Helly, Betti, Ramsey...

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joint work with

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Martin Tancer (Charles University)

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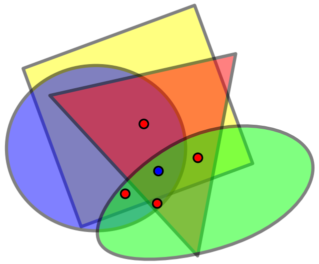
http://arxiv.org/abs/1310.4613

Gaspard - Monge Laboratoire d'informatique

Helly's Theorem. Any finite family of convex sets in $\mathbb{R}^d$ has non-empty intersection if any $d+1$ elements have non-empty intersection.



Classical result in convex geometry

Related to Radon and Caratheodory's theorems...

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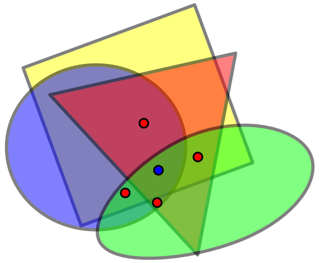

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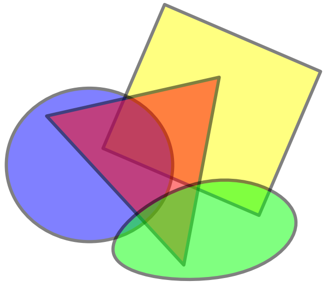
In the contrapositive:

If finitely many convex sets in $\mathbb{R}^d$ have empty intersection, some d of them have empty intersection.



Statement about size of witnesses for empty intersection

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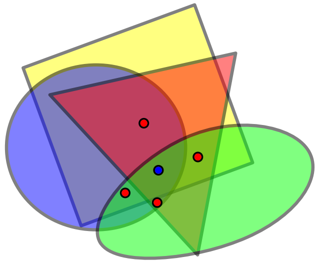

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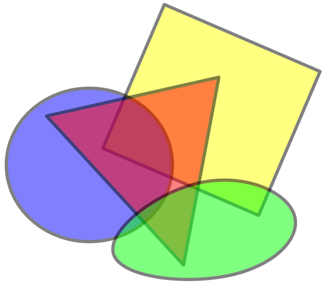
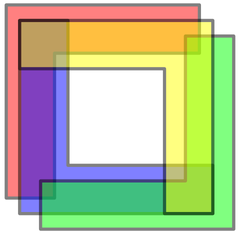


Statement about size of witnesses for empty intersection

Family-based rather than class-based formulation:

The Helly number of a family $\mathcal{F}$ of sets is the maximum size of an inclusion-minimum sub-family of $\mathcal{F}$ with empty intersection.



We implicitly assume that $\mathcal{F}$ has empty intersection

$$\text{Helly}(\mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F}, \cap\mathcal{G} = \emptyset, \forall A \in \mathcal{G}, \cap(\mathcal{G} \setminus \{A\}) \neq \emptyset\}$$

Helly's Theorem. If $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ then $\text{Helly}(\mathcal{F}) \leq d+1$.

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Helly numbers $\supset$ blockers in reconstruction, basis size in combinatorial optimization...

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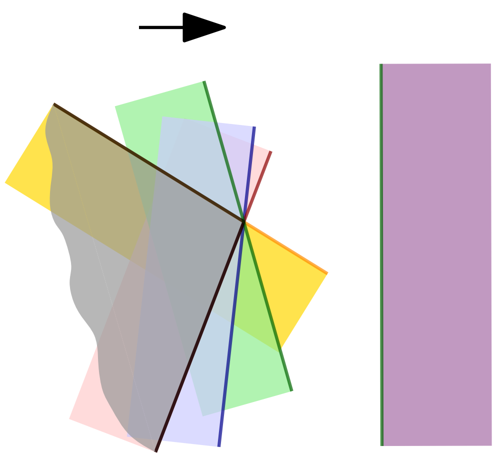
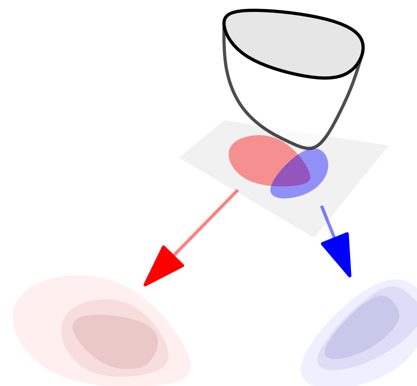
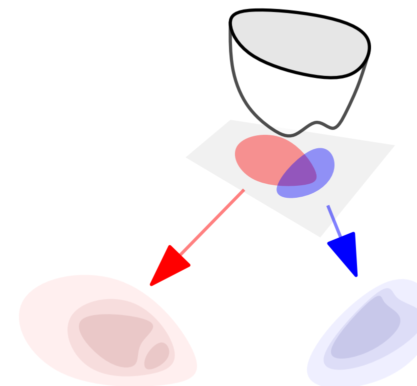


Linear programming



Convex programming



Generalized linear programming

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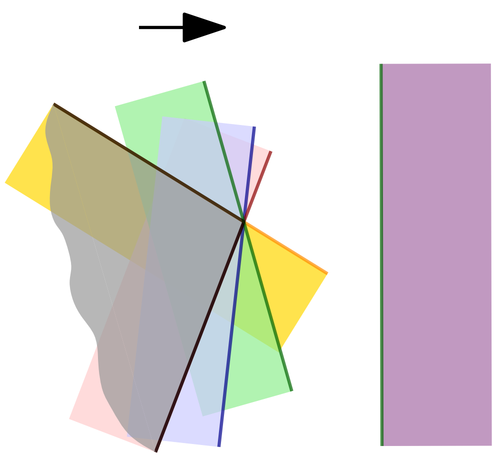

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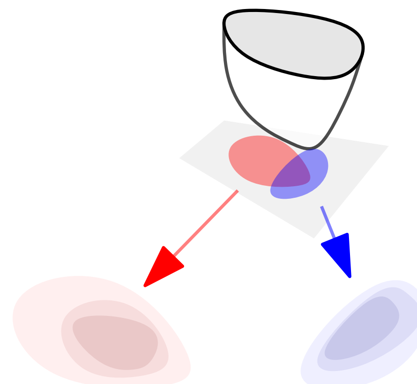

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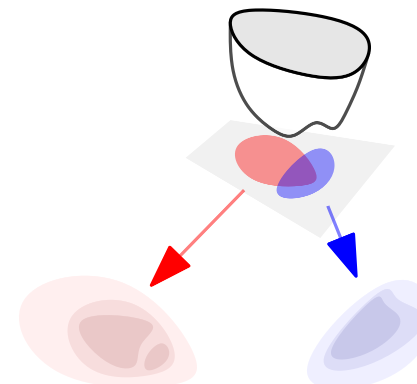
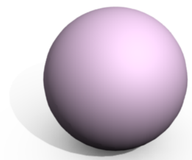


Generalized linear programming

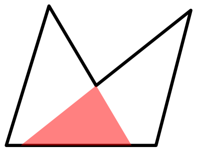
Which families of sets have bounded Helly numbers? What are these bounds?

A whole industry of bounds on Helly numbers (a.k.a "Helly-type theorems").



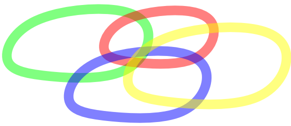
convex sets in $\mathbb{S}^d$

d+2



Star-shapness in the plane

3 [Breen 1985]



Homothets of a convex curve in $\mathbb{R}^2$

4 [Swanepoel 2003]

Convexity spaces [Kolodziejczyk 1991], Matroids [Edmonds 2001], ...

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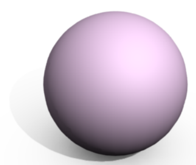

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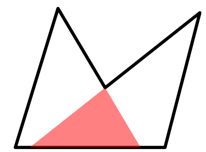

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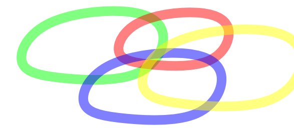

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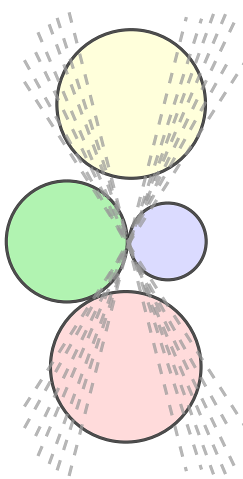
A line transversal to a family is a line that intersects each of its members.

