Riemannian simplices and triangulations

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Geometric simplices in Riemannian manifolds

Motivation: Generic triangulation criteria $|\mathcal{A}| \to M$

- manifold abstract simplicial complex $\mathcal{A}$ with vertices $\mathsf{P} \subset M$
- intrinsic setting
- explicit density requirements
- dimension > 2

Natural way to "fill in" a simplex?

- Convex hull doesn't work
- Instead, define barycentric coordinates on M
- Karcher means

Main idea

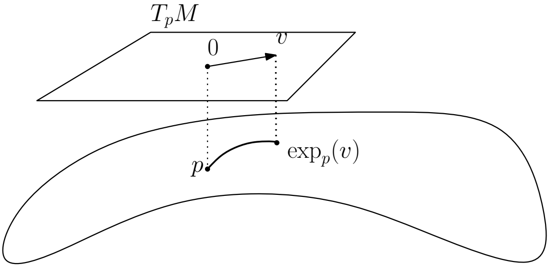
- M an n-dimensional Riemannian manifold
- $B \subset M$ a convex set (this is a size restriction in positive curvature)
- $\sigma^j = \{p_0, \ldots, p_j\} \subset B$ a finite set of vertices
- $\boldsymbol{\Delta}^j$ the standard Euclidean j-simplex

The Barycentric coordinate map

$$\mathcal{B}_{\sigma^j} : \boldsymbol{\Delta}^j \to M$$

The Riemannian simplex $\boldsymbol{\sigma}_M$ is the image of this map.

The exponential map



- Injectivity radius ι_M
- Sectional curvature bounds Λ_-, Λ_+, Λ
- Convexity

The barycentric coordinate map

Energy function

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, p_i)^2$$

barycentric coordinates: $\lambda_i \geq 0$; and $\sum_{i=0}^{j} \lambda_i = 1$

Theorem (Karcher 1977)

If $\{p_0, \ldots, p_j\} \subset B_\rho \subset M$, and B_ρ is an open ball of radius ρ with

$$\rho < \min\left\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda_+}}\right\},$$

then $\mathcal{E}_\lambda$ is convex and has a unique minimum in B_ρ.

$$\mathcal{B}_{\sigma^j} : \mathbf{\Delta}^j \to M$$
$$\lambda \mapsto \operatorname*{argmin}_{x \in \overline{B}_\rho} \mathcal{E}_\lambda(x)$$

Non-degenerate Riemannian simplices

Definition

A Riemannian simplex $\boldsymbol{\sigma}_M$ is *non-degenerate* if the barycentric coordinate map $\mathcal{B}_{\sigma^j} \to M$ is an embedding.

$$\operatorname{grad} \mathcal{E}_\lambda(x) = -\sum_i \lambda_i \exp_x^{-1}(p_i)$$

Notation

$$v_i(x) = \exp_x^{-1}(p_i) \quad \text{and} \quad \sigma(x) = \{v_0(x), \ldots, v_j(x)\} \subset T_x M$$

Proposition

A Riemannian simplex $\boldsymbol{\sigma}_M \subset M$ is non-degenerate if and only if $\sigma(x) \subset T_x M$ is non-degenerate for every $x \in \boldsymbol{\sigma}_M$.

Proof outline

Buser and Karcher (1981)

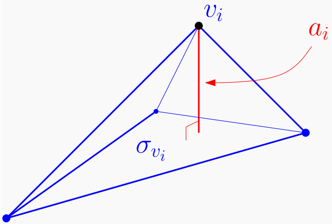
Consider a Riemannian n-simplex $\boldsymbol{\sigma}_M$ defined by $\sigma \subset B_\rho \subset M$.
Coordinate chart $\phi : M \supset B_\rho \to U$ with $\tilde{\sigma} = \phi(\sigma)$
$\tilde{\sigma} = \{\tilde{v}_0, \dots, \tilde{v}_n\}$, with $\tilde{v}_0$ the origin in $U \subseteq \mathbb{R}^n$

Write $\operatorname{grad} \mathcal{E}_\lambda$ as $\nu : U \times B_\rho \to TM$
$\nu(u, x) = -\sum_{i=0}^{n} \lambda_i(u) v_i(x)$
$b : \tilde{\boldsymbol{\sigma}}_{\mathbb{E}} \xrightarrow{\mathcal{L}} \boldsymbol{\Delta}^n \xrightarrow{\mathcal{B}_\sigma} \boldsymbol{\sigma}_M$
$\nu(u, b(u)) = 0$
$\partial_u \nu + \left(\nabla^M \nu\right) db = 0$
$db = -\left(\nabla^M \nu\right)^{-1} \partial_u \nu$

