Persistent harmonic forms

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Studying (co-)chains that are L$^2$ or L$^1$ minimal in their (persistent) homology class
Homology (simplicial)

support : simplicial complex

$$H_k = \ker \partial_k / \text{im } \partial_{k-1}$$

$$\partial_k : C_k \rightarrow C_{k-1}$$

In this talk, the field of coefficients is always $\mathbb{R}$
de Rham Cohomology

support: Riemannian manifold \( M \)

Exterior derivative:

\[
d^k : \mathcal{A}^k \to \mathcal{A}^{k+1}
\]

\[
d^k (f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}
\]

\[
H^n = \ker d^k / \text{im } d^{k-1}
\]

In this talk, the field of coefficients is always \( \mathbb{R} \).
Hilbert structure on $C_p(K)$

The vector space of cochains $C^p(K)$ is the set of (continuous) linear maps from $C_p(K)$ to $\mathbb{R}$.
We introduce an inner product on $C_p(K)$: $\langle \cdot, \cdot \rangle_p : C_p(K) \times C_p(K) \rightarrow \mathbb{R}.$

A $k$-harmonic form is an element $\omega$ of $\ker \partial_k$ (or $\ker d^k$, etc...) whose energy is minimal in its (co-)homology class:

\[
\partial_k \omega = 0 \text{ and } \forall \alpha \in C_{k+1}, \langle \omega, \omega \rangle \leq \langle \omega + \partial_{k+1} \alpha, \omega + \partial_{k+1} \alpha \rangle
\]

This inner product is the « physic » or « geometry » ingredient.
For example DEC is the art of defining it on combinatorial manifolds
For a $p$-chain (or $p$-cochain) $\sigma$, we call the quantity $\frac{1}{2} \langle \sigma, \sigma \rangle_p$ the energy of $\sigma$.

If $C_k$ is a Hilbert space it is then isomorphic to its dual
Harmonic forms

A generic definition:

- Orientable Riemannian manifolds (Hodge)
- Discrete Hodge Theories (also DEC)
- Hodge Theory for metric spaces (S.Smale)
- etc...

Define a dot product on $C_k$ (or $A^k$, etc...):

A $k$-harmonic form is an element $\omega$ of $\ker \partial_k$ (or $\ker d^k$, etc...) whose energy is minimal in its (co-)homology class:

$$\partial_k \omega = 0 \text{ and } \forall \alpha \in C_{k+1}, \langle \omega, \omega \rangle \leq \langle \omega + \partial_{k+1}\alpha, \omega + \partial_{k+1}\alpha \rangle$$

$\Rightarrow$ Laplacian operator $\Delta_k$, Hodge Theorem, Hodge decomposition theorem follows…
Persistent homology groups

We consider a simplicial complex $K$ and a simplicial subcomplex $L \subset K$. We denote:

$C_p(K)$: vector spaces of $p$-chains on $K$ with coefficients in $\mathbb{R}$.

$\partial^K_p : C_p(K) \rightarrow C_{p-1}(K)$: the boundary operator.

$i : C_p(L) \rightarrow C_p(K)$: the inclusion map induced by $L \subset K$

$i^\#$: the corresponding map on homology groups.

Recall that the vector space:

$$H_p(K, L) = \frac{\ker(\partial^K_p)}{\text{Im}(\partial^K_{p+1}) \cap C_p(L)} = \text{Im}(i^\#)$$

is called the persistent $p$-homology group of the pair $(K, L)$.

$\dim(H_p(K, L))$ is the persistent $p^{th}$ Betti number of the pair $(K, L)$. 
Persistent harmonic form (first definition)

From the definition of persistent homology group:

\[ H_p(K, L) = \frac{\ker(\partial_p^L)}{\Im(\partial_{p+1}^K \cap C_p(L))} = \Im(i^\#) \]

One get the first definition of persistent harmonic forms:

We can define the \textit{persistent harmonic forms} of the simplicial pair \((K, L)\) as the cycles of \(\ker(\partial_p^L)\) with minimal energy in their persistent homology class:

\[ \mathcal{H}_p(K, L) = \left\{ \sigma \in \ker(\partial_p^L) \mid \forall \beta \in \Im(\partial_{p+1}^K \cap C_p(L)), \langle \sigma, \sigma \rangle_p \leq \langle \sigma + \beta, \sigma + \beta \rangle_p \right\} \]
Adjoint operators (on cochains)