Helly numbers of sets of line transversals to



disjoint unit disks in $\mathbb{R}^2$: ≤ 5 [Danzer 1957]

disjoint translates of a convex figure in $\mathbb{R}^2$: ≤ 5 [Tverberg 1989]

disjoint translates of a convex polyhedron in $\mathbb{R}^3$: unbounded [Holmsen-Matoušek 2004]

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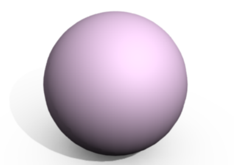

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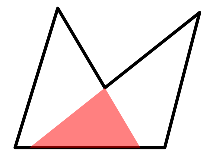

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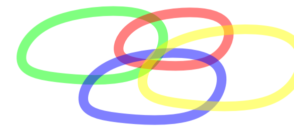

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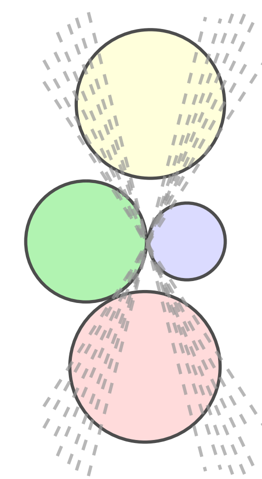
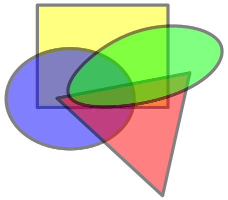


- disjoint unit disks in $\mathbb{R}^2$: ≤ 5 [Danzer 1957]
- disjoint translates of a convex figure in $\mathbb{R}^2$: ≤ 5 [Tverberg 1989] (conj. 1960)
- disjoint translates of a convex polyhedron in $\mathbb{R}^3$: unbounded [Holmsen-Matoušek 2004]
- disjoint unit balls in $\mathbb{R}^d$: $\leq 4d-1$ [Cheong-Holmsen-G-Petitjean 2006] (conj. 1957)

Proofs are technical and somewhat ad hoc.

What systematic conditions could explain these bounds?



Convex sets in $\mathbb{R}^d$

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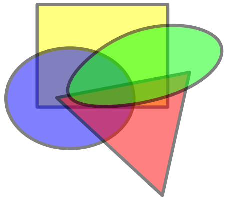

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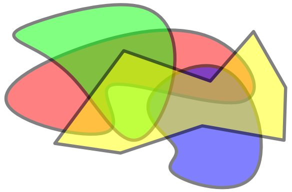

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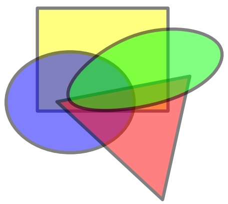

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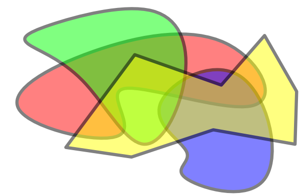

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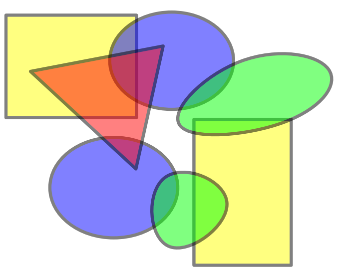
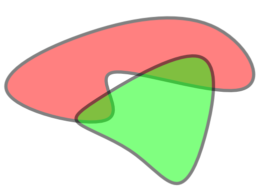


The $\cap$ of any subfamily is a $\cup$ of $\leq r$ disjoint convex sets in $\mathbb{R}^d$

$r(d+1)$, [Amenta 1996]



Subsets of $\mathbb{R}^d$ whose $\cap$ have $\leq r$ connected components, each $(\lceil d/2 \rceil - 1)$-connected.

some fct of r and d [Matoušek 1996]

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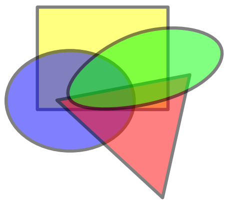

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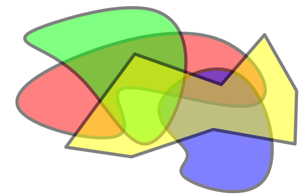

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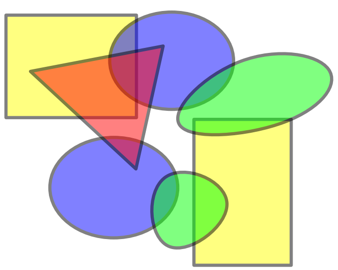

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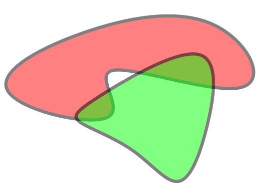

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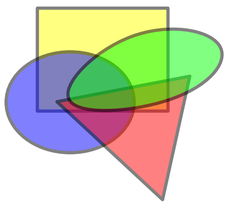

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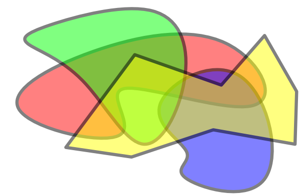

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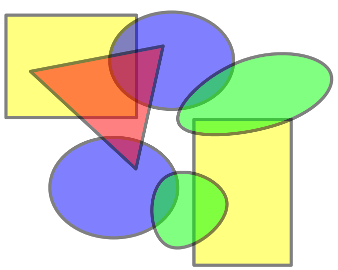

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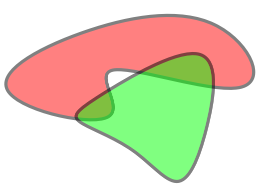

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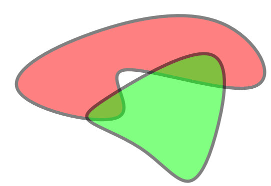

In "reasonable" topological spaces:

$\cap$ of any subfamily has $\leq r$ connected components, each homologically trivial

$r(d_\Gamma + 1)$ [Colin de Verdière-Ginot-G 2012]

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New insight

In "reasonable" d-dimensional spaces:

$\cap$ of any subfamily has
reduced $\mathbb{Z}_2$-Betti numbers $\leq r$ $\Rightarrow$ Helly $\leq$ some Ramsey number of r and d
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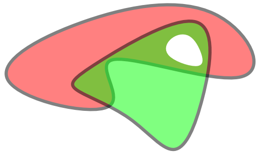

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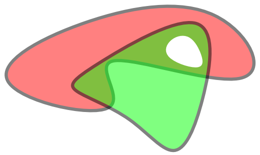

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X is "reasonable" $\Leftrightarrow$ the $\lceil d/2 \rceil$-skeleton of a k-simplex does not embed in X for k large enough.

Any d-dimensional manifold with finite $\lceil d/2 \rceil$th Betti number is "reasonable".
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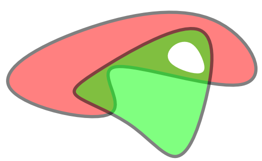

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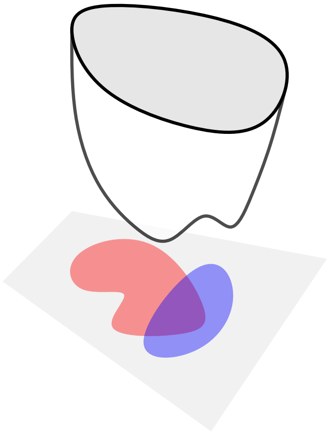
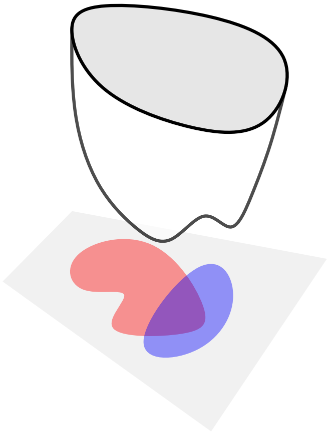
The condition is sharp: allowing one unbounded Betti number in that range allows unbounded Helly numbers.

A consequence on the complexity of optimization problems



$\min_{\cap_i C_i} f$

where $f : \mathbb{R}^d \to \mathbb{R}$ and $C_1, C_2, \ldots, C_n$ subsets of $\mathbb{R}^d$



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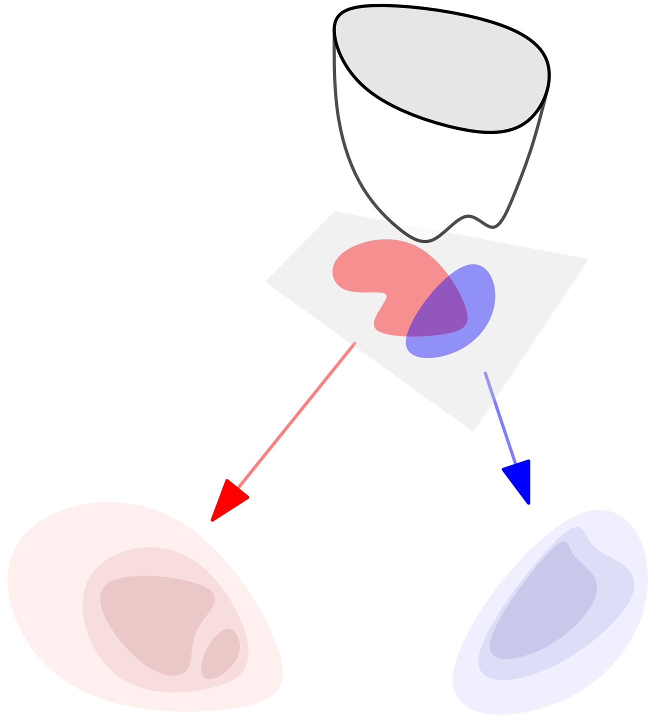

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If the maximum Helly number of the sets $\{C_1, C_2, \ldots, C_n, f^{-1}(]-\infty, t])\}$ is some constant h independent of n then there exist $i_1, i_2, \ldots, i_h$ such that

$$\min_{\cap_i C_i} f = \min_{\cap_{j=1}^h C_{i_j}} f$$

and they can be computed "efficiently".