$\partial_u \nu = -\sum_{i=0}^{n} v_i(x)\, d\lambda_i$
$(\partial_u \nu)\, \xi = -P \tilde{P}^{-1} \xi$
where $\tilde{P}$ has i^{th} column $\tilde{v}_i$,
and P has columns $v_i(x) - v_0(x)$ w.r.t. an arbitrary coord syst on $T_x M$

Euclidean simplex quality

Altitudes



If σ_{v_i}, the face opposite v_i in $\sigma \subset \mathbb{R}^n$, then the *altitude* of v_i, in σ is

$$a_i = d_{\mathbb{R}^m}(v_i, \mathrm{aff}(\sigma_{v_i})).$$

The norm of the gradient of λ_i is the inverse of a_i, i.e., $|d\lambda_i| = a_i^{-1}$.

Definition (Thickness)

The *thickness* of a j-simplex σ with longest edge L is

$$t(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{v_i \in \sigma} \frac{a_i}{jL} & \text{otherwise.} \end{cases}$$

Stability of thickness

Boissonnat,D,Ghosh

Lemma (Thickness under distortion)

Suppose that $\sigma = \{v_0, \ldots, v_k\}$ *and* $\tilde{\sigma} = \{\tilde{v}_0, \ldots, \tilde{v}_k\}$ *are two* k*-simplices in* $\mathbb{R}^n$ *such that*

$$\left| |v_i - v_j| - |\tilde{v}_i - \tilde{v}_j| \right| \leq C_0 L(\sigma)$$

for all $0 \leq i < j \leq k$. *Let* P *be the matrix whose* i^{th} *column is* $v_i - v_0$, *and define* $\tilde{P}$ *similarly.*
If

$$C_0 = \frac{\eta t(\sigma)^2}{4} \qquad \textit{with} \quad 0 \leq \eta \leq 1,$$

then

$$t(\tilde{\sigma}) \geq \frac{4}{5\sqrt{k}}(1 - \eta) t(\sigma).$$

Simplified Rauch Theorem

Suppose the sectional curvatures in M are bounded by $|K| \leq \Lambda$. If $v \in T_pM$ satisfies $|v| = r < \frac{\pi}{2\sqrt{\Lambda}}$, then for any vector $w \in T_v(T_pM) \cong T_pM$, we have

$$\left(1 - \frac{\Lambda r^2}{6}\right)|w| \leq \left|(d\exp_p)_v w\right| \leq \left(1 + \frac{\Lambda r^2}{2}\right)|w|.$$

Exponential transitions

$$\begin{aligned} \exp_x \circ \exp_p^{-1} \colon T_pM &\to T_xM \\ \sigma(p) &\to \sigma(x) \end{aligned}$$

The change in edge lengths is bounded.

Theorem (Non-degeneracy criteria)

Suppose M is a Riemannian manifold with sectional curvatures K bounded by $|K| \leq \Lambda$, and $\boldsymbol{\sigma}_M$ is a Riemannian simplex, with $\boldsymbol{\sigma}_M \subset B_\rho \subset M$, where B_ρ is an open geodesic ball of radius ρ with

$$\rho < \rho_0 = \min \left\{ \frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda}} \right\}.$$

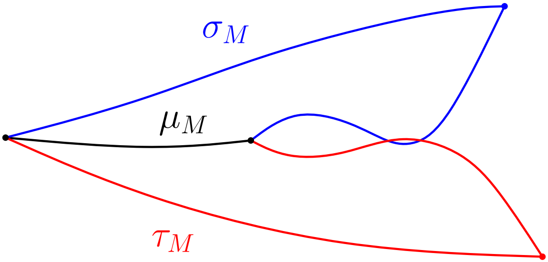
Then $\boldsymbol{\sigma}_M$ is non-degenerate if there is a point $p \in B_\rho$ such that the lifted Euclidean simplex $\sigma(p)$ has thickness satisfying

$$t(\sigma(p)) > 10\sqrt{\Lambda} L(\boldsymbol{\sigma}_M),$$

where $L(\boldsymbol{\sigma}_M)$ is the geodesic length of the longest edge in $\boldsymbol{\sigma}_M$.

Triangulation

In order to be able to guarantee that $\mathcal{A}$ with vertex set $\mathsf{P} \subset M$ is a triangulation, we need to ensure that there are no local conflicts. For example, suppose $\sigma(p)$ and $\tau(p)$ in T_pM are n-simplices that share a common $(n-1)$ facet $\mu(p)$. We would like to guarantee that $\boldsymbol{\sigma}_M \cap \boldsymbol{\tau}_M = \boldsymbol{\mu}_M$.



We don't quite do this.