The operators $\partial^K_p$, $\partial^L_p$ and $i$ induce adjoint operators $\partial^K_p^*$, $\partial^L_p^*$ and $i^*$ defined by:

$$\forall \alpha \in C_p(K), \forall \beta \in C_{p-1}(K), \langle \alpha, \partial^K_p^* \beta \rangle_p = \langle \partial^K_p \alpha, \beta \rangle_{p-1}$$

and:

$$\forall \alpha \in C_p(K), \forall \gamma \in C_p(L), \langle \alpha, i \gamma \rangle_p = \langle i^* \alpha, \gamma \rangle_p$$

$\partial^K_p^*$ and $\partial^L_p^*$ are called coboundary or exterior derivative operators. $i$ is the inclusion operator, while $i^*$ is the restriction operator.

\[ C_p(K) \xrightarrow{\partial^K_p} C_p(K) \xrightarrow{i} C_{p-1}(K) \]

\[ i^* \alpha \xrightarrow{\partial^K_p^*} \beta \xrightarrow{i^*} \alpha \]
Trivial reformulation of persistent homology group

Recall:

\[ H_p(K, L) = \frac{\ker(\partial^L_p)}{\text{Im}(\partial^K_{p+1}) \cap C_p(L)} = \text{Im}(\iota^\#) \]

\[ \tilde{A}_p(K, L) \] is the set of \( p \)-chain of \( K \) whose boundary is in \( C_{p-1}(L) \):

\[ A_p(K, L) = \{ \alpha \in C_p(K) | \partial^K_p \alpha \in C_{p-1}(L) \} = (\partial^K_p)^{-1}(C_{p-1}(L)) \]

\( \overline{\partial}_p \) is simply the restriction of \( \partial^K_p \) to \( A_p(K, L) \). One has therefore the usual relation \( \partial^L_p \overline{\partial}_{p+1} = 0 \).

**Remark**: The persistent homology group of the pair \((K, L)\) can equivalently be defined as:

\[ H_p(K, L) = \frac{\ker(\partial^L_p)}{\text{Im}(\overline{\partial}_{p+1})} \]
(persistent) Laplacian operator

Recall:

\[ \mathcal{H}_p(K, L) = \{ \sigma \in \ker(\partial^L_p) \mid \forall \beta \in \text{Im}(\partial^K_{p+1}) \cap C_p(L), \langle \sigma, \sigma \rangle_p \leq \langle \sigma + \beta, \sigma + \beta \rangle_p \} \]

Even if this is not immediately obvious from this definition, next lemma shows that \( \mathcal{H}_p(K, L) \) is a linear space.

Let us define the persistent laplacian operator \( \Delta_p : C_p(L) \rightarrow C_p(L) \) by:

\[ \Delta_p = \partial^{L*}_p \partial^L_p + \overline{\partial}^*_p \overline{\partial}^*_{p+1} \]

Lemma 1. One has:

\[ \mathcal{H}_p(K, L) = \ker(\partial^L_p) \cap \ker(\overline{\partial}^*_{p+1}) = \ker \Delta_p \]
(persistent)-Hodge Theorems

(persistent) Hodge Theorem

**Lemma 2.** $\mathcal{H}_p(K, L)$ is isomorphic to the persistent homology group $H_p(K, L)$

\[ \mathcal{H}_p(K, L) \xrightarrow{\text{isomorphism}} H_p(K, L) \]

(persistent) Hodge decomposition Theorem

**Lemma 3.** We have the following decomposition as a direct sum:

\[ C_p(L) = \text{Im}(\bar{\partial}_{p+1}) \oplus \mathcal{H}_p(K, L) \oplus \text{Im}(\partial_p^{L*}) \]

Moreover these three subspaces are pairwise orthogonal.
Canonical orthogonal basis of persistent harmonic forms

**Lemma 4.** If $M \subset L \subset K$ are finite simplicial complexes, then:

$$\mathcal{H}_p(K, M) \subset \mathcal{H}_p(L, M)$$

The inclusion in the lemma is trivially injective. A consequence of this lemma is that, given a filtration of finite simplicial complexes $K_0 \subset K_1 \subset \ldots \subset K_n$, such that $K_n$ has trivial homology (for example it could be contractible), then we have the sequence of inclusion:

$$\{0\} = \mathcal{H}_p(K_n, K_0) \subset \mathcal{H}_p(K_{n-1}, K_0) \subset \ldots \mathcal{H}_p(K_1, K_0) \subset \mathcal{H}_p(K_0, K_0)$$

If the filtration is fine enough, the dimension will never increase by more than 1 along this sequence. This allows to build a canonical orthonormal basis.
Application to linear PDEs?