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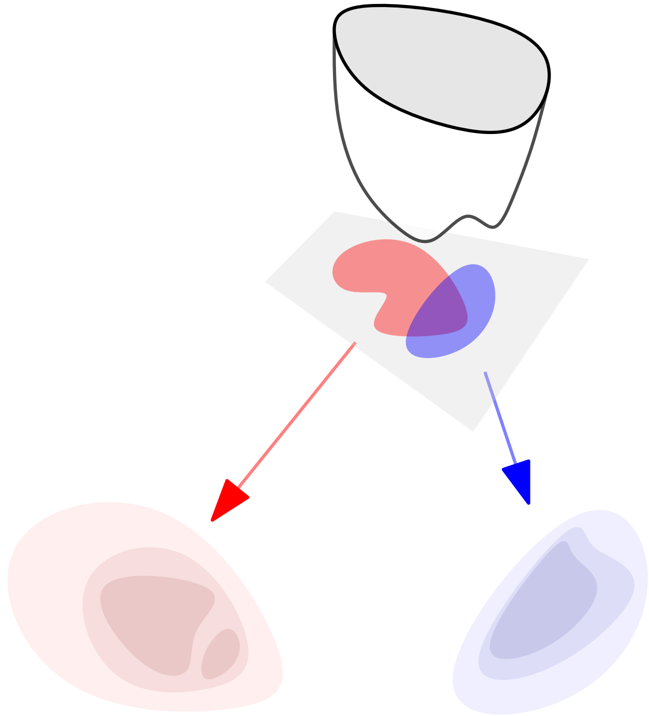

$$\min_{\cap_i C_i} f$$

where $f : \mathbb{R}^d \to \mathbb{R}$ and $C_1, C_2, \ldots, C_n$ subsets of $\mathbb{R}^d$

This problem has a GLP formulation if the local minima of f have distinct values.

If the maximum Helly number of the sets $\{C_1, C_2, \ldots, C_n, f^{-1}(]-\infty, t])\}$ is some constant h independent of n then there exist $i_1, i_2, \ldots, i_h$ such that

$$\min_{\cap_i C_i} f = \min_{\cap_{j=1}^h C_{i_j}} f$$

and they can be computed "efficiently".

"hard problems" $\simeq (\cap C_i) \cap f^{-1}(]-\infty, t])$ of "unbounded topological complexity".

1. Bounds on Helly numbers arise from non-embeddability

2. Ramsey's theorem helps finding non-embeddable structures

3. Non-embeddability can be argued at the level of chain maps

$\tilde{\beta}_i(\cap\mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$	$\Rightarrow$	$\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b

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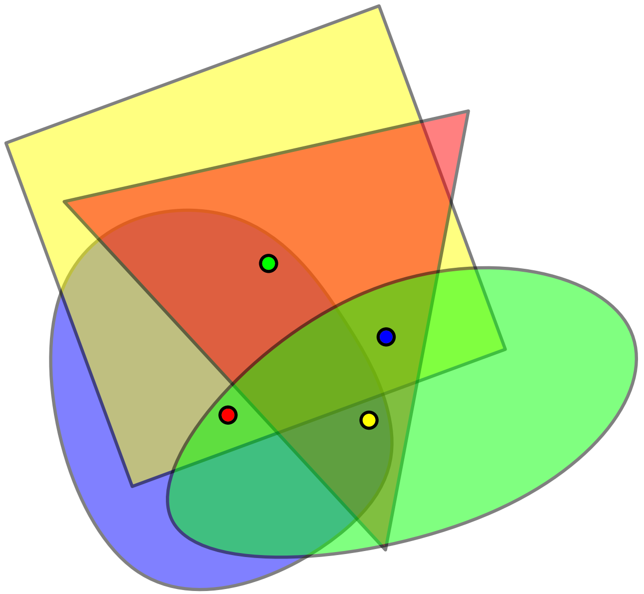

Helly from Radon

Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ be convex sets in $\mathbb{R}^d$ such that $k \geq d + 2$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

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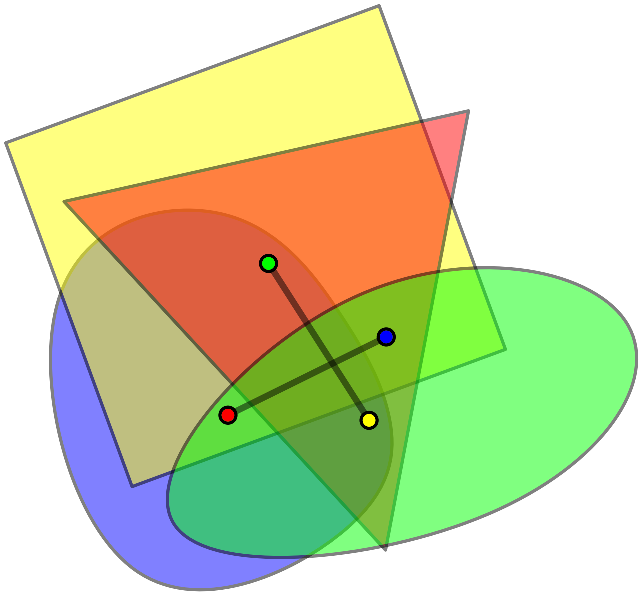

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$h \in (\cap_{i: p_i \notin X} A_i) \cap (\cap_{i: p_i \notin Y} A_i) = \cap \mathcal{F}$

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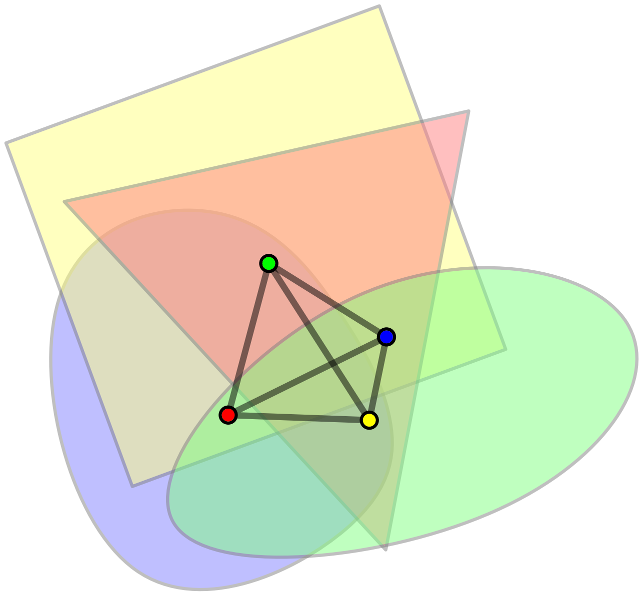

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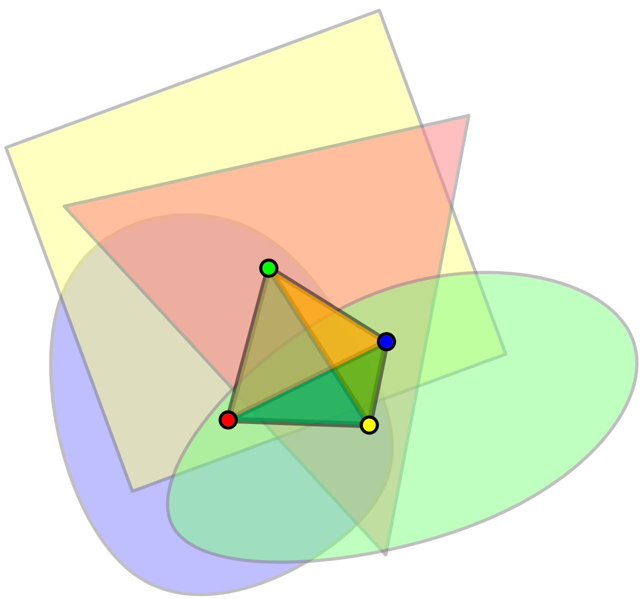

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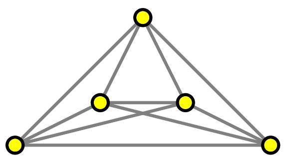
Non-planarity of $K_5 \Rightarrow$ Helly number for path-connected intersections in $\mathbb{R}^2$.