Lemma (Embedding a star)

Suppose $\mathcal{C} = \underline{\mathrm{St}}(p)$ is a t_0-thick, pure n-complex embedded in $\mathbb{R}^n$ such that all of the n-simplices are incident to a single vertex, p, and $p \in \mathrm{int}(|\mathcal{C}|)$ (i.e., $\underline{\mathrm{St}}(p)$ is a full star). If $F : |\mathcal{C}| \to \mathbb{R}^n$ is smooth on $\mathcal{C}$, and satisfies

$$\|dF - \mathrm{Id}\| < nt_0$$

on each n-simplex of $\mathcal{C}$, then F is an embedding.

In order to exploit this, we employed a strengthening of the Rauch Theorem, obtained by Buser and Karcher (1981):

$$\left\|(d\exp_p)_v - T_{xp}\right\| \leq \frac{\Lambda r^2}{2},$$

where T_{xp} is the parallel transport operator.

Theorem (Triangulation criteria)

Suppose M is a compact n-dimensional Riemannian manifold with sectional curvatures K bounded by $|K| \leq \Lambda$, and $\mathcal{A}$ is an abstract simplicial complex with finite vertex set $\mathsf{P} \subset M$. Define a quality parameter $t_0 > 0$, and let

$$h = \min \left\{ \frac{\iota_M}{4}, \frac{\sqrt{n} t_0}{6\sqrt{\Lambda}} \right\}.$$

If

1. *For every $p \in \mathsf{P}$, the vertices of $\underline{\mathrm{St}}(p)$ are contained in $B_M(p;h)$, and the balls $\{B_M(p;h)\}_{p \in \mathsf{P}}$ cover M.*
2. *For every $p \in \mathsf{P}$, the restriction of the inverse of the exponential map $\exp_p^{-1}$ to the vertices of $\underline{\mathrm{St}}(p) \subset \mathcal{A}$ defines a piecewise linear embedding of $|\underline{\mathrm{St}}(p)|$ into T_pM, realising $\underline{\mathrm{St}}(p)$ as a full star such that every simplex $\sigma(p)$ has thickness $t(\sigma(p)) \geq t_0$.*

then $\mathcal{A}$ triangulates M, and the triangulation is given by the barycentric coordinate map on each simplex.

Theorem (Metric distortion)

If the requirements of the Triangulation Theorem, are satisfied with the scale parameter h replaced by

$$h = \min\left\{ \frac{\iota_M}{4}, \frac{t_0}{6\sqrt{\Lambda}} \right\},$$

then $\mathcal{A}$ is naturally equipped with a piecewise flat metric $d_{\mathcal{A}}$ defined by assigning to each edge the geodesic distance in M between its endpoints. If $H : |\mathcal{A}| \to M$ is the triangulation defined by the barycentric coordinate map in this case, then the metric distortion induced by H is quantified as

$$|d_M(H(x), H(y)) - d_{\mathcal{A}}(x, y)| \leq \frac{50\Lambda h^2}{t_0^2} d_{\mathcal{A}}(x, y),$$

for all $x, y \in |\mathcal{A}|$.

Thank You.

Bibliographic notes

Karcher means

- Cartan (1928); Frechet (1948); Karcher (1977); Kendall (1990)
- Buser and Karcher (1981); Peters (1984); Chavel (2006)

Riemannian simplices

- Berger (2003)
- Rustamov (2010); Sander (2012)
- von Deylen (2014)

Theorem (Intrinsic simplex triangulation criteria)

Suppose M is a compact n-dimensional Riemannian manifold with sectional curvatures K bounded by $|K| \leq \Lambda$, and $\mathcal{A}$ is an abstract simplicial complex with finite vertex set $\mathsf{P} \subset M$. Define a quality parameter $t_0 > 0$, and let

$$h = \min\left\{ \frac{\iota_M}{4}, \frac{t_0}{8\sqrt{\Lambda}} \right\}.$$

If

1. *For every simplex $\sigma = \{p_0, \ldots, p_n\} \in \mathcal{A}$, the edge lengths $\ell_{ij} = d_M(p_i, p_j)$ satisfy $\ell_{ij} < h$, and they define a Euclidean simplex $\boldsymbol{\sigma}_{\mathbb{E}}$ with $t(\boldsymbol{\sigma}_{\mathbb{E}}) \geq t_0$.*
2. *The balls $\{B_M(p; h)\}_{p \in \mathsf{P}}$ cover M, and for each $p \in \mathsf{P}$ the secant map of $\exp_p^{-1}$ realises $\underline{\mathrm{St}}(p)$ as a full star.*

then $\mathcal{A}$ triangulates M, and the triangulation is given by the barycentric coordinate map on each simplex.