Reconstruction of Filament Structure

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1 Problem Statement
   Characterization of Dataset
   Formulation

2 Our Approaches
   Reeb Graph
   Geometrical Guarantee
   \( \alpha \)-Reeb Graph
   Topological Guarantee
We are interested in the dataset with underlying linear structure.

Example
Earthquake faults:
Outline

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   $\alpha$–Reeb Graph
   Topological Guarantee
Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces, an \(\epsilon\)-correspondence between them is a subset \(C \subseteq X \times Y\) such that:

- for any \(x \in X\), there exists \(y \in Y\) such that \((x, y) \in C\);
- for any \(y \in Y\), there exists \(x \in X\) such that \((x, y) \in C\);
- for any \((x, y), (x', y') \in C\), \(|d_X(x, x') - d_Y(y, y')| \leq \epsilon\).
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**Gromov-Hausdorff Distance**

\[
d_{GH}(X, Y) = \frac{1}{2} \inf \{ \epsilon : \text{there exists an \(\epsilon\)-correspondence between } X \text{ and } Y \}\n\]
**Problem Statement**

**Input**
A compact path metric space \( (X, d_X) \), which is close to some unknown metric graph \( (G', d_{G'}) \), meaning \( d_{GH}(X, G') < \epsilon \).
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**Equivalence**

\(x \sim y\) IFF \(d(x) = d(y)\) and \(x, y\) are in the same connected component of \(d^{-1}(d(x))\).
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Reeb Graph

\[ G = X \backslash \sim \]
Illustration
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Let $\pi : X \rightarrow G$ be $\pi(x) = [x]$. Then we consider correspondence $(x, \pi(x)) \subset X \times G$. 
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- 1-Lipschitz: $d_X(x, y) \geq d_G(\pi(x), \pi(y))$;
- $d_X(x, y) \leq d_G(\pi(x), \pi(y)) + 2(\beta_1(G) + 1)M$;
- bounding the diameter of level-set, i.e. $M = \sup_{x \in X} \text{diam}(d^{-1}(d(x)))$. 

Since $X$ is close to some underlying graph $G'$, there is an upper bound for the level-set.
Bounding $M$

Since $X$ is close to some underlying graph $G'$, there is an upper bound for the level-set.

**Theorem**

For any $l > \alpha$, the diameter of any connected component $L$ of $d^{-1}[l - \alpha, l + \alpha]$ satisfies:

$$diam(L) \leq 4(2 + N_{E,G'}(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon$$

$N_{E,G'}(\delta)$ is the number of edges of $G'$ with length small than $\delta$. 

Let $\alpha = 0$, $M \leq (8N_{E,G'}(8\epsilon) + 17)\epsilon$. 


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Let $\alpha = 0$, $M \leq (8N_{E,G'}(8\epsilon) + 17)\epsilon$
To conclude, we proved that:

\[ d_{GH}(X, G) < (\beta_1(G) + 1)(17 + 8N_{E,G'}(8\epsilon))\epsilon \]
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We modify the equivalence relation to construct the $\alpha$–Reeb graph.

- Cover the range of $d(x)$ with the intervals $\{l_i\}$ of length at most $2\alpha$, for example, $l_i = (i\alpha, (i + 2)\alpha), i = 0, 1, 2...$;
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- Cover the range of $d(x)$ with the intervals $\{I_i\}$ of length at most $2\alpha$, for example, $I_i = (i\alpha, (i + 2)\alpha)$, $i = 0, 1, 2...$;
- Define equivalence relationship $x \sim_\alpha y$ IFF $d(x) = d(y)$ and $x, y$ are in the same connected component of $d^{-1}(I_i)$;
\[ \alpha - \text{Reeb Graph} \]

We modify the equivalence relation to construct the \[ \alpha \]–Reeb graph.

- Cover the range of \( d(x) \) with the intervals \( \{ l_i \} \) of length at most \( 2\alpha \), for example, \( l_i = (i\alpha, (i + 2)\alpha) \), \( i = 0, 1, 2... \);
- Define equivalence relationship \( x \sim_\alpha y \) IFF \( d(x) = d(y) \) and \( x, y \) are in the same connected component of \( d^{-1}(l_i) \);
- \( G_\alpha = X / \sim_\alpha \).
Illustration

\[ H \quad d \quad \mathcal{T} \]

\[ \tilde{G} \]

disjoint union of copies of intervals

\[ \alpha \text{-Reeb graph} \]
Geometrical Guarantee

Similarly, we have:

\[ d_{GH}(X, G_\alpha) < (\beta_1(G_\alpha) + 1)(4(2 + N_{E,G'}(4(\alpha + 2\epsilon))))(\alpha + 2\epsilon) + \epsilon) \]
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Homotopy Equivalence Result

Theorem
Let $r \in X$, $\alpha > 60\varepsilon$ and $I\{[0, 2\alpha), (i\alpha, (i + 2)\alpha)|1 \leq i \leq m\}$ covers the segment $[0, \text{Diam}(X)]$. If no edges of $G'$ is shorter than $L$ and no loops of $G'$ is shorter than $2L$ with $L \geq (17\alpha + 9\varepsilon)$, then we have $G_\alpha$ and $G'$ are homotopy equivalent.
Nerve Complex

Let $\mathcal{U} = \bigcup_{i \in A} U_i$ be a finite open covering of topological space $T$. Then the nerve complex of $\mathcal{U}$, $K$, is a simplicial complex whose vertex set is $A$, and where a family $\{i_0, i_1, ..., i_k\}$ spans a $k$–simplex in $K$ IFF $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_k} \neq \emptyset$. 
Illustration

$H$  $d$  $\mathcal{I}$

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disjoint union of copies of intervals

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Nerve Lemma

Furthermore, if all finite intersections of the open sets in the open covering are either empty or contractible, then $T$ is homotopy equivalent to $K$. 
Illustration

\[ H \] disjoint union of copies of intervals

\[ \tilde{G} \] disjoint union of copies of intervals

\[ \alpha\text{-Reeb graph} \]
Correspondence between $X$ and $G'$

Let $(r, g_r) \in C(X, G')$, $b(g) = d_{G'}(g_r, g)$. Define $C(U) = \{ g : (x, g) \in C, \forall x \in U \subseteq X \}$ ($C(V)$ is defined in the same way for $V \subseteq G'$).
Correspondence between $X$ and $G'$

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Path Correspondence

For $x_1, x_2 \in X$, if there is a path in $d^{-1}(l, u)$ connecting them, then for $g_i, (x_i, g_i) \in C$, we can always find a path in $b^{-1}(l - 2\epsilon, u + 2\epsilon)$ connecting $g_1, g_2$. 
New Open Covering of $X$

$\tilde{V}_1 \in \tilde{V}_1$

$\tilde{V}_2 \in \tilde{V}_2$

$\tilde{V}_3 \in \tilde{V}_3$

$N(V_0)$
Thanks for your attention.

Questions?