Reconstruction of Filament Structure

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Joint work with Frédéric CHAZAL and Jian SUN

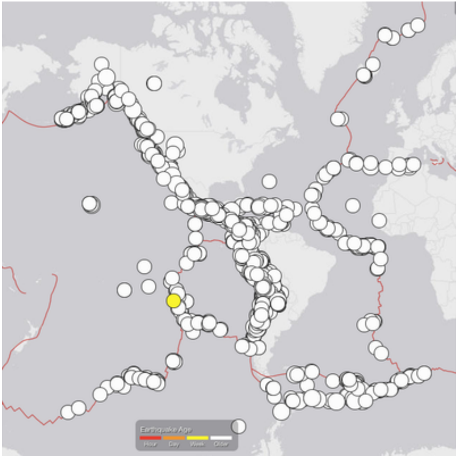
27/10/2014

Outline

We are interested in the dataset with underlying linear structure.

Example

Earthquake faults:



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$\epsilon-$correspondence

Let (X, d_X) and (Y, d_Y) be two metric spaces,
an $\epsilon-$correspondence between them is a subset $C \in X \times Y$ such that:

- for any $x \in X$, there exists $y \in Y$ such that $(x, y) \in C$;
- for any $y \in Y$, there exists $x \in X$ such that $(x, y) \in C$;
- for any $(x, y), (x', y') \in C$, $|d_X(x, x') - d_Y(y, y')| \leq \epsilon$.

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Gromov-Hausdorff Distance

$$d_{GH}(X, Y) = \frac{1}{2} inf\{\epsilon : \text{there exists an } \epsilon-\text{correspondence between } X \text{ and } Y\}$$

Problem Statement

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A compact path metric space (X, d_X), which is close to some unknown metric graph $(G', d_{G'})$, meaning $d_{GH}(X, G') < \epsilon$.

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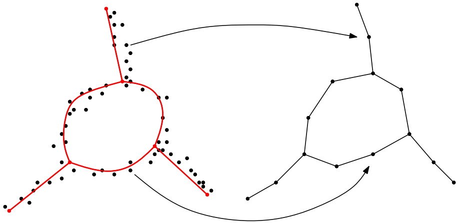

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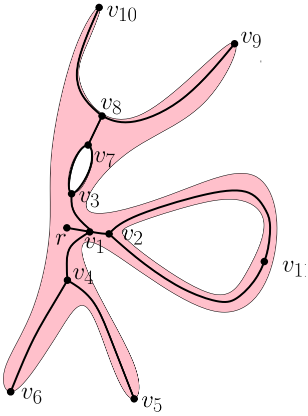
Reeb Graph

$$G = X \backslash \sim$$

Illustration



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- $d_X(x, y) \leq d_G(\pi(x), \pi(y)) + 2(\beta_1(G) + 1)M$;
- bounding the diameter of level-set, i.e. $M = sup_{x \in X} diam(d^{-1}(d(x)))$.

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Theorem

For any $l > \alpha$, the diameter of any connected component L of $d^{-1}[l - \alpha, l + \alpha]$ satisfies:

$$diam(L) \leq 4(2 + N_{E,G'}(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon$$

$N_{E,G'}(\delta)$ is the number of edges of G' with length small than δ.

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Let $\alpha = 0$, $M \leq (8N_{E,G'}(8\epsilon) + 17)\epsilon$

Geometrical guarantee

To conclude, we proved that:

$$d_{GH}(X, G) < (\beta_1(G) + 1)(17 + 8N_{E,G'}(8\epsilon))\epsilon$$

Outline

$\alpha-$Reeb Graph

We modify the equivalence relation to construct the $\alpha-$Reeb graph.

- Cover the range of $d(x)$ with the intervals $\{I_i\}$ of length at most 2α, for example, $I_i = (i\alpha, (i+2)\alpha), i = 0, 1, 2 \ldots;$

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- Define equivalence relationship $x \sim_\alpha y$ IFF $d(x) = d(y)$ and x, y are in the same connected component of $d^{-1}(I_i)$;
- $G_\alpha = X/\sim_\alpha$.

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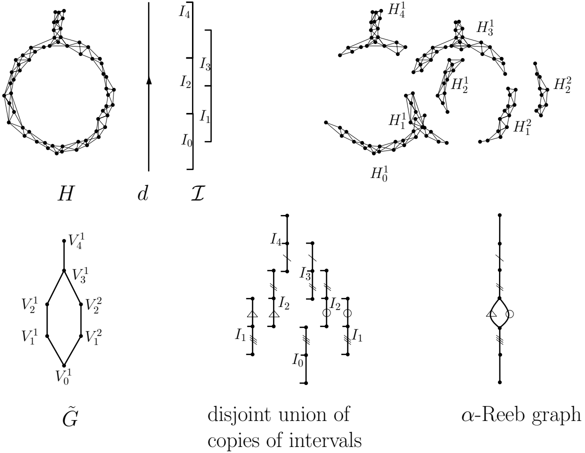