$\Delta_n^{(1)} \not\hookrightarrow \mathbb{R}^2$ for $n \geq 5$



Corollary. If $\mathcal{F}$ is a family of sets in $\mathbb{R}^2$ such that the intersection of any subfamily is empty or path-connected then $\mathrm{Helly}(\mathcal{F}) \leq 4$.

Proof: Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ such that $k \geq 5$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

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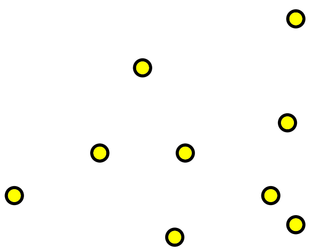

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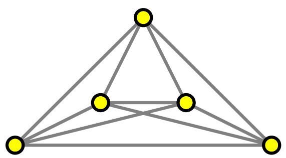

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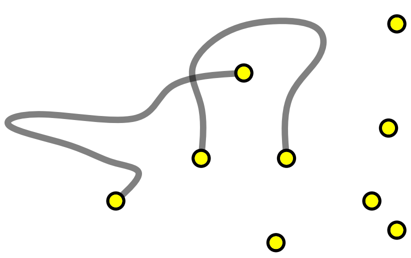

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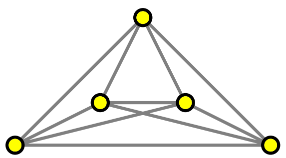

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Two edges $p_a p_b$ and $p_u p_v$ cross, with $\{a, b\} \cap \{u, v\} = \emptyset$.

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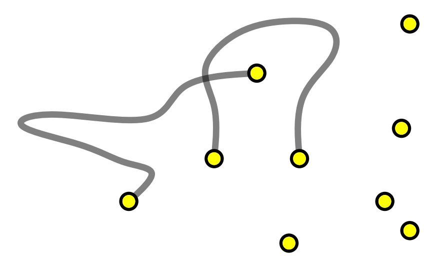

$\not\hookrightarrow$	Assumption on nonempty intersections	Bound on the Helly number
Topological Radon: $\Delta_{d+1}^{(d)} \not\hookrightarrow \mathbb{R}^d$ [Bajmóczy-Bárány 1979]	d-connected	$d+1$
$\Delta_{2\lceil d/2\rceil+2}^{(\lceil d/2\rceil)} \not\hookrightarrow \mathbb{R}^d$ [Van Kampen 1931, Flores 1932]	$\lceil d/2\rceil$-connected	$2\lceil d/2\rceil+2$

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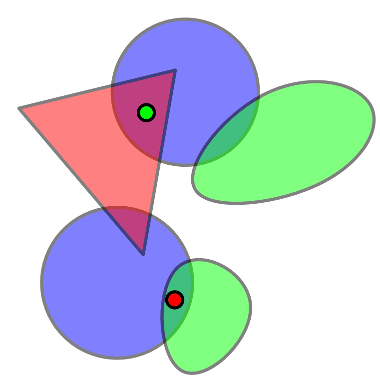
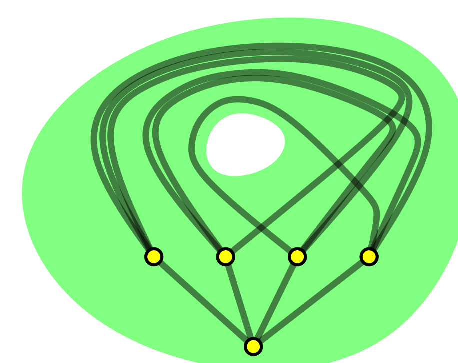
Can we allow some disconnection?





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Ramsey's theorem. For any x, y and z there exists $R = R_{x,y,z} \in \mathbb{N}$ such that any coloring of the complete x-uniform hypergraph on at least R vertices by y colors contains z vertices inducing a monochromatic sub-hypergraph.

complete x-uniform hypergraph: all subsets of size x of a finite set

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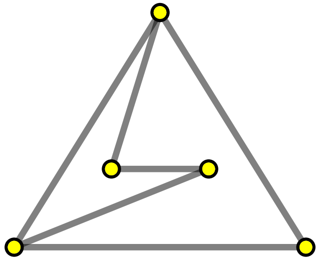
Lemma. Let G be a graph on n vertices where any 3 vertices span at least one edge. If $n \geq R_{3,3,9}$ then G contains K_5 as an induced subgraph.



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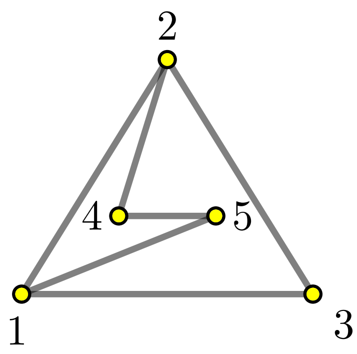

Proof: Number the vertices $1, 2, \ldots, n$.

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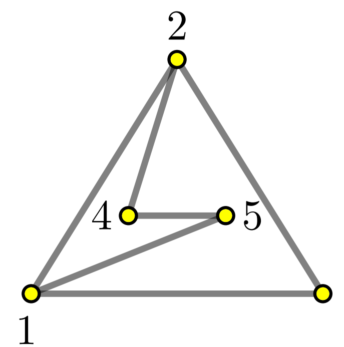

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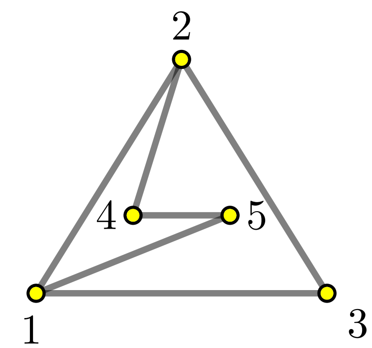
This colors the complete 3-uniform hypergraph by $\{1,2\}, \{1,3\}$ and $\{2,3\}$.

For $n \geq R_{3,3,9}$ some 9 vertices span triples all colored by the same pair $\{a, b\}$.



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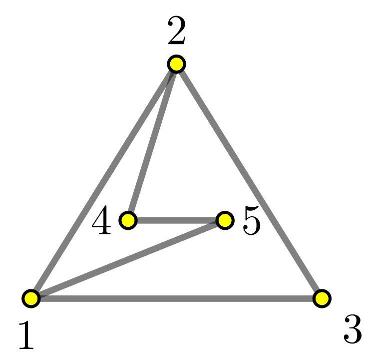
If $\{a, b\} = \{1, 2\}$ then the vertices with rank $\{1, 2, 3, 4, 5\}$ span a K_5.

$\cdots$ $\{2, 3\}$ $\cdots$ $\{2, 3, 4, 5, 6\}$ $\cdots$

$\cdots$ $\{1, 3\}$ $\cdots$ $\{1, 3, 5, 7, 9\}$ $\cdots$ □



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In the graph that was drawn, 5 vertices must span a complete graph.

The intersection point of these edges lies in $\cap \mathcal{F}$. $\square$

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We actually proved:

Lemma. Let G be a graph on n vertices where any 3 vertices span at least one edge. If $n \geq R_{3,3,9}$ then G contains 5 vertices such that for any two there exists a triple in which they span an edge.

We need a stronger statement where triples use different "dummy" vertices

The same idea works in higher dimension using that $\Delta_{2\lceil d/2\rceil+2}^{(\lceil d/2\rceil)} \not\hookrightarrow \mathbb{R}^d$.

Assuming intersections are k-connected, each "constrained" drawing of K_n extends into a "constrained" drawing of $\Delta_{n-1}^{(k)}$.

Every $p_{i_u}p_{i_v}$ is drawn in $\cap_{i\neq i_u,i_v,i_{u,v}} A_i$

Every $p_{i_u}p_{i_v}p_w$ is drawn in $\cap_{i\neq i_u,i_v,i_{u,v},i_w,i_{u,w},i_{v,w}} A_i$, etc...

Vertex-disjoint faces are drawn missing disjoint sets of A_i's

$\Rightarrow$ If $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily has at most 2 connected components, each $(\lceil d/2\rceil - 1)$-connected, then $\mathrm{Helly}(\mathcal{F}) \leq f(d)$.

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Vertex-disjoint faces are drawn missing disjoint sets of A_i's

$\Rightarrow$ If $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily has at most 2 connected components, each $(\lceil d/2\rceil - 1)$-connected, then $\mathrm{Helly}(\mathcal{F}) \leq f(d)$.

This was essentially the proof of:

Theorem. [Matoušek 1996] If $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily has at most r connected components, each $\lceil d/2\rceil$-connected, then $\mathrm{Helly}(\mathcal{F}) \leq f(r,d)$.

