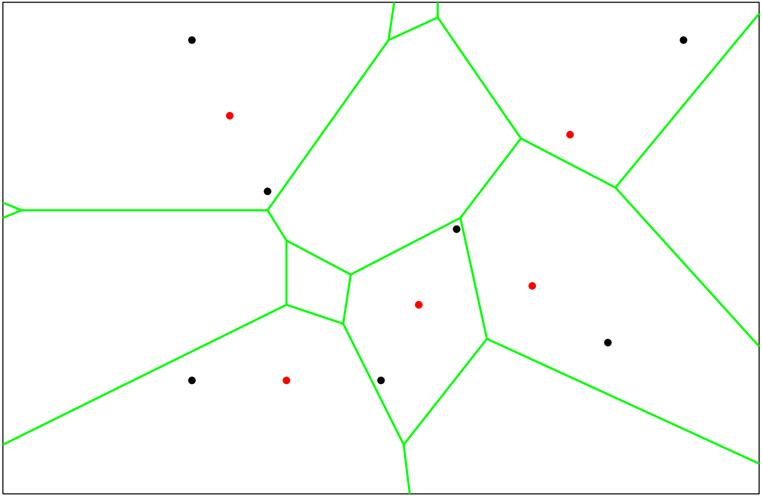

Approximation supported by the points

Using a power distance supported by input points.

Definition

$$d^{P}_{\mu,m}(x) = \min_{p \in P} \sqrt{d_{\mathbb{X}}(x,p)^2 + d_{\mu,m}(p)^2}$$

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Theorem

In Euclidean space:

$$\frac{1}{\sqrt{2}} d_{\mu,m} \leq d^P_{\mu,m} \leq \sqrt{3} d_{\mu,m}$$

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In any metric space:

$$\frac{1}{\sqrt{2}} d_{\mu,m} \leq d^P_{\mu,m} \leq \sqrt{5} d_{\mu,m}$$

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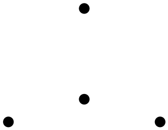
Approximation to the distance to the empirical measure and weighted Rips filtration:

- Complexity: $O(n^{d+1})$
- Robustness to outliers: yes

Sparse Rips



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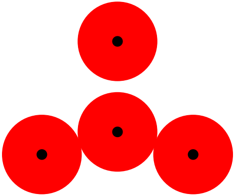

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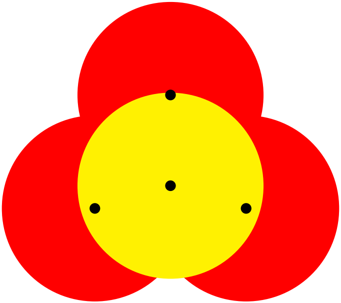

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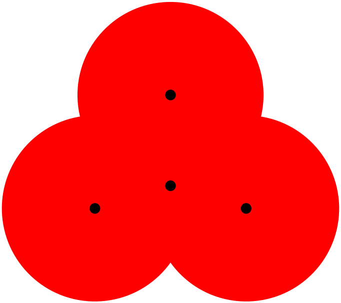
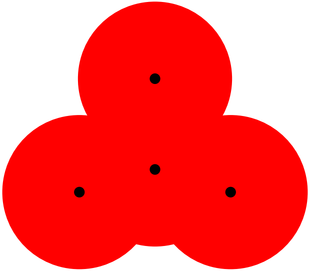


Topological noise

Naively removing points can create topological noise.



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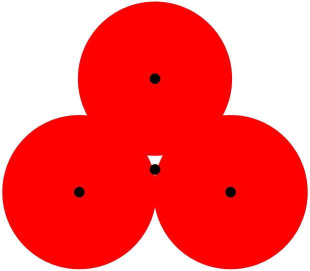

Construction (I)

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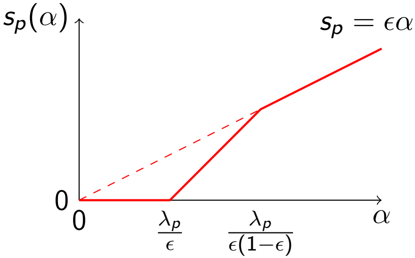
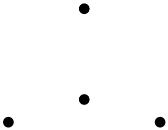
- Fix $\bar{N}_\gamma = \{p \in P | \lambda_p \geq \gamma\}$.
- Perturbated metric : $f_\alpha(p,q) = d_{\mathbb{X}}(p,q) + s_p(\alpha) + s_q(\alpha)$.