Geometrical Guarantee

Similarly, we have:

$$d_{GH}(X, G_\alpha) < (\beta_1(G_\alpha) + 1)(4(2 + N_{E,G'}(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon)$$

Outline

Homotopy Equivalence Result

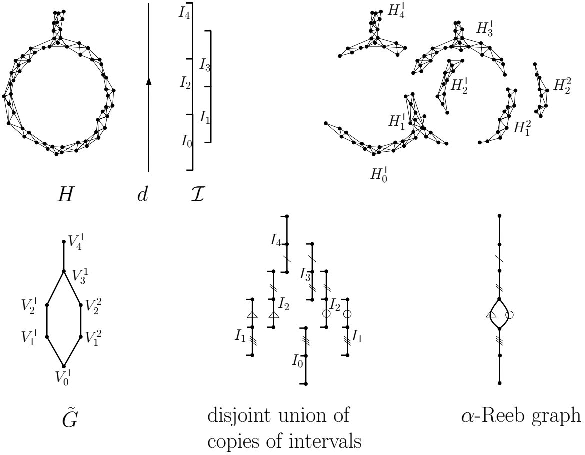
Theorem

Let $r \in X$, $\alpha > 60\epsilon$ and $\mathcal{I}\{[0, 2\alpha), (i\alpha, (i+2)\alpha) | 1 \leq i \leq m\}$ covers the segment [0, Diam(X)]. If no edges of G' is shorter than L and no loops of G' is shorter than $2L$ with $L \geq (17\alpha + 9\epsilon)$, then we have G_α and G' are homotopy equivalent.

Nerve Complex

Let $\mathcal{U} = \cup_{i \in A} U_i$ be a finite open covering of topological space T. Then the nerve complex of $\mathcal{U}$, K, is a simplicial complex whose vertex set is A, and where a family $\{i_0, i_1, ..., i_k\}$ spans a $k-$simplex in K IFF $U_{i_0} \cap U_{i_1} \cap ... \cap U_{i_k} \neq \emptyset$.

Illustration



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Nerve Lemma

Further more, if all finite intersections of the open sets in the open covering are either empty or contractible, then T is homotopy equivalent to K.

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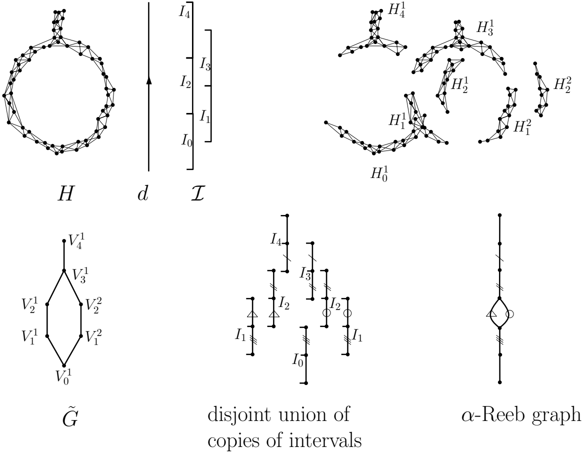

Correspondence between X and G'

Let $(r, g_r) \in C(X, G')$, $b(g) = d_{G'}(g_r, g)$. Define $C(U) = \{g : (x, g) \in C, \forall x \in U \subset X\}$($C(V)$ is defined in the same way for $V \subset G'$).

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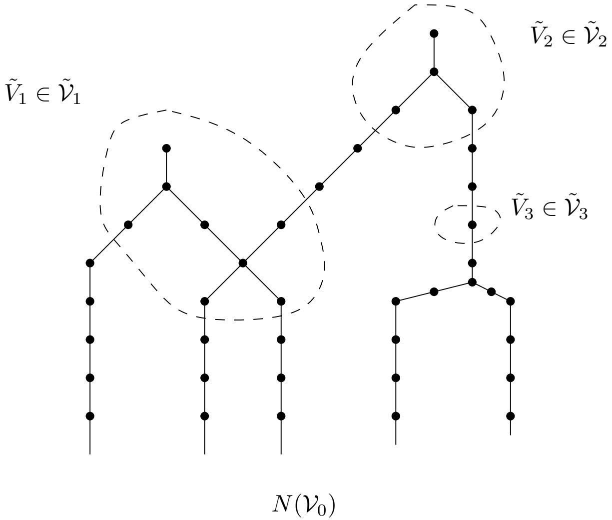
Path Correspondence

For $x_1, x_2 \in X$, if there is a path in $d^{-1}(l, u)$ connecting them, then for $g_i, (x_i, g_i) \in C$, we can always find a path in $b^{-1}(l - 2\epsilon, u + 2\epsilon)$ connecting g_1, g_2.

New Open Covering of X



THE END

Thanks for your attention.

Questions?