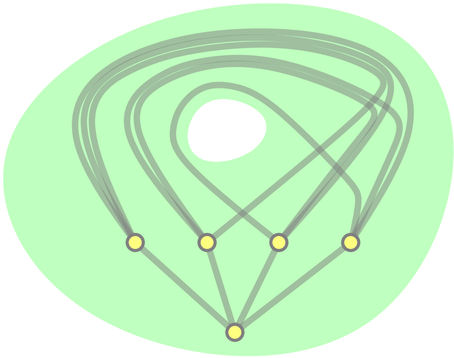
Uses a generalization of the "selection trick"

1. Bounds on Helly numbers arise from non-embeddability

2. Ramsey's theorem helps finding non-embeddable structures

3. Non-embeddability can be argued at the level of chain maps

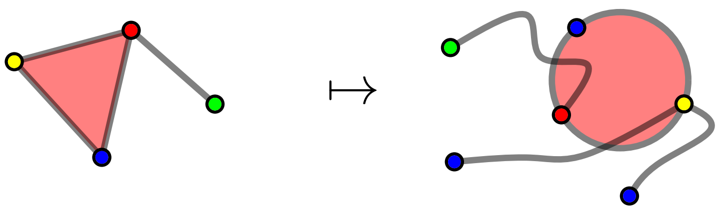
$\tilde{\beta}_i(\cap \mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$ $\Rightarrow$ $\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b



A chain map is non-trivial if it maps every vertex to a sum of an odd number of points.

A continuous map $f : |K| \to \mathbb{R}^d$ induces a non-trivial chain map $f_\sharp : C_*(K) \to C_*(\mathbb{R}^d)$.

A chain map is an homological almost embedding if it is non-trivial and for disjoint simplices $\sigma, \tau \in K$, $\gamma(\sigma)$ and $\gamma(\tau)$ have disjoint supports.



If f is an almost-embedding then $f_\sharp$ is an homological almost embedding.

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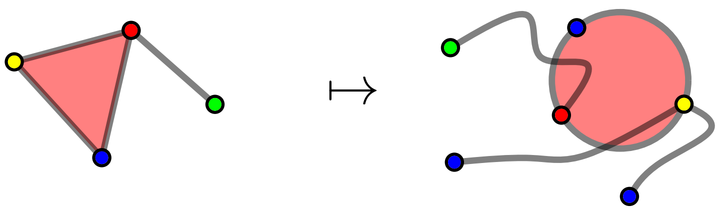

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Theorem. There is no homological almost embedding from $C_*\left(\Delta_{d+1}^{(d)}\right)$ or from $C_*\left(\Delta_{d+2}^{(\lceil d/2 \rceil)}\right)$ into $C_*(\mathbb{R}^d)$.

Homological versions of the Radon and Van Kampen-Flores theorems

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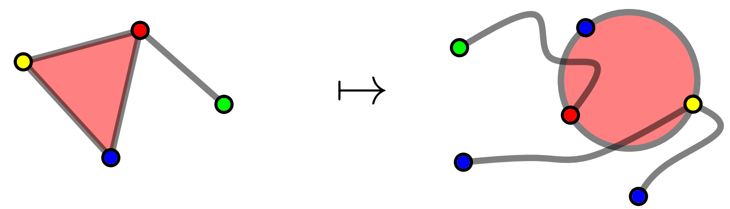

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Homological versions of the Radon and Van Kampen-Flores theorems

Proof shows that the Van Kampen obstruction to embeddability into $\mathbb{R}^d$ also forbids homological almost embeddings.

Technique: adapt the classical proof...

$\mathbb{Z}_2$ spaces, equivariant maps, deleted products, Gauss map, Van Kampen obstruction

... using equivariant chain homotopy [Wagner 2011]

We can repeat the previous homotopic arguments in a homological language:

Corollary. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. If for any $\mathcal{G} \subseteq \mathcal{F}$, $\cap\mathcal{G}$ is empty or has $\tilde{\beta}_i(\cap\mathcal{G}, \mathbb{Z}_2) = 0$ for $i = 0, 1, \ldots, \lceil d/2 \rceil - 1$ then $\mathrm{Helly}(\mathcal{F}) \leq d + 2$.

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Proof: Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ such that $k \geq d + 3$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

Construct a non-trivial chain map $\gamma : C_* \left(\Delta_{d+2}^{(\lceil d/2 \rceil)}\right) \to C_*(\mathbb{R}^d)$ "constrained by $\mathcal{F}$".

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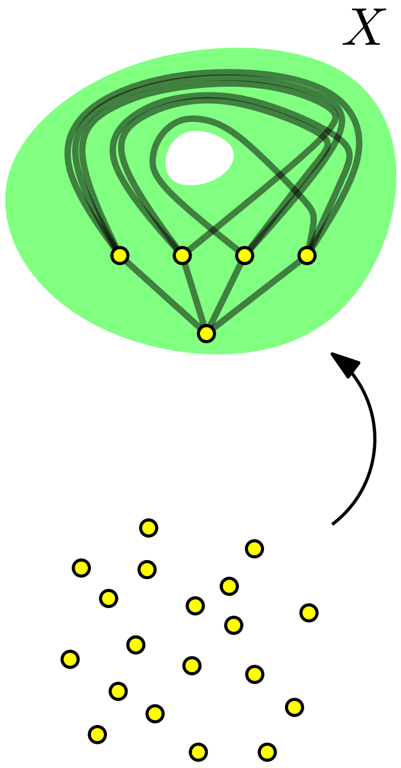
What more can we do?

Consider a chain map $\gamma : C_*(K_n) \to C_*(X)$ where X is an annulus.

X has two $\mathbb{Z}_2$-homology class in dimension 1.



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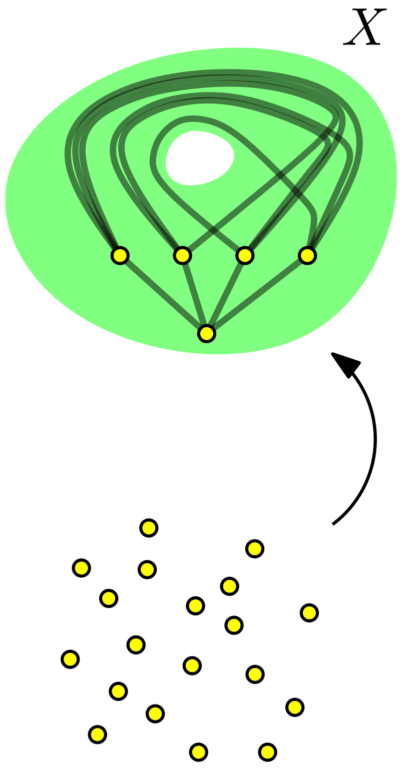
Pick four vertices $v_1, v_2, v_3, v_4 \in K_n$

$$\partial v_1 v_2 v_3 + \partial v_1 v_2 v_4 = v_1 v_3 + v_3 v_2 + v_2 v_4 + v_4 v_1$$

so one of these three cycles is a boundary.



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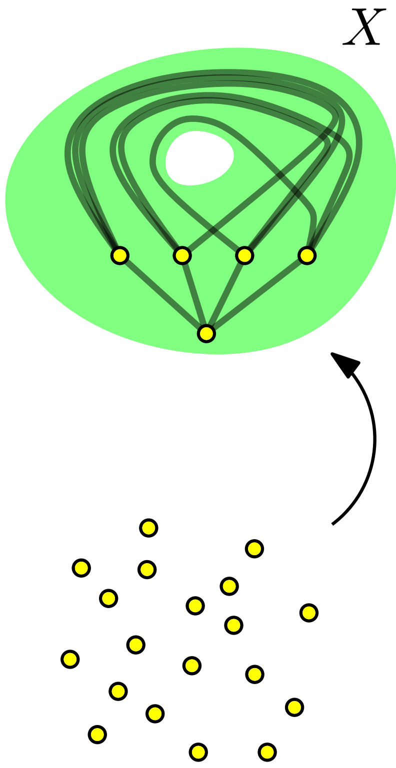
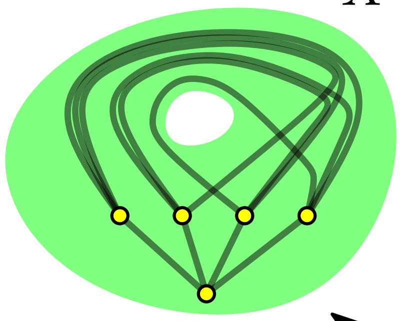


Lemma. Let $f : C_*(K_n) \to C_*(X)$ be a chain map and let $s \in \mathbb{N}$. For n large enough there exists a PL-embedding $g : K_s \to K_n$ such that for any $u, v, w \in K_s$, $f \circ g_\sharp(\partial uvw)$ is a boundary.



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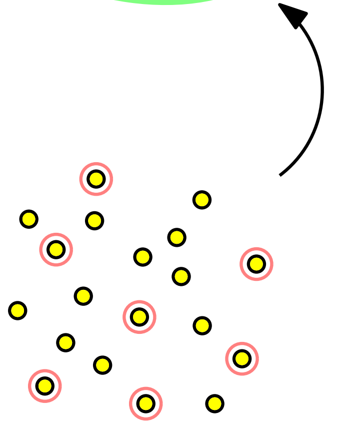

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Proof: Color every triangle xyz of K_n by the homology class of $\gamma(\partial xyz)$ in X. Use Ramsey's theorem to find t vertices so that all triangles have the same homology class under γ.