References DEC: (Desbrun,…

Incompressible fluids:

\[ K \text{ is the meshed domain and } L \subset K \text{ the meshed domain boundary.} \]
\[ C_1(K, L) = C_1(K)/C_1(L) \text{ and } C_0(K, L) = C_0(K)/C_0(L) \text{ the relative chains and} \]
\[ \partial_{K,L} : C_1(K, L) \to C_0(K, L) \text{ the corresponding boundary operator.} \]

Taking the right inner product (finite elements, DEC) and minimizing the energy of 1-forms in their relative (co-)homology class we get a set of irrotational flow (incompressible Euler) as «relative harmonic form». The circulation around the wing gives the lift.
Application to linear PDEs?

In the previous example, the homology of the domain is crucial.
• What if the domain of the PDE is known through approximations (point sample) ?
• If we are able to build a pair of complex capturing the homology of the domain, could we approximate the solution of the PDE by « relative persistent harmonic forms » ?
• The inner product can be inherited from DEC if the complexes are embedded (alpha-complex). But is it possible to design a reasonable inner product if the complexes are not embedded ? (Cech, Rips) ?
$L^2$ minima are not sparse

Minimizing $L^2$ norm

=> relative harmonic form:

\[ \sum_j \frac{1}{R_j} I_j^2 \]

(Electric power)

$K$ is a 2-dimensional simplicial complex,
$L$ is a 0-dimensional simplicial complex
The relative Homology group $H_1(K,L)$ has dimension 1.
L^1 minima are sparse

\( K \) is a 2-dimensional simplicial complex, 
\( L \) is a 0-dimensional simplicial complex 
The relative Homology group \( H_1(K,L) \) has dimension 1.

Minimizing \( L^1 \) norm : 
=> shortest path

\[ \sum_j l_j |I_j| \] (length)

generically 1-manifold ??

Sparsity
Convergence of $L^1$ minimal cycles

$M$ is a connected orientable compact embedded $k$-manifold such that:

$\forall t \in [0, 4\epsilon]$, the inclusion $M \rightarrow M^t$ is a homotopy equivalence.

For a cloud $P$ such that $d_H(P, M) < \eta$, we denote:

$$(K, L) = (\text{Cech}(P, 3\epsilon_0), \text{Cech}(P, \epsilon_0))$$

If $\eta \leq \epsilon_0$, the simplicial pair $(K, L)$ captures the homology of $M$, in particular:

$H_k(K, L)$ captures the fundamental class of $M$ and we have $\dim(H_k(K, L)) = 1$
Convergence of $L^1$ minimal cycles

If $\alpha$ is a $k$-chain on a Čech (or Rips) complex, we define the $L^1$ norm $\| \alpha \|_1$ as:

$$\| \alpha \|_1 = \sum_{\sigma \in k\text{-simplex}} \text{Vol}(\sigma) |\alpha(\sigma)|$$

Example: $k=1$, $M$ is a circle

$$\dim(H_k(K, L)) = 1$$

If $k=1$ : we get non trivial cycle of minimal length.

If $k=1$, this cycle generically induces an homeomorphic manifold.
Convergence of $L^1$ minimal cycles

We consider a sequence of point clouds $P_i, i \in \mathbb{N}$, such that $d_H(P_i, M) < \eta_i$, and the $k$-persistent homology group of the pair $(C(P_i, 3\epsilon_0), C(P_i, \epsilon_i))$, with:

$$\lim_{i \to \infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i \to \infty} \frac{\eta_i}{\epsilon_i} = 0$$

Let $\gamma_i$ be a $k$-cycle of $C_k(C(P_i, \epsilon_i))$ $L^1$ minimal in its persistent homology class. Generically, $\gamma_i$ is unique, up to a multiplicative constant (\text{\Rightarrow} need for normalization).

In which sense could we say (or hope) that $\gamma_i$ converges toward $M$ as $i \to \infty$?
Convergence of $L^1$ minimal cycles

Rectifiability assumption on the manifold $M$:

There is a constant $C > 0$ and a sequence of triangulations $T_i$ of $M$ such that:

- the radius of any simplex of $T_i$ is less than $\frac{1}{2} \epsilon_i$,
- maximal simplices of $T_i$ contain a ball of radius $C \epsilon_i$
- $\lim_{i \to \infty} \text{Vol}(T_i) = \text{Vol}(M)$

Since $\lim_{i \to \infty} \frac{\eta_i}{\epsilon_i} = 0$, for each triangulation $T_i$, there is a cycle $\sigma_i \in C_k(C(P_i, \epsilon_i))$ “close enough” to $T_i$ with:

$$\lim_{i \to \infty} \frac{|\text{Vol}(T_i) - \|\sigma_i\|_1|}{\text{Vol}(T_i)} = 0$$