Effects of perturbated metric



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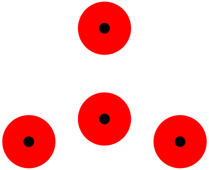

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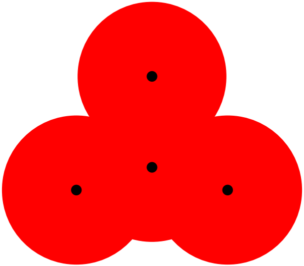

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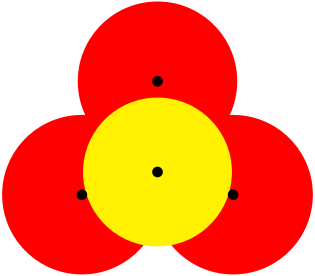

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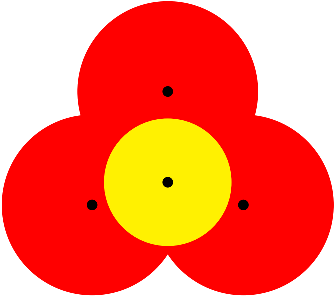

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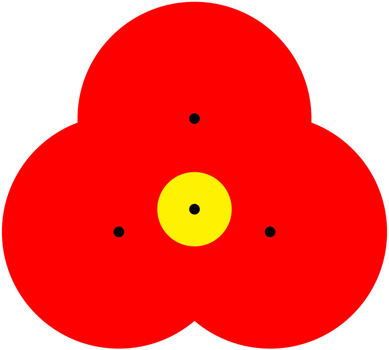

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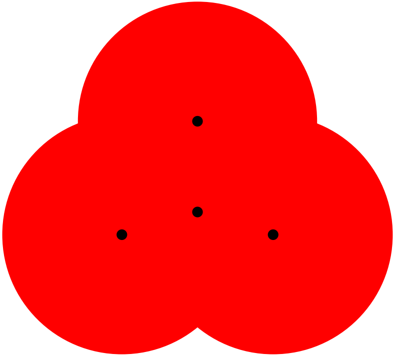

Construction (II)

Definition

The sparse Rips complex is given by:

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Construction (II)

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Definition

The sparse Rips filtration is given by:

$$S_\beta = \bigcup_{\alpha \leq \beta} Q_\alpha.$$

Properties of sparse Rips

Theorem

$\{S_\alpha\}$ contains $O(C^l n)$ simplexes where l is the intrinsic dimension of the underlying object.

Theorem

$\{S_\alpha\}$ is $\frac{1}{1-\epsilon}$-interleaved with the Rips filtration $\{R_\alpha\}$.

Using the weighted metric

We used a weighted Rips filtration to compute the persistence diagram:

$$R_\alpha = \left\{\sigma \subset P | \forall p, q \in P,\ d_{\mathbb{X}}(p, q) \leq \sqrt{\alpha^2 - w_p^2} + \sqrt{\alpha^2 - w_q^2}\right\}$$

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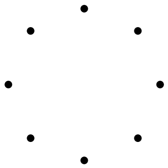

Adapting to weighted Rips

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The sparse weighted RIps is defined by:

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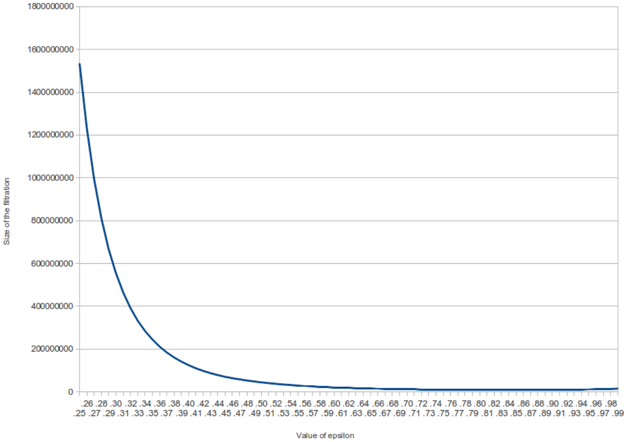
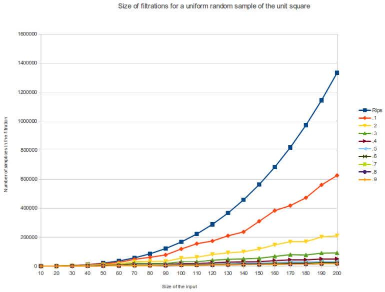
Theorem

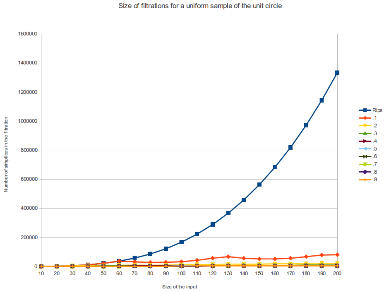
R_α and T_α are κ-interleaved where $\kappa = 1 + \frac{\sqrt{1+t^2\epsilon}}{1-\epsilon}$, id est :

SIze of the filtration depending on ϵ

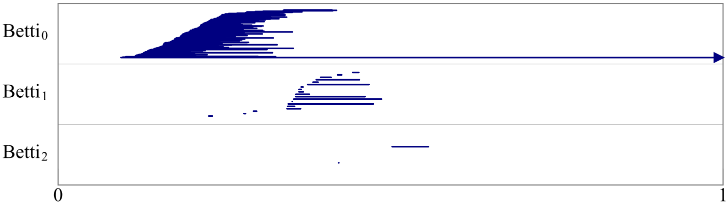


Intrinsic dimension influence





Cube skeleton with Gaussian noise



Take home

- The distance to a measure can handle outliers.

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- The distance to a measure can handle outliers.
- The sparse wieghted Rips filtration and the approximation to the distance to a measure make computing its peristence diagram tractable.

Efficient and Robust Persistent Homology for Measures

Mickaël Buchet

joint work with F. Chazal, S. Oudot and D. Sheehy