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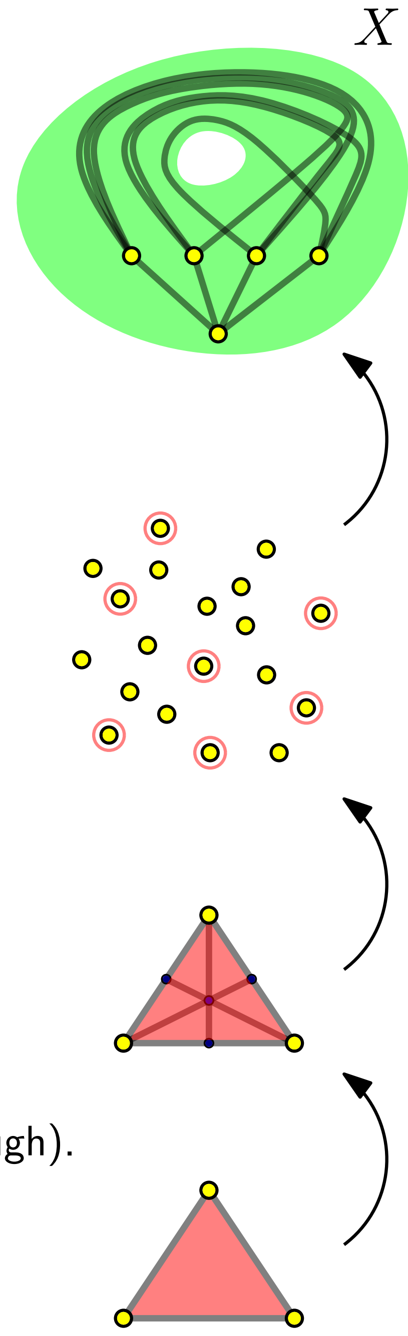
Then map the 1-skeleton of sd $\Delta^{(2)}_{s-1}$ to these t vertices (assuming t is large enough).



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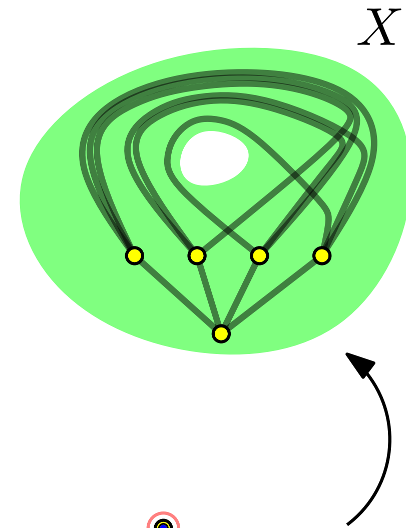

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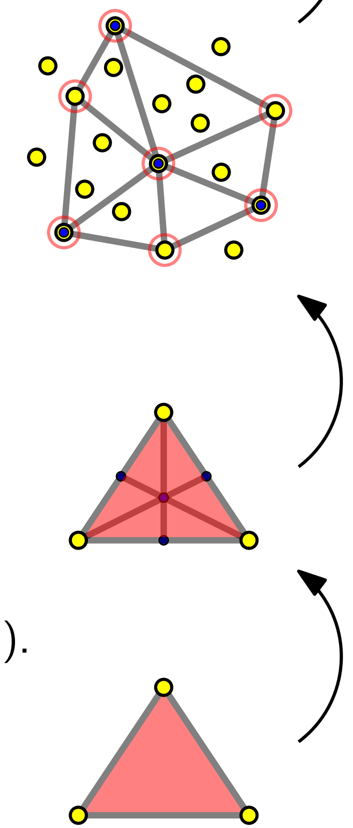

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Then map the 1-skeleton of sd $\Delta^{(2)}_{s-1}$ to these t vertices (assuming t is large enough).
Every triangle in K_s is the sum of 6 triangles in sd K_s.
A sum of an even number of times the same homology class is a $\mathbb{Z}_2$-boundary. $\square$

Applies in any dimension, provided the number of $\mathbb{Z}_2$-homology classes of the target space is bounded.

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5. Ramsey's theorem helps finding non-embeddable structures

6. Non-embeddability can be argued at the level of chain maps

$\tilde{\beta}_i(\cap \mathcal{G}) \le b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \le \lceil d/2 \rceil - 1$ $\Rightarrow$ $\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b

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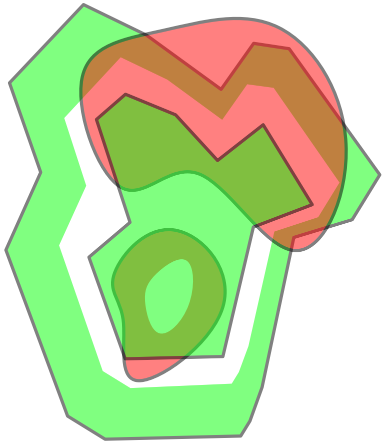

Assume that $\forall \mathcal{G} \subseteq \mathcal{F}$, $\cap \mathcal{G}$ has at most r connected components and $\tilde{\beta}_1(\cap \mathcal{G}, \mathbb{Z}_2) \leq r$

Goal: build a chain map from $C_*\left(\Delta_7^{(2)}\right)$ into $C_*(\mathbb{R}^d)$ such that disjoint faces σ, τ are mapped to chains supported in $\cap_{i \notin \Phi(\sigma)} A_i$ and $\cap_{i \notin \Phi(\tau)} A_i$ with $\Phi(\sigma) \cap \Phi(\tau) = \emptyset$

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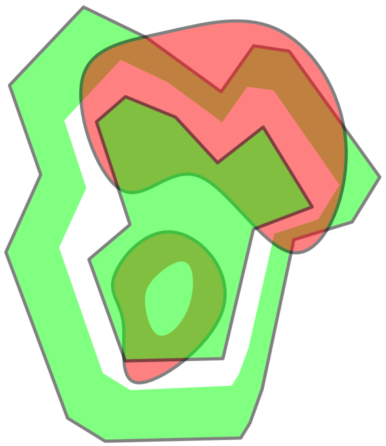

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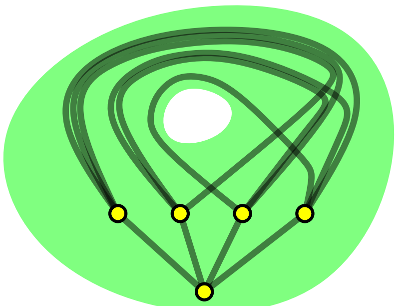

Ramsey $\rightsquigarrow T \subset [k]$ such that the positions of P_J for all $J \subset T$ are identical

Injection lemma $\rightsquigarrow$ a chain map $\gamma_1 : C_*(K_n) \to C_*\left(\cap_{i \notin T} A_i\right)$

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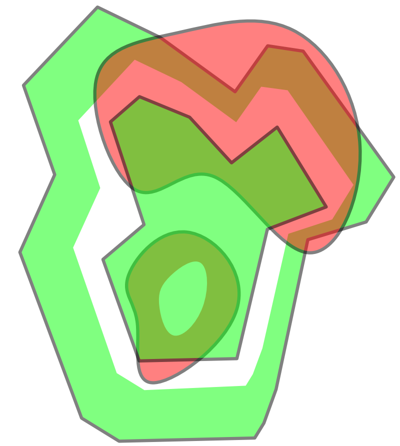

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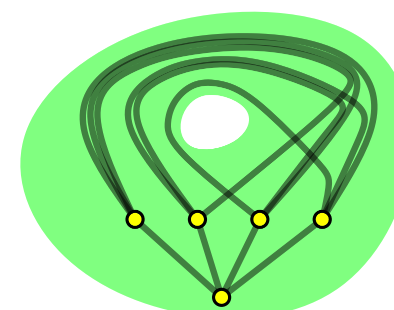

Ramsey $\rightsquigarrow$ $T \subset [k]$ such that the positions of P_J for all $J \subset T$ are identical

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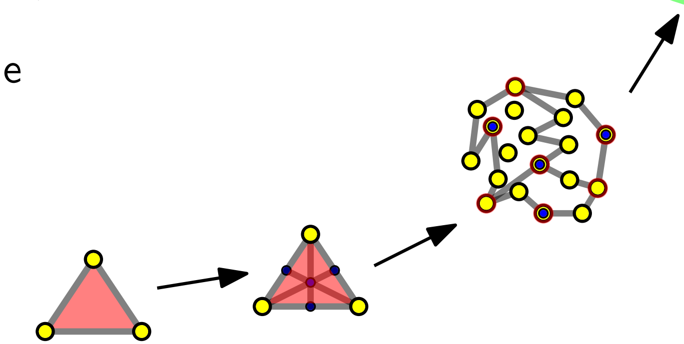
Filling lemma $\rightsquigarrow$ $\gamma_1' : C_*(K_s) \to C_*\left(\cap_{i \notin T} A_i\right)$ such that the image of every triangle is a boundary in $\cap_{i \notin T} A_i$.