Which gives:

$$\limsup \|\gamma_i\|_1 \leq \text{Vol}(M)$$
Embedded polyhedral chains and embedded oriented surfaces as Whitney chains and currents

There is an obvious way to integrate a \( k \)-differential form of the euclidean ambient space \( \mathbb{R}^n \) on an oriented \( k \)-simplex. By additivity, a cycle \( \gamma_i \in C_k(C(P_i, \epsilon_i)) \) is continuous linear form on the set of differential forms, in other words, a \( k \)-current denoted \( \Gamma_i \) with mass \( M(\Gamma_i) = \|\gamma_i\|_1 \). Similarly, it is possible to integrate \( k \)-differential forms on the oriented embedded manifold \( M \) which therefore define a \( k \)-current denoted \( \Gamma_M \) with mass \( M(\Gamma_M) = \text{Vol}(M) \).
A k-current is something on which one can integrate differential forms. Formally it is the dual of the space of k-differential forms. Boundary is defined as the adjoint of exterior derivative operator (think of Stokes or Archimedes Theorems).

**Boundary:**

\[
\int_{\partial x} \varphi = \int_{\partial x} \varphi
\]

Mass and flat norms

\[
M(T) = \sup \{ T(\phi), \| \phi \|_\infty \leq 1 \}
\]

\[
F(T) = \sup \{ T(\phi), \| \phi \|_\infty \leq 1 \text{ and } \| d\phi \|_\infty \leq 1 \}
\]

\[
= \min \{ M(A) + M(B), T = A + \partial B \}
\]
Convergence of $L^1$ minimal cycles

Lipchitz deformation retract assumption on the manifold $M$:

There are constants $C_1, C_2$ and a deformation retract $\psi : M^{4\epsilon_0} \to M$ such that:

- $\forall x_1, x_2 \in M^{4\epsilon_0}, \ d(\psi(x_1), \psi(x_2)) \leq C_1 d(x_1, x_2)$
- $\forall x \in M^{4\epsilon_0}, \ d(x, \psi(x)) \leq C_2 d(x, M)$

With the assumption above, if we consider the homotopy $h : [0, 1] \times M^{4\epsilon_0} \to M^{4\epsilon_0}$ defined by $h(t, x) = (1 - t)x + t\psi(x)$ we get a $(k + 1)$-current $H_i = h([0, 1] \times \Gamma_i)$ that “span the space between” $\Gamma_i$ and $\Gamma_M$, formally:

$$\partial H_i = \Gamma_M - \Gamma_i$$

Since $\lim \sup M(\Gamma_i) \leq \text{Vol}(M)$ and from the properties of $\psi$ we get that:

$$\lim_{i \to \infty} M(H_i) = 0, \ \text{in other words:} \ \lim_{i \to \infty} F(\Gamma_M - \Gamma_i) = 0$$
Convergence of $L^1$ minimal cycles

One has (flat norm convergence entails weak convergence):

$$\lim_{i \to \infty} F(\Gamma_M - \Gamma_i) = 0 \Rightarrow \lim_{i \to \infty} \inf M(\Gamma_i) \geq M(\Gamma_M) = Vol(M)$$

This together with $\limsup_{i \to \infty} M(\Gamma_i) \leq M(\Gamma_M)$ gives us:

$$\lim_{i \to \infty} ||\gamma_i||_1 = \lim_{i \to \infty} M(\Gamma_i) = M(\Gamma_M) = Vol(M)$$
Convergence of $L^1$ minimal cycles

Perspectives:
- Understanding first rectifiability condition (the existence of regular triangulations) beyond piecewise smooth manifolds.
- Understanding second rectifiability condition (Lipchitz deformation retract) beyond pieces smooth and positive mu-reach.
- Normalization in practice?
- Sparcity $\Rightarrow$ Homeomorphic manifold in the limit:

$$\exists i, \forall j > i, \sigma_j \text{ defines a manifold homeomorphic to } M?$$

By “defines” one means that the coefficients of $k$-simplices of $C(P_j, \epsilon_j)$ for $\sigma_j$ are in $\{0, 1\}$ (or in $\{0, \lambda\}$, with $\lambda \neq 0$ if the normalization is not done) and selecting the set of simplifies with non-zero values in $\sigma_j$ gives the manifold.
Convergence of $L^1$ minimal cycles

Perspectives:

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Application to Smale $L^2$ Hodge theory?

Dimension of $\ker \Delta_p$ is the $k^{th}$ persistent Betti number. It is finite under mild conditions. Are they mild conditions for the laplacian operator to be closed? Why not revisit Smale Hodge Theory in the context of persistence?