We can define $\gamma_2 : C_*(\Delta_{s-1}^{(2)}) \to C_*\left(\cap_{i \notin T} A_i\right)$



Filling lemma. Let $f : C_*(K_n) \to C_*(X)$ be a chain map and let $s \in \mathbb{N}$. For n large enough there exists a PL-embedding $g : K_s \to K_n$ such that for any $u, v, w \in K_s$, $f \circ g_\sharp(\partial uvw)$ is a boundary.

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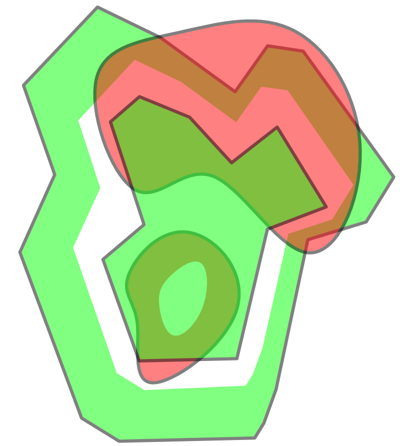

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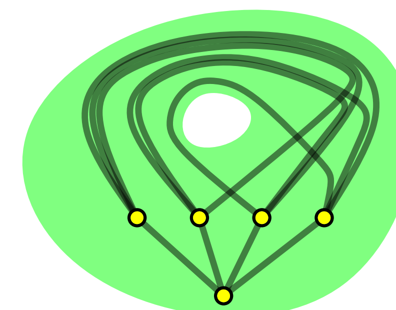

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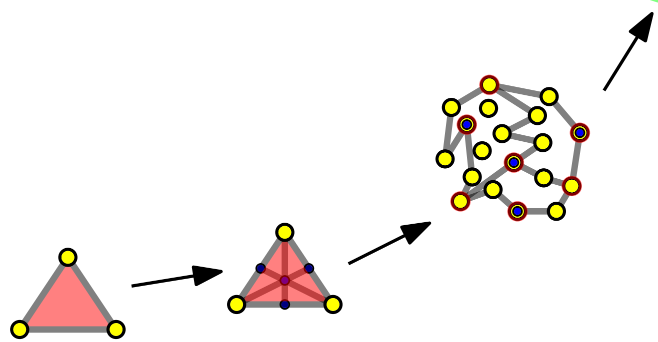
We can define $\gamma_2 : C_*(\Delta_{s-1}^{(2)}) \to C_*\left(\cap_{i \notin T} A_i\right)$



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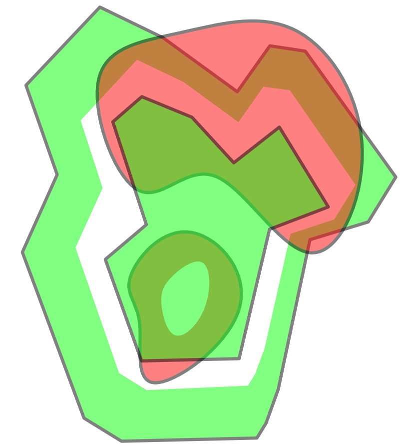

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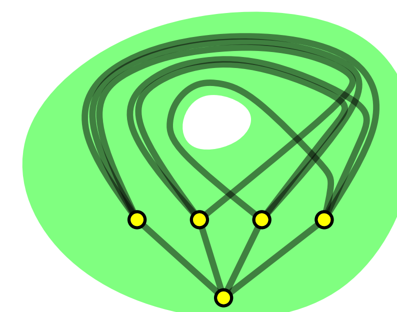
For any $J \subset [k]$ of size $r+1$ there is a pair $P_J = \{u, v\} \subset J$ such that $\gamma(u) + \gamma(v)$ is a boundary in $\cap_{i \notin J} A_i$



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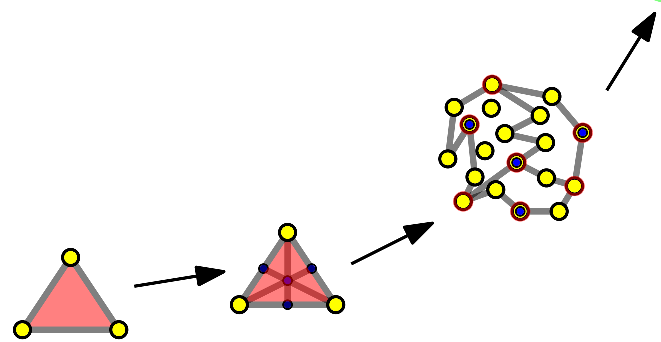

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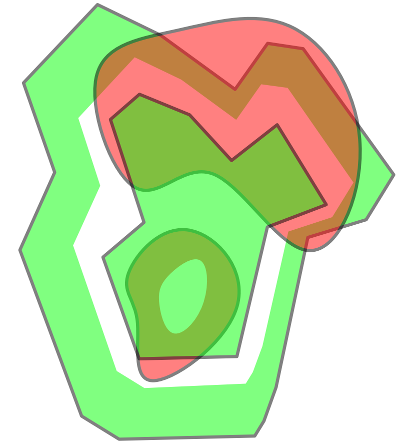

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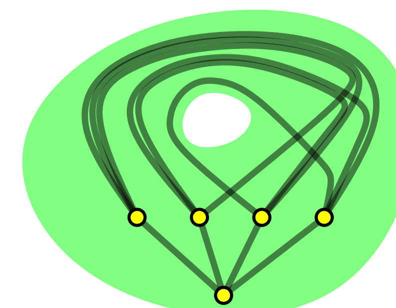

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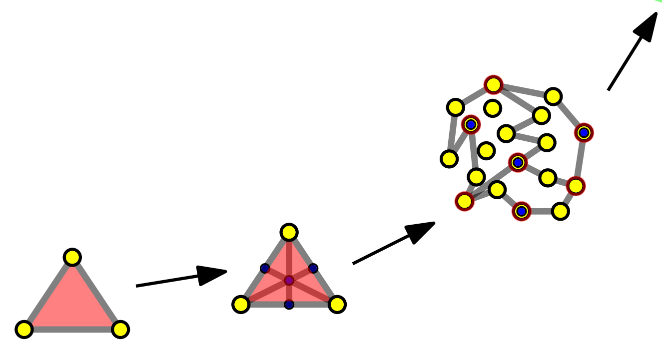

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Pigeonhole $\rightsquigarrow$ any $(r+1)$-elements subset $J \subseteq [k]$ has a pair of points that forms a boundary in $\cap_{i \notin J} A_i$.

Color the $(r+1)$-uniform hypergraph on $[k]$ by the $\binom{r+1}{2}$ relative positions of these pairs.

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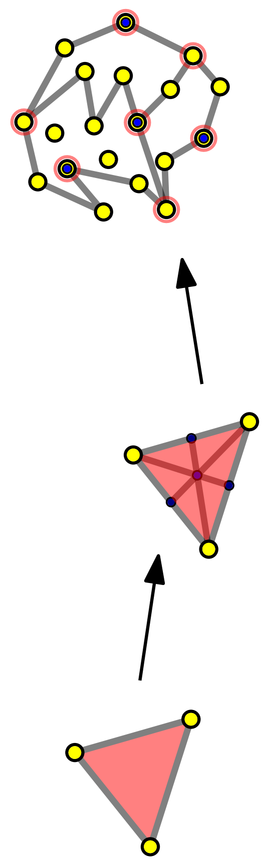

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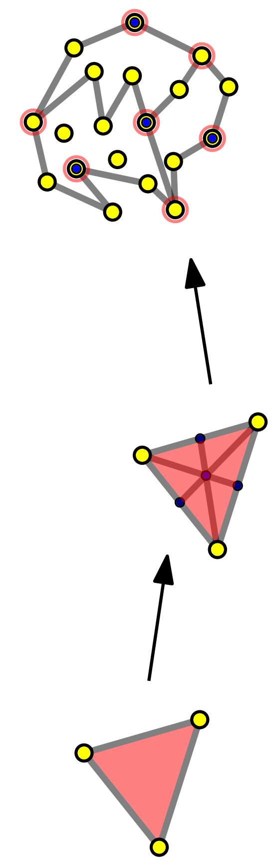

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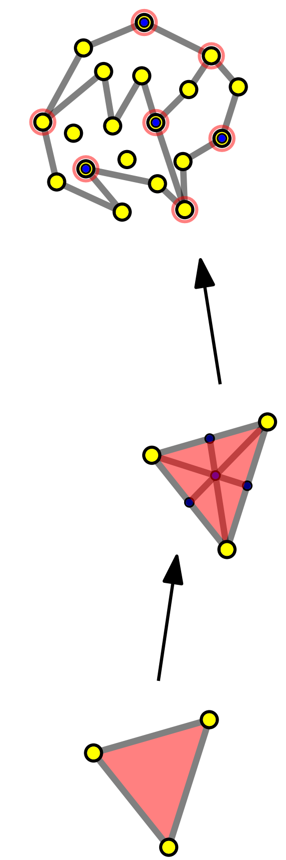

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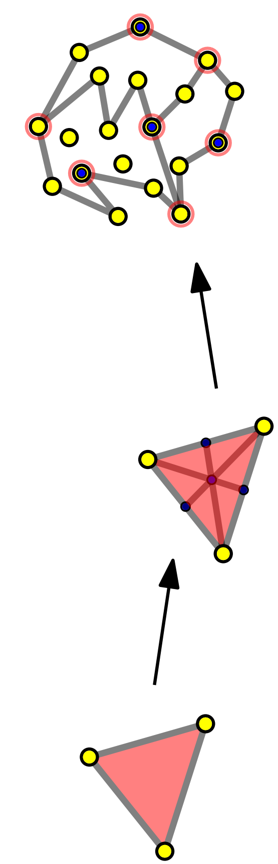
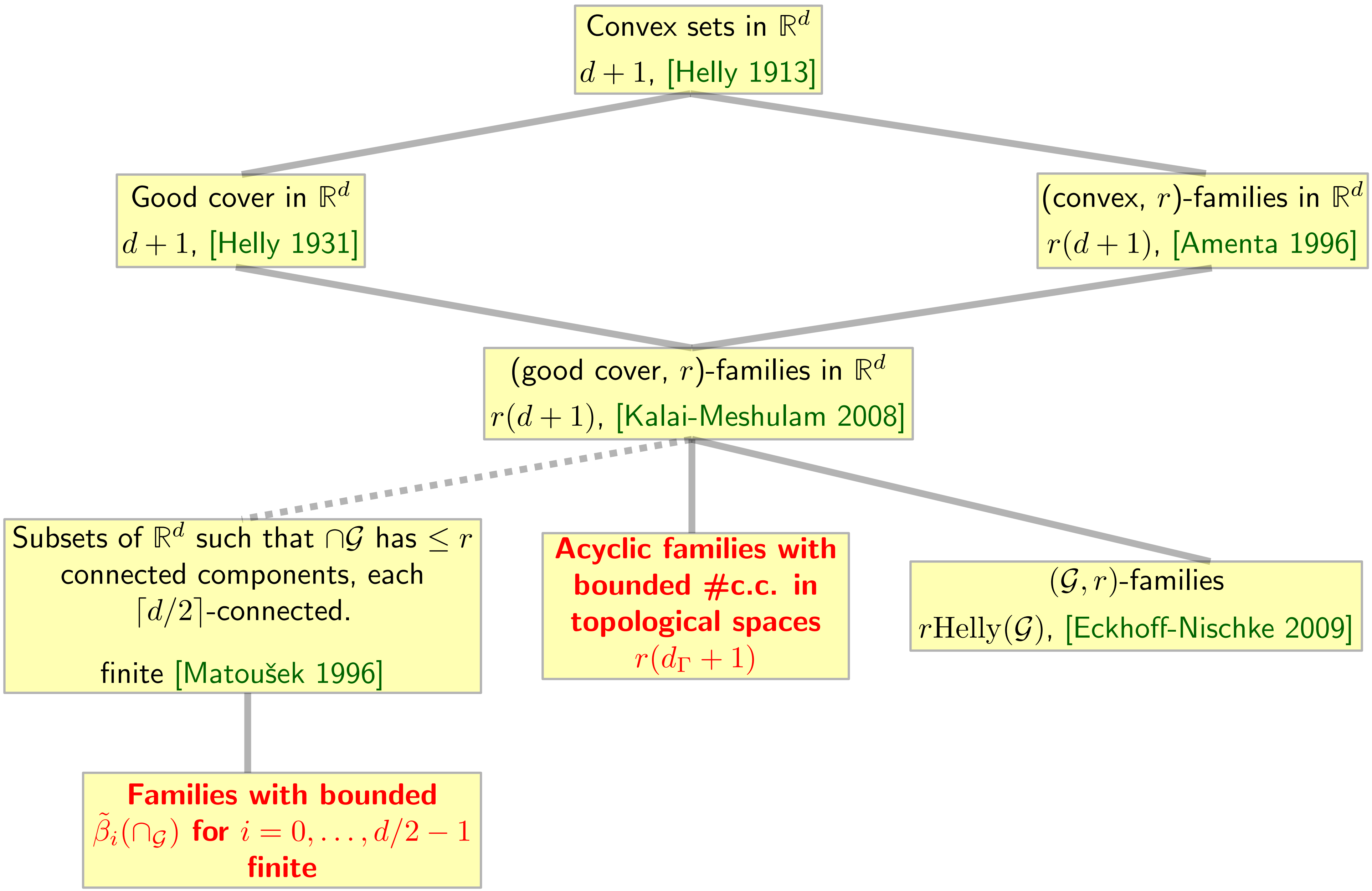
Recurse...



Perspectives



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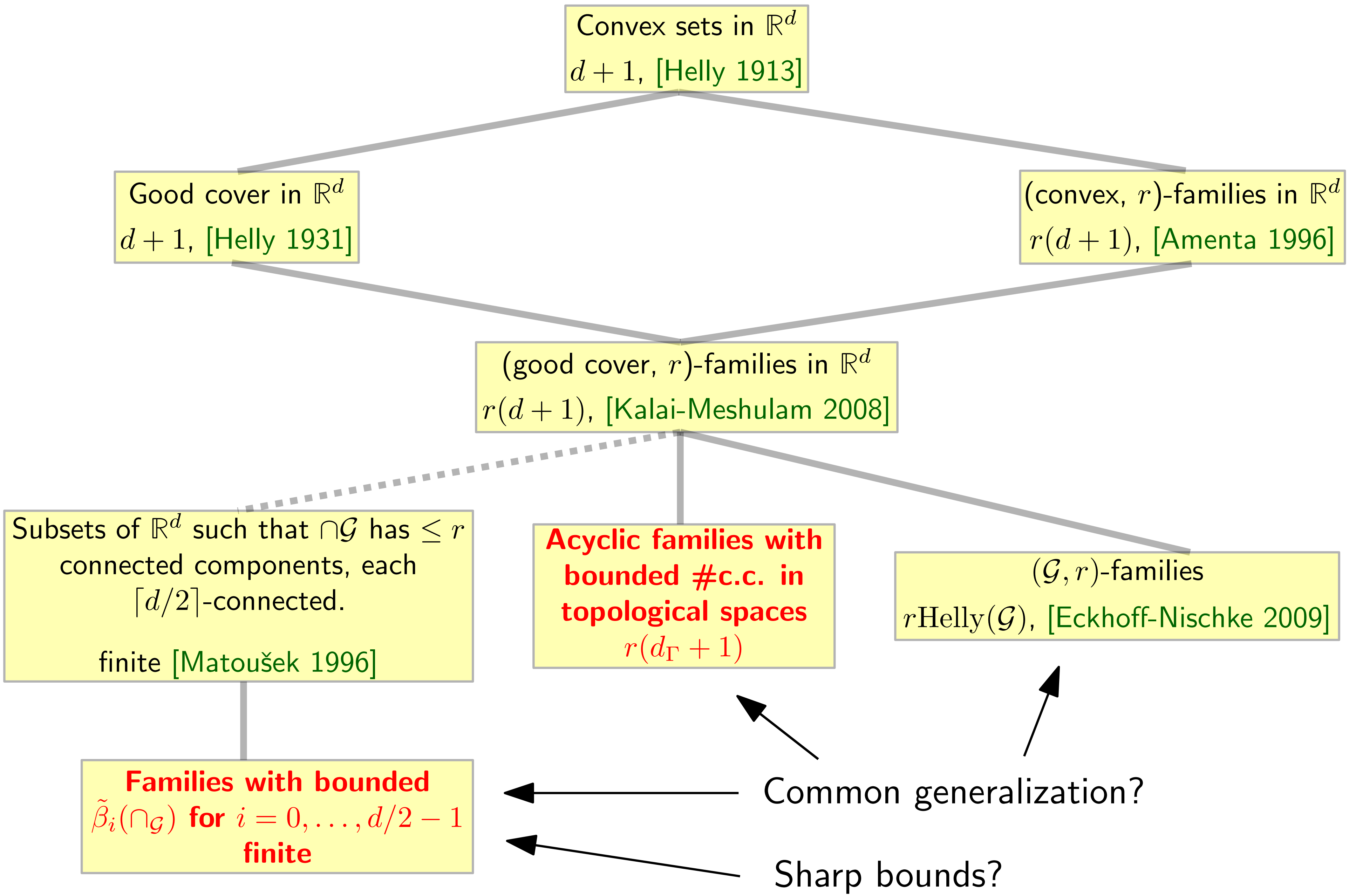

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